Higher spin operators in conformal field theories

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Physics
Adviser: Simone Giombi

November 2018

Abstract

In this thesis we study higher spin operators in conformal field theories with weakly broken higher spin symmetry. Exact higher spin symmetry is known to be very constraining, containing the usual conformal group as a subgroup, and actually enforcing the theory to be free. Moreover, even in the presence of some weak, perturbative breaking of this symmetry, it still constrains the correlation functions.

In particular, it simplifies the calculation of anomalous dimensions of those currents to the lowest order in perturbation theory, reducing it to a two-point function of the non-conservation operator corresponding to the current. We apply this method to a variety of vector models, both bosonic and fermionic. Specifically, we reproduce some known results for Wilson-Fisher model in $4 - \epsilon$ expansion, as well as the $1/N$ expansion of the model and use those to interpolate with good accuracy to $d = 3$ Ising model. We also get some new results for non-linear sigma model in $2 + \epsilon$ expansion and cubic models in $6 - \epsilon$. We also apply this technique to fermionic models, the Gross-Neveu-Yukawa model in $4 - \epsilon$ expansion and $1/N$ expansion. Further, we use a combination of direct Feynman diagram calculation and analytic bootstrap methods to calculate the anomalous dimension of some composite operators.

In the last chapter, we study vector models in $d = 3$ with a Chern-Simons interaction, which were an object of close study recently as a testing ground of a whole family of boson-fermion dualities. In particular, these dualities imply a matching of anomalous dimensions and three-point functions of higher spin currents between bosonic and fermionic theories under a certain mapping of the Chern-Simons coupling and the rank of the gauge group. We confirm those predictions using both the non-conservation operator formalism and a direct Feynman diagram calculation.

Acknowledgements

I wish to express my gratitude to people guiding me throughout my journey as a physicist. First and foremost to my advisers: Valentin Ivanovich Zakharov, who introduced me to research and a scientist's mindset and Simone Giombi, guiding me through the many calculations which constitute this thesis. I'm also indebted to my school, Lyceum School 2, which not only pushed me to develop my love to Math and Physics but also provided me with lifelong friends (Sanya, Senya, Gika, Larin, you know I'm talking about you). All my colleagues in ITEP and Jadwin Hall: I've enjoyed talking to you about Physics and perhaps more often about other things as much as it looked I have. My father Pavel: you supported me when I needed it the most. I try to look up to you in all I do.

To Dasha, Pavlik and Nika.

Contents

Chapter 1

Introduction

Much of the physics of 20-th (and, at least so far, 21-st) century was concentrated on the study of critical phenomena. It was marked by two fundamental breakthroughs: the classical field theoretic model of symmetry breaking proposed by Landau and Ginzburg [1], which provided a non-quantized, thermodynamic description of second-order phase transitions (or in a more moderate language, critical points); and the discovery of renormalization group by Wilson [2], where a path-integral, fully quantum-field-theoretical treatment was given. Together with a plethora of related work, these concepts provide an understanding of a concept of universality, that is different physical systems behaving in the same way nearby the critical point.

A very handwavy explanation as to what conformal theories have to do with critical phenomena can be formulated as follows. In a typical thermodynamic setting, critical points occur at the end of a continuous line of first-order phase transitions, where some characteristic energy scale of the transition, such as the latent heat for water-vapor phase transition, vanishes. Amusingly enough, at this point the correlation length ξ of, say, density fluctuations becomes infinite. These two facts are of course related, owing to the relation $l \sim 1/E$ in natural units. In other words, in the vicinity of critical points physical observables (such as the correlation functions, from which other observables can be derived) can no longer depend on ξ, but have to capture something characteristic of phase transition itself. More concretely, away from a critical point, a typical correlation function of a local quantity $O(x)$ behaves like

$$\langle O(x)O(y)\rangle \propto e^{-|x-y|/\xi} \tag{1.0.1}$$

whereas in the absence of ξ the only functional form you would be forced to write would be

$$\langle O(x)O(y)\rangle \propto \frac{1}{|x-y|^{\alpha}} \tag{1.0.2}$$

This argument establishes relevance of studying field theories with scale invariance. From a Lagrangian perspective, that would seem to be any theory without dimensionful parameters. At the quantum level, however, there is a subtlety: an energy scale can be dynamically generated. Without diving into details for now, we just mention that scale invariance requires the vanishing of beta functions of the theory:

$$\frac{\partial g_i}{\partial \log \Lambda} = 0 \tag{1.0.3}$$

for all couplings g_i. This connects to the second point mentioned above: that scale invariant theories represent ending points of the renormalization group flows, where couplings no longer evolve.

1.1 Conformal invariance

With the observations made above, we now point out that empirically scale invariant critical theories also exhibit full conformal invariance (under certain assumptions such as unitarity). With those assumptions there exists a proof of that statement in $d = 2$ [3,4]. In other dimensions the situation is more complicated, with varying amount of evidence or proof available. Perhaps the most compelling argument was given for $d = 4$ in [5]. We refer the reader to the overview [6] as well as original papers for $d = 3$ [7,8] and $d = 6$ [9]. For now we will take that symmetry enhancement for granted and describe the construction of the conformal group, following [10] and [11].

Conformal transformations of d-dimensional $\mathbb{R}^d$ space with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ are defined as coordinate transformations $x \to x'$ which preserve the metric up to an overall conformal factor $\Lambda(x)$:

$$\eta_{\rho\sigma}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \Lambda(x)\eta_{\mu\nu} \tag{1.1.1}$$

It's straightforward to observe that scale transformation $x \to \lambda x$ trivially satisfies this equation with $\Lambda(x) = \lambda^{-2}$. The Poincaré group transformations also do, with $\Lambda(x) = 1$.

A key new (relative to scale and Poincaré) transformation which completes the conformal group is the so called special conformal transformation (SCT):

$$x'^{\mu} = \frac{x^{\mu} - x^2 b^{\mu}}{1 - 2(b \cdot x) + b^2 x^2} \tag{1.1.2}$$

with the scale factor

$$\Lambda(x) = (1 - 2(b \cdot x) + b^2 x^2)^2 \tag{1.1.3}$$

To rewrite in a more intuitive way:

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu} \tag{1.1.4}$$

which means shows that SCT is just a composition of inversion, translation and another inversion. This remark simplifies the analysis of constraints imposed by the conformal group on the correlation functions: for instance, the coordinate dependence of 2-, 3- and 4-point functions will be universal up to (possible) functions of inversion-invariant coordinate combinations. Under the inversion, the distance between two points $x_{12} = |x_1 - x_2|$ transforms as:

$$x'^2_{12} = \frac{x^2_{12}}{x^2_1 x^2_2} \tag{1.1.5}$$

It is also possible to write down the action of the conformal group on local fields. Omitting the details, imposing that a field belongs to an irreducible representation of the Lorentz group (for Euclidean signature this would be just $SO(d)$ rotations) [1], we construct the so-called quasi-primary fields. Under a conformal transformation these fields transform as:

$$\phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\Delta}{d}} \phi(x) \tag{1.1.6}$$

Here we restrict our attention to scalar fields, we will return to fields with spin later. Notice that Δ in this definition coincides with scaling dimension: under $x' = \lambda x$ transformation we get:

$$\phi(\lambda x) = \lambda^{-\Delta} \phi(x) \tag{1.1.7}$$

Using those transformation laws and imposing conformal symmetry, it is possible to restrict correlation

[1] Simply put, it means that the field has the amount of indices equal to its spin, eg it's a vector field and not a derivative of a scalar.

functions. For the two point function of two scalar primaries of dimensions Δ_1, Δ_2 we get:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}\delta_{\Delta_1\Delta_2}}{x_{12}^{\Delta_1+\Delta_2}} \tag{1.1.8}$$

which is only non-zero for $\Delta_1 = \Delta_2$. For three-point functions we get

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.1.9}$$

We will refer in future to C_{12} and C_{123} as two and three-point coefficients. Note that it's possible to "normalize away" C_{12}, since only the combination

$$\frac{C_{123}^2}{C_{12}C_{23}C_{31}} \tag{1.1.10}$$

is physical. Finally, for a four-point function one gets

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} F(u,v) \tag{1.1.11}$$

Here u and v are the conformally invariant cross-ratios [2] given by

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v \equiv \frac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2} \tag{1.1.12}$$

It is possible to impose constraints on higher-point functions. However, it is less relevant as the operator-product expansion allows to reduce these two a combination of two- and three-point functions. Most of the dynamics of a conformal field theory (CFT) is encoded in such an expansion of a four-point function. It is also possible to impose similar constraints for fields with spin. We come back to it in later chapters. We conclude those introductory remarks by noticing that promoting conformal invariance from a classical to a quantum case is not automatic: it breaks down for theories with non-vanishing beta-functions. It doesn't mean that one is restricted to studying non-interacting conformal field theories however. The most physically interesting case is the study of critical points, where two or more terms in the beta function cancel out at some value of couplings, giving rise to a fully interacting CFT. Finally, the concept of radial quantization described in all classical textbooks on CFT (eg [10]) allows to define a Hilbert space of states in a CFT and thus provides a mapping between local fields

[2] Inversion (and thus conformal) invariance follows from (1.1.5).

and operators. We will therefore use those two terms interchangeably further on.

1.2 Higher spin operators

We will now turn our attention to operators with spin. This is accomplished by allowing the quasi-primary fields mentioned in the previous chapter to transform in some non-trivial representation R of $SO(d)$. We then conclude that the spectrum of a d-dimensional conformal field theory [12] consists of local operators labeled by conformal dimension Δ, representation R, and possibly a representation r_G of an internal global symmetry group. The precise determination of the spectrum of an interacting CFT is of fundamental importance. Together with the knowledge of the OPE coefficients, it essentially amounts to a solution of the CFT.

It is well-known that in a unitary CFT$_d$, dimensions of primary operators satisfy certain inequalities known as unitarity bounds [13, 14]. For a spin s operator $J_{\mu_1\mu_2\cdots\mu_s}$ in the symmetric traceless representation of $SO(d)$, the unitarity bound is

$$\Delta_s \geq d-2+s\,, \qquad s\geq 1\,. \tag{1.2.1}$$

For a scalar operator $\mathcal{O}$, it reads

$$\Delta_0 \geq \frac{d}{2}-1 \tag{1.2.2}$$

and one may derive similar bounds for more general representations of $SO(d)$. When these inequalities are saturated, the corresponding operator satisfies some differential equation, and it belongs to a short representation of the conformal algebra. In the case of a scalar operator, the shortening condition is simply the wave equation $\partial^2\mathcal{O}=0$, i.e. $\mathcal{O}$ is a free scalar field. For a spin s operator ($s\geq 1$), saturation of the bound (1.2.1) implies that it is a conserved current

$$\partial^\mu J_{\mu\mu_2\cdots\mu_s}=0\,. \tag{1.2.3}$$

The cases $s=1$ (an exactly conserved current with $\Delta=d-1$) and $s=2$ (the conserved stress tensor with $\Delta=d$) are familiar in any CFT. Conserved currents of higher spins $s>2$ are explicitly realized in free field theories. For example, in a free scalar CFT they take the schematic form

$$J_{\mu_1\cdots\mu_s} = \sum_{k=0}^{s} c_{sk}\partial_{\{\mu_1}\cdots\partial_{\mu_k}\phi\partial_{\mu_{k+1}}\cdots\partial_{\mu_s\}}\phi\,, \tag{1.2.4}$$

where brackets denote traceless symmetrization and the coefficients may be determined by the conservation equation, as we review in Section 2.1. It is evident that these operators have exact dimension $\Delta_s = d-2+s$ in the free theory. As usual, conserved currents correspond to symmetries of the theory. The presence of exactly conserved currents of all spins implies that the CFT has an infinite dimensional higher spin symmetry which includes the conformal symmetry as a subalgebra. Higher spin symmetries turn out to be very constraining. One may prove that if a CFT possesses a spin 4 conserved current, then an infinite tower of conserved higher spin operators is present, and all correlation functions of local operators coincide with those of a free CFT [15–19].

In an interacting CFT, the higher spin operators are not exactly conserved and acquire an anomalous dimension

$$\Delta_s = d-2+s+\gamma_s\,. \tag{1.2.5}$$

An interesting class of models are those for which the higher spin symmetries are slightly broken. By this we mean that there is an expansion parameter g, playing the role of a coupling constant, such that for small g the anomalous dimensions $\gamma_s(g)$ are small, and in the $g \to 0$ limit one recovers exact conservation of the currents. Explicit examples are weakly coupled fixed points of the Wilson-Fisher type [20], where g corresponds to a power of ϵ, or certain large N CFT's, where g is related to a power of $1/N$. At the operator level, this implies that the non-conservation equation for the spin s operator takes the form

$$\partial\cdot J_s = gK_{s-1} \tag{1.2.6}$$

where K_{s-1} is an operator of spin $s-1$, and we factored out g to highlight the fact that at $g=0$ the current is conserved. The slightly broken higher spin symmetries (1.2.6) can still be used to put non-trivial constraints on the correlation functions [21]. The equation (1.2.6) also gives an efficient way to determine the anomalous dimensions γ_s to leading order in the small parameter g [22, 23]. As we show below, using (1.2.6) and conformal symmetry one readily finds that $\gamma_s(g) \propto g^2$, where the proportionality constant is simply obtained by computing the two point functions $\langle K_{s-1}K_{s-1}\rangle$ at $g=0$. This method is similar in spirit to the one recently advocated in [24], where the leading anomalous dimension of ϕ at the Wilson-Fisher fixed point in $d = 4 - \epsilon$ was reproduced by using conformal symmetry, without explicit input from perturbation theory.

In addition to their intrinsic interest and their relevance in statistical mechanics, the higher-spin operators we study in the thesis also play an important role in the context of the AdS/CFT correspondence. According to a well understood entry of the AdS/CFT dictionary, exactly conserved currents of spin s in CFT$_d$ are dual to massless spin s gauge fields in AdS$_{d+1}$. Interacting higher spin gauge theories in AdS$_{d+1}$ were explicitly constructed by Vasiliev [25–28], and a class of them were naturally conjectured [29] to be dual to the singlet sector of the free $O(N)$ vector model (we will describe the AdS/CFT interpretation for fermionic and Chern-Simons models in chapter 3 and 4 respectively). The exactly conserved currents J_s with $s = 2, 4, 6, \ldots$ are dual to the corresponding massless gauge fields in the Vasiliev theory, and the scalar operator $J_0 = \phi^i\phi^i$ to a bulk scalar field with $m^2 = -2(d-2)/\ell_{AdS}^2$. As further conjectured in [29], one may extend this duality to the interacting case, obtained by adding to the free theory the "double-trace" interaction $\lambda(\phi^i\phi^i)^2$. For $d < 4$ there is a flow to an interacting IR fixed point[3] which is conjectured to be dual to the same Vasiliev theory but with alternate boundary conditions on the bulk scalar field [31]. A distinguishing feature of large N interacting vector models is that the descendant operator K_{s-1} appearing in the non-conservation equation is a "double-trace" operator, schematically

$$\partial \cdot J_s = \frac{1}{\sqrt{N}} \sum JJ \,. \tag{1.2.7}$$

This implies that the anomalous dimensions are $\gamma_s \sim O(1/N)$, which corresponds to a quantum breaking of the higher spin gauge symmetry in the bulk: the higher spin gauge fields acquire masses through loop corrections[4] when the bulk scalar is quantized with the alternate boundary conditions. In representation theory language, the equation (1.2.6) means that the short representation of the conformal algebra with $(\Delta = d-2+s, s)$ combines with the representation $(\Delta = d-1+s, s-1)$ to form a long multiplet with $(\Delta > d-2+s, s)$. In the bulk, this phenomenon corresponds to a higher spin version of the Higgs mechanism [32]: the gauge field swallows a spin $s-1$ Higgs field to yield a massive spin s field. The fact that the operator on the right-hand side of (1.2.7) is double-trace implies that the Higgs field is a composite two-particle state, and the breaking is subleading at large N. This

[3]For $4 < d < 6$ there is a flow to a (presumably metastable) perturbatively unitary, UV fixed point [30].

[4]In the higher spin/CFT dualities, the bulk Newton's constant G_N scales as $1/N$. The mass of a spin s field in AdS$_{d+1}$ is related to the dual conformal dimension by $(\Delta_s + s - 2)(\Delta_s + 2 - d - s) = m_s^2\ell_{AdS}^2$, which implies $m_s^2\ell_{AdS}^2 \sim 1/N$ for $\gamma_s \sim 1/N$. To leading order at large N, $m_s^2\ell_{AdS}^2 \approx (2s+d-4)\gamma_s$.

is different from theories of Yang-Mills type, where K_{s-1} in (1.2.6) is a single trace operator, and the anomalous dimensions are non-zero already at planar level. In the bulk, this would correspond to a tree-level Higgs mechanism.

1.3 Outline

We begin by setting up some general notation and the method used to fix the anomalous dimensions of the higher-spin currents in particular models. In Chapter 2, based on [33], co-authored with S. Giombi, we apply that machinery to a family of bosonic $O(N)$ vector models in dimensions between 2 and 6. In Chapter 3, based on [34] with E. Skvortsov and S. Giombi, we study various fermionic models. Finally, in Chapter 4, based on [35] with S. Giombi, E. Skvortsov, S. Prakash and V. Gurucharan, we calculate the $1/N$ anomalous dimensions for Chern-Simons vector models in $d = 3$ and use them to confirm a particular case of $d = 3$ boson-fermion dualities.

1.4 General method

In this section we will setup the definitions and notations which will then be applied to the particular models. We will also describe the derivation of the master formula which allows to calculate the lowest-order value of the anomalous dimensions without doing any loop calculations.

1.4.1 Constructing higher spin currents

We first introduce some useful technology for the manipulation of symmetric tensors. For a given a rank s tensor $J_{\mu_1\mu_2\cdots\mu_s}$ in the symmetric traceless representation, we may introduce an auxiliary "polarization vector" z^μ, which can be taken to be null ($z^2 = 0$), and construct the index-free projected tensor

$$\hat{J}_s \equiv J_{\mu_1\cdots\mu_s} z^{\mu_1} \cdots z^{\mu_s}\,, \qquad z^2 = 0\,. \tag{1.4.1}$$

It is evident that the multiplication by z^μ selects only the symmetric traceless part of $J_{\mu_1\cdots\mu_s}$. One may always go back to the full tensor by "stripping off" the null vectors and subtracting traces. In practice, this can be done efficiently with with the help of the following differential operator in

z-space [23, 36–38] (sometimes called Thomas derivative):

$$D_z^\mu \equiv \left(\frac{d}{2}-1\right)\partial_{z_\mu} + z^\nu \partial_{z_\nu}\partial_{z_\mu} - \frac{1}{2} z^\mu \partial_{z_\nu}\partial_{z_\nu}\,. \tag{1.4.2}$$

Acting once with this operator removes a z^μ, thus freeing one index of the tensor, while taking into account the constraint $z^2 = 0$. The unprojected $J_{\mu_1\cdots\mu_s}$ can thus be recovered via

$$J_{\mu_1\mu_2\cdots\mu_s} \propto D^z_{\mu_1} D^z_{\mu_2} \cdots D^z_{\mu_s} \hat{J}_s\,. \tag{1.4.3}$$

The symmetrization and tracelessness of the operator obtained this way is ensured by the properties

$$[D_z^\mu, D_z^\nu] = 0, \qquad D_z^\mu D^z_\mu = 0. \tag{1.4.4}$$

Similarly, the conservation equation (1.2.3) of the spin s operator may be written compactly in this notation as

$$\partial_\mu D_z^\mu \hat{J}_s = 0. \tag{1.4.5}$$

This equation is satisfied in any free field theory (bosonic or fermionic).

1.4.2 Anomalous dimensions of the weakly broken currents

Let us consider a CFT with a parameter g playing the role of a coupling constant, such that in the $g = 0$ limit there are exactly conserved currents J_s. When a non-zero coupling g is turned on, the currents will be no longer conserved for general s and acquire anomalous dimensions

$$\Delta_s = d - 2 + s + \gamma_s(g)\,. \tag{1.4.6}$$

The non-conservation of the currents means that a non-zero operator of spin $s-1$ must appear on the right hand side of (1.2.3), or equivalently (1.4.5)

$$\partial_\mu D_z^\mu \hat{J}_s = g\hat{K}_{s-1}\,, \tag{1.4.7}$$

where we have pulled out an explicit factor of g in front of the descendant to stress that the right hand side vanishes when $g = 0$. Here g is assumed to be a small expansion parameter, and may be either a power of ϵ in the Wilson-Fisher type models, or a power of $1/N$ in the large N approach. We now proceed by noting that in a CFT the form of the two-point function of the spin s operators is fixed by conformal symmetry to be

$$\langle \hat{J}_s(x_1)\hat{J}_{s'}(x_2)\rangle = \delta_{ss'}C(g)\frac{\hat{I}^s}{(x_{12}^2)^{\Delta_s}} \tag{1.4.8}$$

where

$$\hat{I} = I_{\mu\nu}z_1^\mu z_2^\nu\,, \qquad I_{\mu\nu} = \eta_{\mu\nu} - 2\frac{x_{12}^\mu x_{12}^\nu}{x_{12}^2}\,. \tag{1.4.9}$$

Acting on this two-point function with $\partial_\mu D_z^\mu$ on both operators (with different projection vectors z_1 and z_2), one gets, using the form of the non-conservation equation (1.4.7)

$$\partial_{1\mu}D^\mu_{z_1}\partial_{2\mu}D^\mu_{z_2}\langle \hat{J}_s(x_1)\hat{J}_s(x_2)\rangle = g^2\langle \hat{K}_{s-1}(x_1)\hat{K}_{s-1}(x_2)\rangle. \tag{1.4.10}$$

On the other hand, differentiating the right hand side of (1.4.8), setting $z_1 = z_2$ at the end, and dividing by the two-point function of $J's$, one finds the relation [22, 23]

$$g^2\hat{x}^2\frac{\langle \hat{K}_{s-1}(x_1)\hat{K}_{s-1}(x_2)\rangle}{\langle \hat{J}_s(x_1)\hat{J}_s(x_2)\rangle} = -\gamma_s(g^2)s(s+d/2-2)\big[(s+d/2-1)(s+d-3) + \gamma_s(g^2)(s^2+sd/2-2s+d/2-1)\big]. \tag{1.4.11}$$

The right-hand side being proportional to γ_s is not a coincidence and follows from the conservation of the higher-spin current at zero coupling (1.4.5). From a CFT standpoint, (1.4.11) is an exact relation. In practice, when doing perturbation theory in g, one computes the correlators on the left hand side in powers of the coupling. It is then evident that (1.4.11) allows to gain an order in perturbation theory. To obtain the anomalous dimensions of the broken currents to leading order in g, one has simply to evaluate ratio of correlators in the free theory, $g = 0$. In particular, this only involves finite tree-level correlators, avoiding the issues of regularization and renormalization.

1.4.3 Universal large spin behavior

All of the results in this thesis can be compared to the universal prediction of large spin behavior of operators of the form $\phi\hat{\partial}^s\phi$ done in [39, 40]. More concretely, the leading order $s \to \infty$ asymptotic for the twist, $\tau_s = \Delta_s - s$ and consequently the anomalous dimension of higher-spin currents is established:

$$\tau_s = 2\tau_\phi - \frac{c_{\tau_{min}}}{s^{\tau_{min}}} + \ldots , \tag{1.4.12}$$

The coefficient $c_{\tau_{min}}$ is given by the formula (3.18) of [39] which reads in terms of two- and three-point function coefficients

$$c_{\tau_{min}} = \frac{\Gamma(\tau_{min} + 2s_{min})}{2^{s_{min}-1}\Gamma(\frac{\tau_{min}+2s_{min}}{2})^2} \frac{\Gamma(\Delta)^2}{\Gamma(\Delta - \frac{\tau_{min}}{2})^2} \frac{C^2_{OOO_{\tau_{min}}}}{C^2_{OO} C_{O_{\tau_{min}} O_{\tau_{min}}}} \tag{1.4.13}$$

where C_{OO} and $C_{O_{\tau_{min}} O_{\tau_{min}}}$ are the two-point function coefficients of O and $O_{\tau_{min}}$ operators and $C_{OOO_{\tau_{min}}}$ is the three-point function coefficient (the fraction in (1.4.13) is the one appearing in the conformal block expansion of the four-point function of O's, and the derivation of (1.4.13) uses the contribution to of $O_{\tau_{min}}$ block to the four-point function in a certain limit.)

Let us briefly review the derivation of this result, closely following the method used in [41]. We study the four-point function of four scalar operators O:

$$v^{\Delta}\left(1 + \sum_{\tau,s} a_{\tau,s} u^{\tau/2} f_{\tau,s}(u,v)\right) = u^{\Delta}\left(1 + \sum_{\tau,s} a_{\tau,s} v^{\tau/2} f_{\tau,s}(v,u)\right) \tag{1.4.14}$$

where the contribution of the identity operator is isolated and the conformal blocks are written as $g_{\tau,s}(u,v) = u^{\tau/2} f_{\tau,s}(u,v)$ to emphasize the small-u behavior (functions $f_{\tau,s}$ stay finite as $u \to 0$).

Now we will study this equation in a double-scaling limit, where we first send v to 0 and then u. As shown in [39, 40], the presence of the identity operator in the crossed channel gives rise to double-trace higher spin operators constructed from O, $[O,O]_s$, with twists $\tau_s = 2\Delta_O + \gamma_s$ where the anomalous dimension γ_s vanishes as $s \to \infty$. The right-hand side will in turn be dominated by the contribution of the operator with minimum twist τ_{min}, such that the most singular contributions as $v \to 0$ of (1.4.14) take the form:

$$\sum_s a_s u^{\gamma_s/2} f^{(s)}_{coll}(v)|_{v\to 0} = a_{\tau_{min}} v^{\tau_{min}/2-\Delta} f_{\tau_{min}}(v,u)|_{v\to 0} \tag{1.4.15}$$

where a are the normalized three-point coefficients (as in the last term of (1.4.13)) and the collinear conformal blocks $f^{(s)}_{coll}$ are given by (see eg [42]):

$$a_s = \frac{2\Gamma(\Delta + s + \frac{\gamma_s}{2})^2\Gamma(2\Delta + 2 + \frac{\gamma_s}{2} - 1)}{\Gamma(s + \frac{\gamma_s}{2} + 1)\Gamma(\Delta)^2\Gamma(2\Delta + 2s + \gamma_s - 1)} \tag{1.4.16}$$

$$f^{(s)}_{coll}(v) = (1-v)^s {}_2F_1(\Delta + s + \gamma_s/2, \Delta + s + \gamma_s/2, 2\Delta + 2s + \gamma_s; 1-v) \tag{1.4.17}$$

A key observation is that each conformal block only diverges logarithmically in the second argument (v on LHS, u on RHS). Thus we need a sum of logarithmically divergent terms to reproduce a power-like $v^{\tau_{min}/2-\Delta}$ on RHS. This is only possible if infinitely many terms on LHS contribute. Now we proceed with small u expansion, extracting only $\log u$ pieces:

$$\sum_s a_s \frac{\gamma_s}{2} \log u f^{(s)}_{coll}(v)|_{v\to 0} = a_{\tau_{min}} v^{\tau_{min}/2-\Delta} f_{\tau_{min}}(v,u)|_{\substack{v\to 0 \\ \log u, u\to 0}} \tag{1.4.18}$$

The RHS is known to be (see [41, 43]):

$$f_\tau(v,u)|_{\substack{v\to 0 \\ \log u, u\to 0}} = -\frac{\Gamma(\tau+2s)}{(-2)^s\Gamma(\frac{\tau+2s}{2})^2} \log u + O(v) \tag{1.4.19}$$

As a last step, we convert the sum over the spin s in (1.4.18) into an integral over the variable $j = s\sqrt{v}$ at large s, using the following form for the hypergeometric function:

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} \tag{1.4.20}$$

so that (1.4.18) takes the form:

$$\int_0^\infty dj K^\Delta(j,v) \frac{\gamma(\frac{j}{\sqrt{v}})}{2} = -\frac{\Gamma(\tau_{min} + 2s_{min})}{(-2)^s\Gamma(\frac{\tau_{min}+2s_{min}}{2})^2} v^{\tau_{min}/2}(1 + O(v)) \tag{1.4.21}$$

where the kernel $K^\Delta(j,v)$ is complicated expression given in [41]. This equation determines γ_s order by order in $1/s$ by applying the inverse of the kernel $K^\Delta(j,v)$ to the RHS. However, for the purpose of determining the leading order behavior, it's sufficient to point out that

$$\int_0^\infty dj K^\Delta(j,v) \left(\frac{v}{j^2}\right)^n = v^n \frac{\Gamma(\Delta-n)^2}{\Gamma(\Delta)^2} + O(v^{n+1}) \tag{1.4.22}$$

It's evident then that we can just set $n = \tau_{min}/2$ in the equation above, such that

$$\gamma_s = -\frac{c_{\tau_{min}}}{s^{\tau_{min}}} \tag{1.4.23}$$

and then upon collecting all the coefficients the formula (1.4.13) is reproduced exactly (the $(-1)^{s_{min}}$ is not important for the case of four identical scalar operators, as only even spins can be exchanged in the four-point function). Notice that it's possible to calculate all the terms in the $\log u$ part of the conformal block (1.4.19), as well as extra terms in (1.4.22). This allows to systematically extract all terms in the $1/s$ expansion of γ_s. For this and other details of the calculation see [41].

Chapter 2

Anomalous dimensions in the bosonic vector models

2.1 Introduction and Summary

In this chapter, we apply the general method to the explicit example of interacting scalar field theories with $O(N)$ symmetry in various dimensions. These include the familiar Wilson-Fisher fixed point of the ϕ^4 theory in $d = 4 - \epsilon$, the large N expansion of the critical $O(N)$ model in arbitrary dimension d, the perturbative IR fixed points of the cubic $O(N)$ models in $d = 6 - \epsilon$ [30], and the UV fixed point of the non-linear sigma model in $d = 2 + \epsilon$. In all these examples, we determine the explicit structure of the non-conservation equation (1.2.6) and use it to find the leading order anomalous dimensions of higher spin operators in the singlet, symmetric traceless and antisymmetric representations of $O(N)$. Many of our findings were obtained before by different methods [44, 45], but the results in the cubic models in $d = 6 - \epsilon$ and in the nonlinear sigma model in $d = 2 + \epsilon$ are new as far as we know. In all examples, we pay particular attention to the large spin behavior of the anomalous dimensions, finding precise agreement with general expectations [39, 40, 46–48]. Combining information from the $d = 4 - \epsilon$ and $d = 2 + \epsilon$ expansions, as well as some input from the large spin limit, in Section 7 we also obtain some estimates for the dimension of the singlet higher spin operators in the $d = 3$ $O(N)$ models for a few low values of spin s and N.

2.1.1 The higher spin currents in free field theory

Let us now construct the explicit conserved higher spin currents in the free CFT of N real massless scalar fields. They satisfy the free wave equation

$$\partial^2 \phi^i = 0, \qquad i = 1, \ldots, N \tag{2.1.1}$$

and there is a $O(N)$ global symmetry under which ϕ^i transforms in the fundamental representation. This free CFT admits an infinite tower of exactly conserved higher spin operators (1.2.4), which are bilinears in the scalars with a total of s derivatives acting on the fields. Projecting indices with the null vector z^μ, these operators can be written as

$$\hat{J}^{ij} = \sum_{k=0}^{s} c_{sk} \hat{\partial}^{s-k} \phi^i \hat{\partial}^k \phi^j \tag{2.1.2}$$

where we have introduced the projected derivative $\hat{\partial} = \partial_\mu z^\mu$, and c_{sk} are coefficients that will be fixed shortly. Of course, one can separate this operator into irreducible representations of $O(N)$, as discussed in more detail below. It is convenient to rewrite (2.1.2) in the following form

$$\begin{aligned} \hat{J}^{ij}_s &= f_s(\hat{\partial}_1, \hat{\partial}_2) \phi^i(x_1) \phi^j(x_2) \Big|_{x_1, x_2 \to x} \\ f_s(u,v) &= \sum_{k=0}^{s} c_{sk} u^{s-k} v^k \,, \qquad u = \hat{\partial}_1, v = \hat{\partial}_2. \end{aligned} \tag{2.1.3}$$

where we have encoded the coefficients c_{sk} into the function $f_s(u,v)$. Now the conservation equation (1.4.5) may be turned into a differential equation for the function f_s, which, upon using the free equation of motion $\partial^2 \phi = 0$, reduces to[1]

$$\big((d/2-1)(\partial_u + \partial_v) + u\partial_u^2 + v\partial_v^2\big) f_s = 0. \tag{2.1.4}$$

The following ansatz for f_s is convenient

$$f_s = (u+v)^s \phi_s\Big(\frac{u-v}{u+v}\Big), \tag{2.1.5}$$

[1] The same equation may be obtained by requiring that J_s is a conformal primary, see e.g. [37].

and results in the ordinary differential equation

$$\Big((1-t^2)\frac{d^2}{dt^2}-(d-2)t\frac{d}{dt}+s(s+d-3)\Big)\phi_s(t)=0\,. \tag{2.1.6}$$

The solution to this equation is given by the order s Gegenbauer polynomials, $\phi_s(t)=C_s^{d/2-3/2}(t)$, which are even (odd) for even (odd) s. Hence, up to the overall normalization, one gets the following expressions for the conserved higher spin currents

$$\hat{J}_s^{ij}=(\hat{\partial}_1+\hat{\partial}_2)^s C_s^{d/2-3/2}\Big(\frac{\hat{\partial}_1-\hat{\partial}_2}{\hat{\partial}_1+\hat{\partial}_2}\Big)\phi^i(x_1)\phi^j(x_2)\Big|_{x_1,x_2\to x}. \tag{2.1.7}$$

One may also write

$$(u+v)^s C_s^{d/2-3/2}\left(\frac{u-v}{u+v}\right)= \tag{2.1.8}$$
$$=\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}+s-1\right)\Gamma(d+s-3)}{2^{d-4}\Gamma\left(\frac{d-3}{2}\right)}\sum_{k=0}^{s}\frac{(-1)^k u^{s-k}v^k}{k!(s-k)!\,\Gamma\left(k+\frac{d}{2}-1\right)\Gamma\left(s-k+\frac{d}{2}-1\right)}$$

from which one can read-off the coefficients c_{sk} in (2.1.2) if desired. The overall normalization is arbitrary at this level. Note that one feature of the form (2.1.7) is that it vanishes at $d=3$, see the factor in front of the sum in (2.1.8). This vanishing is not meaningful, one could always remove it by normalizing the currents differently. For the explicit calculations below, we find it more convenient to use the form (2.1.7) in terms of Gegenbauer polynomials.

The higher spin operators may be decomposed into symmetric traceless, antisymmetric and singlet of $O(N)$

$$J_s^{ij}=J_s^{(ij)}+J_s^{[ij]}+J_s \tag{2.1.9}$$

where $J_s\equiv J_s^{ii}$ denotes the singlet current. It is evident by symmetry that the singlet and symmetric traceless representations only exist for even spin, and the antisymmetric one for odd spins. For $s=1$, the antisymmetric operator $J_1^{[ij]}$ is nothing but the familiar conserved current corresponding to the $O(N)$ global symmetry. The presence of the conserved currents of all spins implies that the free CFT has an infinite dimensional exact higher spin symmetry. The generators can be constructed in a canonical way as follows. First, by contracting a spin s current with a spin $s-1$ conformal Killing tensor $\zeta^{\mu_1\cdots\mu_{s-1}}$,[2] we may obtain an ordinary current $J^{\zeta}_{\mu,s}=J_{\mu\mu_2\cdots\mu_s}\zeta^{\mu_2\cdots\mu_s}$, which is conserved as it

[2]A conformal Killing tensor is a symmetric tensor satisfying $\partial_{(\mu_1}\zeta_{\mu_2\cdots\mu_s)}=\frac{s-1}{d+2s-4}g_{(\mu_1\mu_2}\partial^{\nu}\zeta_{\mu_3\cdots\mu_s)\nu}$.

is easily checked. From this, one can get a conserved charge Q_s^ζ in the usual way. For instance, for $s = 2$, the singlet current J_2 is proportional to the traceless stress tensor of the CFT, and contracting this with the linearly independent conformal Killing vectors one gets the $(d+2)(d+1)/2$ generators of the conformal algebra. In the interacting theory, all of the currents (2.1.9), except for $J_1^{[ij]}$ and J_2, will be broken. In particular, while the free CFT has $N(N+1)/2$ conserved "stress tensors", only one of them remains conserved when interactions are switched on.

For what follows, it will be useful to work out the normalization of the two point function of the currents (2.1.7) in arbitrary dimensions d. Since the currents are bilinear in ϕ there will be two propagators, which are differentiated by the hatted derivatives at both points. The calculation is drastically simplified by using the Schwinger parametrization of the propagator

$$\langle \phi^i(x)\phi^j(0)\rangle = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\frac{\delta^{ij}}{(x^2)^{d/2-1}} = \delta^{ij}\int_0^\infty \frac{d\alpha}{4\pi^{d/2}}\alpha^{d/2-2}e^{-\alpha x^2}\,. \tag{2.1.10}$$

Owing to the fact that $\hat{\partial}\hat{x} = 0$, since $z^2 = 0$, all hatted derivatives are replaced by $-2\alpha\hat{x}$ if acting at point x and $+2\alpha\hat{x}$ at point 0, so that instead of spin sums we have integrals of Gegenbauer polynomials over the parameters α_1 and α_2 for the first and second propagator respectively. Separating the $O(N)$ indices, we may write

$$\langle \hat{J}_s^{ij}(x)\hat{J}_s^{kl}(0)\rangle = (\delta^{ik}\delta^{jl} + (-1)^s\delta^{il}\delta^{jk})\frac{\mathcal{N}_s(\hat{x})^{2s}}{(x^2)^{d+2s-2}} \tag{2.1.11}$$

The $(-1)^s$ comes from the property of $C_s^{d/2-3/2}(-x) = (-1)^s C_s^{d/2-3/2}(x)$. The spacetime and z^μ dependence is of course as required by conformal symmetry for a spin s conserved operator. The normalization factor $\mathcal{N}_s$ is given by the following expression

$$\begin{aligned}\frac{(-1)^s 2^{2s}}{(4\pi^{d/2})^2}\int_0^\infty\int_0^\infty & d\alpha_1 d\alpha_2 \alpha_1^{d/2-2}\alpha_2^{d/2-2}e^{-\alpha_1-\alpha_2}(\alpha_1+\alpha_2)^{2s}C_s^{d/2-3/2}\Big(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\Big)C_s^{d/2-3/2}\Big(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\Big)\\ &= \frac{(-1)^s 2^{2s}}{(4\pi^{d/2})^2}\int_0^\infty dp\frac{p^{d-4+2s+1}}{2^{d-3}}e^{-p}\int_{-1}^1 dq(1-q^2)^{d/2-2}(C_s^{d/2-3/2}(q))^2\end{aligned} \tag{2.1.12}$$

from which we get

$$\mathcal{N}_s = \frac{(-1)^s 2^{2s}}{(4\pi^{d/2})^2}\frac{\pi\Gamma(d+2s-3)\Gamma(d+s-3)}{2^{2d-8}s!(\Gamma(d/2-3/2))^2} \tag{2.1.13}$$

The norms corresponding to the irreducible representations of $O(N)$ are then

$$\langle \hat{J}_s(x)\hat{J}_s(0)\rangle = N(1+(-1)^s)\frac{\mathcal{N}_s(\hat{x})^{2s}}{(x^2)^{d+2s-2}} \tag{2.1.14}$$

for the singlet,

$$\langle \hat{J}_s^{(ij)}(x)\hat{J}_s^{(kl)}(0)\rangle = \frac{(1+(-1)^s)}{2}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}-\frac{2}{N}\delta^{ij}\delta^{kl})\frac{\mathcal{N}_s(\hat{x})^{2s}}{(x^2)^{d+2s-2}} \tag{2.1.15}$$

for the symmetric traceless, and

$$\langle \hat{J}_s^{[ij]}(x)\hat{J}_s^{[kl]}(0)\rangle = \frac{(1-(-1)^s)}{2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})\frac{\mathcal{N}_s(\hat{x})^{2s}}{(x^2)^{d+2s-2}} \tag{2.1.16}$$

for the antisymmetric.

2.1.2 Anomalous dimensions of the weakly broken currents

The considerations above are general and apply to any CFT with weakly broken higher spin operators (1.4.7). For the explicit examples discussed in the rest of the chapter, it will be useful to determine the general form of the descendants K_{s-1} in the scalar theories. Applying the divergence operator to the higher spin currents (2.1.7), we find in terms of the function $f_s(u,v)$:

$$\begin{gathered}\partial_\mu D_z^\mu \hat{J}_s^{ij} = \big[h_s(\hat{\partial}_1,\hat{\partial}_2)\partial_1^2 + (-1)^s h_s(\hat{\partial}_2,\hat{\partial}_1)\partial_2^2\big]\phi^i(x_1)\phi^j(x_2)\Big|_{x_1,x_2\to x}, \\ h_s(u,v) \equiv (d/2-1)\partial_u f_s + \frac{u-v}{2}\partial^2_{uu} f_s + v\partial^2_{uv} f_s\,. \end{gathered} \tag{2.1.17}$$

Of course, this is zero in the free theory where $\partial_1^2 = \partial_2^2 = 0$. In the interacting theory, (2.1.17) allows to determine the form of the descendent once the equation of motion for ϕ is known in the specific model of interest. The function $h_s(u,v)$ can be evaluated more explicitly using the recurrence relations between the Gegenbauer polynomials, and one finds

$$h_s(u,v) = (u+v)^{s-1}(d-3)\big[(d/2-1)C_{s-1}^{d/2-1/2}\Big(\frac{u-v}{u+v}\Big) - \frac{2(d-1)v}{u+v}C_{s-2}^{d/2+1/2}\Big(\frac{u-v}{u+v}\Big)\big]\,. \tag{2.1.18}$$

As discussed above, the vanishing at $d=3$ is superficial and is a consequence of the normalization of the currents (2.1.7).

Note that the methods described in this section can also be used to fix the leading order anomalous dimension of a nearly free field. For instance, in the case of a scalar field, the equation of motion takes the form

$$\partial^2\phi = gV \tag{2.1.19}$$

where V is some operator of spin zero and bare dimension $d/2+1$. By an analogous calculation as the one described above for the higher spin operators, one can show that to leading order $\gamma_\phi \propto g^2$, where the proportionality constant is related to the two point function $\langle VV\rangle$ at $g=0$. We will use this method in the next Section to reproduce the well-known anomalous dimension of ϕ at the Wilson-Fisher fixed point, see also [24]. The analogous calculations in the large N approach and in the nonlinear sigma model in $d=2+\epsilon$ are given in Section 4 and 6 respectively.

2.2 $O(N)$ model in $d=4-\epsilon$

We now apply the general formulae obtained in the previous section to the case of the critical $O(N)$ ϕ^4 model in $d=4-\epsilon$ dimensions, with action

$$S = \int d^dx\Big(\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{\lambda}{4}(\phi^i\phi^i)^2\Big). \tag{2.2.1}$$

The one-loop beta function is well known and reads

$$\beta(\lambda) = -\epsilon\lambda + \frac{(N+8)\lambda^2}{8\pi^2} \tag{2.2.2}$$

and thus there is a IR critical point at $\lambda_* = \frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$. Before moving on to the higher spin operators, let us show how to reproduce the leading order anomalous dimension of ϕ using the classical equations of motion, following the methods reviewed above, see also [24]. In the free theory, when $\lambda=0$, the elementary field ϕ^i has canonical dimension $\Delta_0 = d/2-1$, thus saturating the unitarity bound and obeying $\partial^2\phi^i = 0$. The tree-level two-point function of ϕ^i is given by

$$\langle\phi^i(x_1)\phi^j(x_2)\rangle = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\frac{\delta^{ij}}{(x_{12}^2)^{d/2-1}}. \tag{2.2.3}$$

When we turn on the interaction, the equation of motion is modified to

$$\partial^2 \phi^i = \lambda \phi^i \phi^j \phi^j, \tag{2.2.4}$$

and the two-point function receives corrections. At the conformal point, the exact two-point function is constrained by conformal symmetry to be

$$\langle \phi^i(x_1) \phi^j(x_2) \rangle = \delta^{ij} \frac{C(\lambda)}{(x_{12}^2)^{d/2-1+\gamma_\phi}}. \tag{2.2.5}$$

Applying the equation of motion twice, i.e. taking the $\partial_1^2 \partial_2^2$ on both sides and taking the ratio one gets:

$$\lambda_*^2 (x_{12}^2)^2 \frac{\langle \phi^i \phi^k \phi^k(x_1) \phi^j \phi^l \phi^l(x_2) \rangle}{\langle \phi^i(x_1) \phi^j(x_2) \rangle} = 4\gamma_\phi(\gamma_\phi + 1)(d - 2 + 2\gamma_\phi)(d + 2\gamma_\phi). \tag{2.2.6}$$

The fact that the right-hand side is proportional to γ_ϕ is expected and is due to the shortening condition at zero coupling, $\partial^2 \phi^i = 0$. To get the leading order in ϵ for γ_ϕ we notice that in the left hand side $\lambda_*^2 \sim \epsilon^2$, so in the two-point function ratio we can just plug $d = 4$ propagators

$$\langle \phi^i(x_1) \phi^j(x_2) \rangle = \frac{1}{4\pi^2} \frac{\delta^{ij}}{x_{12}^2}. \tag{2.2.7}$$

In the right-hand side we get

$$4\gamma_\phi(\gamma_\phi + 1)(2 - \epsilon + \gamma_\phi)(4 - \epsilon + 2\gamma_\phi) = 32\gamma_\phi + O(\epsilon^3) \tag{2.2.8}$$

since it is evident that $\gamma_\phi \sim \epsilon^2$. Now, the two-point function in the numerator of (2.2.6), evaluated at tree level, yields

$$\langle \phi^i \phi^k \phi^k(x_1) \phi^j \phi^l \phi^l(x_2) \rangle_0 = \frac{1}{(4\pi^2)^3} (2N + 4) \delta^{ij} \frac{1}{(x_{12}^2)^3} \tag{2.2.9}$$

Finally, taking the ratio by the free propagator and equating the right-hand side (2.2.8), we recover the well-known result

$$\gamma_\phi = \frac{\lambda_*^2}{(4\pi^2)^2} \frac{N+2}{16} = \frac{N+2}{4(N+8)^2} \epsilon^2. \tag{2.2.10}$$

One can contrast this to the usual calculation, which is technically quite different. There, the leading order correction to the two-point function of the ϕ^i field is given by extracting the logarithmic divergence of the standard two-loop diagram in Fig. 1.

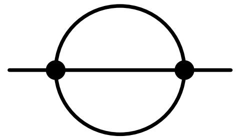


Figure 2.1: The 2-loop diagram yielding the leading order anomalous dimension of the ϕ^i field in the $O(N)$ model in the standard approach.

We may now proceed to studying the higher spin currents using the same method. We use the definition of the currents (2.1.7) and the descendant (2.1.17). To lowest order the ϵ dependence is fixed by the critical coupling λ_*, so we can use $d = 4$ everywhere. The currents are then:

$$\hat{J}^{ij}_s = (\hat{\partial}_1 + \hat{\partial}_2)^s C^{1/2}_s\Big(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2}\Big)\phi^i(x_1)\phi^j(x_2)\Big|_{x_1,x_2\to x}, \tag{2.2.11}$$

and the descendant:

$$\hat{K}^{ij}_{s-1}(x) = \Big(h_s(\hat{\partial}_1 + \hat{\partial}_3 + \hat{\partial}_4, \hat{\partial}_2) + (-1)^s h_s(\hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4, \hat{\partial}_1)\Big)\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^k(x_4)\Big|_{x_{1,2,3,4}\to x},$$
$$h_s(u,v) = (u+v)^{s-1}\big[C^{3/2}_{s-1}\Big(\frac{u-v}{u+v}\Big) - \frac{6v}{u+v}C^{5/2}_{s-2}\Big(\frac{u-v}{u+v}\Big)\big] \tag{2.2.12}$$

Note that this form is redundant in the sense that we could combine $\hat{\partial}_3 + \hat{\partial}_4$ into $\hat{\partial}_3$ acting on $\phi^i\phi^i(x_3)$, but it makes all the symmetries of the diagrams we will need to calculate explicit. A few examples might be useful. For instance, for $s = 1$ the only non-zero current is the antisymmetric one, for which the descendant vanishes as it should since it's the current in the adjoint of $O(N)$. For $s = 2$ we have non-zero currents for the symmetric traceless and the singlet representations, and the descendant is

$$\hat{K}^{ij}_1 = 2(\hat{\partial}_3 + \hat{\partial}_4 - \hat{\partial}_1 - \hat{\partial}_2)\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^k(x_4)\Big|_{x_{1,2,3,4}\to x} = -2\hat{\partial}(\phi^i\phi^j)\phi^k\phi^k + 2\phi^i\phi^j\hat{\partial}(\phi^k\phi^k) \tag{2.2.13}$$

This vanishes for the singlet as it is the conserved stress-energy tensor. It does not vanish for the symmetric traceless representation, as the corresponding operator acquires an anomalous dimension in the interacting theory. As another example, the spin 3 descendant of the spin 4 singlet current is

$$\hat{K}_3 = 20(\phi^i\phi^i\hat{\partial}^3(\phi^k\phi^k) - 6\hat{\partial}(\phi^i\phi^i)\hat{\partial}^2(\phi^k\phi^k) + 33(\phi^i\hat{\partial}^2\phi^i - 30\hat{\partial}\phi^i\hat{\partial}\phi^i)\hat{\partial}(\phi^k\phi^k)$$
$$-3\hat{\partial}(3\phi^i\hat{\partial}^2\phi^i - 4\hat{\partial}\phi^i\hat{\partial}\phi^i)\phi^k\phi^k \tag{2.2.14}$$

The master formula (1.4.11) at the leading order yields the following for γ_s:

$$\gamma_s = -\frac{\lambda_*^2}{s^2(s+1)^2}\frac{\hat{x}^2\langle\hat{K}_{s-1}(x)\hat{K}_{s-1}(0)\rangle}{\langle\hat{J}_s(x)\hat{J}_s(0)\rangle}\,. \tag{2.2.15}$$

For the two-point function of the currents one has in $d=4$, according to (2.1.12):

$$\langle\hat{J}_s\hat{J}_s\rangle = 2N\frac{1}{(4\pi^2)^2}\frac{(2\hat{x})^{2s}}{(x^2)^{2s+2}}\frac{\pi\Gamma(2s+1)\Gamma(s+1)}{s!(\Gamma(1/2))^2} = 2N\frac{(2s)!}{(4\pi^2)^2}\frac{(2\hat{x})^{2s}}{(x^2)^{2s+2}} \tag{2.2.16}$$

for the singlet and similarly

$$\langle\hat{J}_s^{(ij)}\hat{J}_s^{(kl)}\rangle = (\delta^{ik}\delta^{jl}+\delta^{ik}\delta^{jl}-\frac{2}{N}\delta^{ij}\delta^{kl})\frac{(2s)!}{(4\pi^2)^2}\frac{(2\hat{x})^{2s}}{(x^2)^{2s+2}} \tag{2.2.17}$$

$$\langle\hat{J}_s^{[ij]}\hat{J}_s^{[kl]}\rangle = (\delta^{ik}\delta^{jl}-\delta^{ik}\delta^{jl})\frac{(2s)!}{(4\pi^2)^2}\frac{-(2\hat{x})^{2s}}{(x^2)^{2s+2}} \tag{2.2.18}$$

where we used the fact that the singlet and symmetric traceless representations exist for even spins only, and the antisymmetric one for odd spins.

To obtain the anomalous dimensions via eq. (2.2.15), we have to compute the two-point function of the descendant at tree level. Each descendant (2.2.12) consists of a differential operator acting on four ϕ fields. We simply have to compute the free field Wick contractions between the fields (contractions of fields on the same descendant are of course excluded)

$$\langle\phi^i(x_1)\phi^j(x_2)\phi^m(x_3)\phi^m(x_4)\,,\phi^k(y_1)\phi^l(y_2)\phi^n(y_3)\phi^n(y_4)\rangle_0 \tag{2.2.19}$$

and then act with the differential operator in (2.2.12) on the resulting product of free propagators, setting $x_{1,2,3,4}\to x$ and $y_{1,2,3,4}\to 0$ at the end. It is straightforward to do this for any given spin: the problem is purely algebraic and there are no integrals to compute. However, to obtain a general result as a function of spin, it is convenient to use the Schwinger representation (2.1.10) of the propagator and carry out the resulting integrals of products of Gegenbauer polynomials. Some technical details of this are collected in Appendix 2.7. The final result takes the following form. For even spins, based on symmetry we get the structure

$$\langle\hat{K}_{s-1}^{ij}\hat{K}_{s-1}^{kl}\rangle = (A_sN+C_s)\frac{\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}}{2}+B_s\delta^{ij}\delta^{kl}\,,\qquad s=2,4,6,\dots \tag{2.2.20}$$

and similarly for odd spins

$$\langle \hat{K}^{ij}_{s-1} \hat{K}^{kl}_{s-1} \rangle = (A'_s N + C'_s) \frac{\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}}{2}\,, \qquad s = 1, 3, 5, \ldots \tag{2.2.21}$$

The A_s (and A'_s) terms come from contracting the first pair of ϕ fields at different points with each other and the second pair as well (hence the $O(N)$ indices form a closed loop); the B_s term is from contracting the pairs across (pair one with pair two and vice-versa); the C_s (and C'_s) term is from contracting one ϕ from the first pair with one from the second one (so that the $O(N)$ indices thread the diagram without loops). The final result for the coefficients $A_s, B_s, C_s, A'_s, C'_s$ is:

$$\begin{aligned}
A_s &= -(s-1)s(s+1)(s+2)(2s)! \frac{(2\hat{x})^{2s-2}}{(x^2)^{2s+2}(4\pi^2)^4} \\
B_s &= 4s(s+1)(2s)! \frac{(2\hat{x})^{2s-2}}{(x^2)^{2s+2}(4\pi^2)^4} \\
C_s &= -2(s-2)s(s+1)(s+3)(2s)! \frac{(2\hat{x})^{2s-2}}{(x^2)^{2s+2}(4\pi^2)^4} \\
A'_s &= -A_s\,, \qquad C'_s = 2A_s
\end{aligned} \tag{2.2.22}$$

From (2.2.20) and (2.2.21), we can readily extract the singlet, symmetric traceless and antisymmetric parts. They are

$$\begin{aligned}
\langle \hat{K}_{s-1} \hat{K}_{s-1} \rangle &= (A_s + B_s)N^2 + C_s N \\
\langle \hat{K}^{(ij)}_{s-1} \hat{K}^{(kl)}_{s-1} \rangle &= \frac{A_s N + C_s}{2} \left(\delta^{ik}\delta^{jl} + \delta^{ik}\delta^{jl} - \frac{2}{N}\delta^{ij}\delta^{kl} \right) \\
\langle \hat{K}^{[ij]}_{s-1} \hat{K}^{[kl]}_{s-1} \rangle &= -\frac{A_s}{2}(N+2)\left(\delta^{ik}\delta^{jl} - \delta^{ik}\delta^{jl} \right)\,.
\end{aligned} \tag{2.2.23}$$

It is now straightforward to extract the one-loop anomalous dimensions, using the general formula (2.2.15) and the normalization of the currents (2.2.16), (2.2.18). For the singlet operators, we get[3]

$$\gamma_s = \frac{\epsilon^2 (N+2)}{2(N+8)^2} \left(1 - \frac{6}{s(s+1)} \right) \tag{2.2.24}$$

This vanishes for $s = 2$ as it should, corresponding to the conservation of the stress-energy tensor.

[3]It is amusing that the spin dependent factor in brackets in (2.2.24) is the same as the central charge of the unitary minimal models $M(s, s+1)$, which have $c = 0, 1/2, 7/10, \ldots$ for $s = 2, 3, 4, \ldots$. Similarly, the result (2.2.26) for the antisymmetric representation is proportional to the central charge $c = \frac{3}{2}(1 - 8/(p(p+2)))$ of the $\mathcal{N} = 1$ supersymmetric minimal models for $p = 2s$. One may wonder if there is a deeper significance to these apparent coincidences. We thank Igor Klebanov for bringing this to our attention.

For the symmetric traceless operators, we get

$$\gamma_{s(ij)} = \frac{\epsilon^2(N+2)}{2(N+8)^2}\Big(1 - \frac{2(N+6)}{(N+2)s(s+1)}\Big) \tag{2.2.25}$$

and for the antisymmetric ones

$$\gamma_{s[ij]} = \frac{\epsilon^2(N+2)}{2(N+8)^2}\Big(1 - \frac{2}{s(s+1)}\Big) . \tag{2.2.26}$$

The latter vanishes for $s = 1$, corresponding to the exact conservation of the current in the adjoint of $O(N)$. All these results are in agreement with [44].

It is worth mentioning the $s \to \infty$ behavior analysis of these results done in [39, 40], in light of (1.4.12). First of all, we see that the limiting value is equal to $\frac{\epsilon^2(N+2)}{2(N+8)^2} = 2\gamma_\phi$ as follows from expanding (1.4.12) to order ϵ. Second, the leading correction behaves as $\frac{1}{s^2}$, which is a manifestation of having a tower of operators with twist 2, which at this order in ϵ are the higher-spin currents and the $\phi^i\phi^i$ operator (see [39] for more details). As mentioned, the coefficient $c_{\tau_{min}}$ is determined by the certain three-point functions of these operators with ϕ. We will go into more detail about this in the next two sections.

2.3 The large N critical $O(N)$ model

To develop the $1/N$ expansion of the ϕ^4 theory, one may introduce a Hubbard-Stratonovich auxiliary field σ, so that the action (2.2.1) may be rewritten as

$$S = \int d^d x \Big(\frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda}\Big). \tag{2.3.1}$$

In the IR limit for $d < 4$, the last term becomes unimportant and can be dropped.[4] To develop perturbation theory, it is convenient to rescale σ so that the action becomes

$$S = \int d^d x \Big(\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi^i + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i\Big). \tag{2.3.2}$$

[4] For $d > 4$, the last term can be dropped in the UV limit, corresponding to a non-trivial UV fixed point.

The σ field then acquires an effective non-local propagator upon integrating out ϕ^i[5]

$$\langle\sigma(x_1)\sigma(x_2)\rangle = \frac{C_{\sigma\sigma}}{(x_{12}^2)^2}\,, \qquad C_{\sigma\sigma} = \frac{2^{d+2}\Gamma(\frac{d}{2}-\frac{1}{2})\sin(\pi d/2)}{\pi^{\frac{3}{2}}\Gamma(\frac{d}{2}-2)} \tag{2.3.3}$$

so that σ, which replaces the scalar operator $\phi^i\phi^i$, is a primary operator of dimension $2+O(1/N)$ at the interacting fixed point. Systematic perturbation theory can be developed using this effective propagator, the canonical propagator (2.2.3) for ϕ^i and the $\sigma\phi^i\phi^i$ vertex, with $1/\sqrt{N}$ playing the role of the coupling constant.

The equation of motion for ϕ is

$$\partial^2\phi^i = \frac{1}{\sqrt{N}}\sigma\phi^i \tag{2.3.4}$$

and the equation of motion for σ is formally $\phi^i\phi^i = 0$ after we drop the last term of (2.3.2) in the IR limit. The role of this equation is to "subtract" from the theory the operator $\phi^i\phi^i$, which is replaced by σ. This fact will play an important role in our calculation below.

Before turning to the higher spin currents let us calculate as a warmup the anomalous dimension γ_ϕ of the ϕ field, without computing Feynman diagrams. Using the equations of motion (2.3.4) and acting on the ϕ two-point function with $\partial_1^2\partial_2^2$, we get

$$\frac{1}{N}(x_{12}^2)^2\frac{\langle\phi^i\sigma(x_1)\phi^j\sigma(x_2)\rangle}{\langle\phi^i(x_1)\phi^j(x_2)\rangle} = 2\gamma_\phi(\gamma_\phi+1)(d-2+2\gamma_\phi)(d+2\gamma_\phi). \tag{2.3.5}$$

From this, the leading order value of γ_ϕ immediately follows

$$\gamma_\phi = \frac{C_{\sigma\sigma}}{4Nd(d-2)} = \frac{2\sin(\pi d/2)\Gamma(d-2)}{N\pi\Gamma(\frac{d}{2}-2)\Gamma(\frac{d}{2}+1)} \tag{2.3.6}$$

which is a well-known result. It is quite remarkable how simple the calculation is, provided we know $C_{\sigma\sigma}$. It is also helpful that one completely avoids (to the lowest order) the issues of regularization and renormalization, which are actually somewhat thorny in the $1/N$ expansion.

Let us now turn to the higher spin currents. These have the same form (2.1.7), and using (2.1.17)

[5] The quadratic term in the resulting σ effective action is just proportional to the two-point function $\langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_0$ in the free theory. The σ propagator is obtained by Fourier transforming to momentum space, inverting, and transforming back to x-space.

and the equations of motion (2.3.4), one ends up with

$$\partial_\mu D_z^\mu \hat{J}_s^{ij} = \frac{1}{\sqrt{N}} \hat{K}_{s-1}^{ij} \tag{2.3.7}$$

where

$$\hat{K}_{s-1}^{ij} = \Big(h_s(\hat{\partial}_1 + \hat{\partial}_3, \hat{\partial}_2) + (-1)^s h_s(\hat{\partial}_2 + \hat{\partial}_3, \hat{\partial}_1) \Big) \phi^i(x_1)\phi^j(x_2)\sigma(x_3) \Big|_{x_{1,2,3}\to x}, \tag{2.3.8}$$

and the function $h_s(u,v)$ is given in eq. (2.1.18).[6] It is possible, and convenient for what follows, to decompose the descendant in products of the conformal primaries $\hat{J}_s$, σ and their derivatives. We find

$$\hat{K}_{s-1}^{ij} = \sum_{s'=0}^{s-2} \sum_{k=0}^{s-s'-1} C_{s'k} \hat{\partial}^{s-s'-k-1} \hat{J}_{s'}^{ij} \hat{\partial}^k \sigma \tag{2.3.9}$$

where the coefficients $C_{s'k}$ are given explicitly by

$$C_{s'k} = \begin{cases} (s-s')(2s'+d-3)\binom{s-s'-1}{k}\binom{-s-s'+k-d+3}{k+1}, & s-s' \text{ even} \\ 0\,, & s-s' \text{ odd} \end{cases} \tag{2.3.10}$$

An important point is that so far we have only used the ϕ equation of motion (2.3.4), and not the equation for σ whose role is to formally project out $J_0 = \phi^i\phi^i$ from the theory. This implies that in fact the form of the descendant (2.3.9) only applies as written to the non-singlet currents. For the singlets, one obtains the correct descendant by the prescription that the term $s'=0$ should be dropped from the sum. As an example, for $s=2$ we get from (2.3.9)

$$\hat{K}_1^{ij} = (d-1)(d-3)\big((d-2)\hat{J}_0^{ij}\hat{\partial}\sigma - 2(\hat{\partial}\hat{J}_0^{ij})\sigma\big) \tag{2.3.11}$$

For the symmetric traceless $K_1^{(ij)}$, on the right hand side we have $\hat{J}_0^{(ij)} = \phi^{(i}\phi^{j)}$ and the descendant is non-vanishing. However, for the singlet we would have $J_0 = \phi^i\phi^i$, which should be thrown away. This leads to $\hat{K}_1 = 0$, as it should be according to the conservation of the stress-energy tensor. In a similar way, the spin 3 descendant of the spin 4 singlet current is

$$\hat{K}_3 = (d+3)(d+1)\big((d+2)\hat{J}_2\hat{\partial}\sigma - 2(\hat{\partial}\hat{J}_2)\sigma\big) \tag{2.3.12}$$

[6]Note that the vanishing of h_s at $d=3$ is of course not meaningful, and just follows from the normalization of the currents that we have chosen. Such factors of $(d-3)$ cancel out in the ratio $\langle K_{s-1}K_{s-1}\rangle/\langle J_s J_s\rangle$.

At $d=3$, we see that $\hat{K}_3 \propto \hat{J}_2 \hat{\partial}\sigma - 2/5(\hat{\partial}\hat{J}_2)\sigma$, in agreement with [21].

We can now compute the anomalous dimensions using (1.4.11). Let us first discuss the case of the non-singlet currents, where we can use directly the form (2.3.8), equivalent to (2.3.9). The descendant two-point function can be computed similarly to the previous section and one ends up with

$$\gamma_{s(ij)} = \gamma_{s[ij]} = 2\gamma_\phi \frac{(s-1)(d+s-2)}{(d/2+s-2)(d/2+s-1)}, \tag{2.3.13}$$

where γ_ϕ is the anomalous dimension of ϕ field. This is the correct result, in agreement [45]. Let us now turn to the case of the singlet currents, which is slightly more involved due to the J_0 projection discussed above. The correct singlet descendant is given by (here as usual we denote by $J_{s'} = J^{ii}_{s'}$ the singlet operators)

$$\begin{aligned} \hat{K}_{s-1} &= \sum_{s'=2}^{s-2} \sum_{k=0}^{s-s'-1} C_{s'k} \hat{\partial}^{s-s'-k-1} \hat{J}_{s'} \hat{\partial}^k \sigma = \hat{K}^{\text{naive}}_{s-1} - \hat{K}^0_{s-1} \\ \hat{K}^0_{s-1} &= \sum_{k=0}^{s-1} C_{0k} \hat{\partial}^{s-k-1} \hat{J}_0 \hat{\partial}^k \sigma \end{aligned} \tag{2.3.14}$$

where $\hat{K}^{\text{naive}}_{s-1}$ coincides with (2.3.8), with $O(N)$ indices traced. Its two-point function leads to a contribution equal to (2.3.13) to the anomalous dimension. To subtract the contribution of the term $\hat{K}^0_{s-1}$ proportional to J_0, we note that

$$\langle \hat{K}^{\text{naive}}_{s-1} \hat{K}^{\text{naive}}_{s-1} \rangle = \langle \hat{K}_{s-1} \hat{K}_{s-1} \rangle + \langle \hat{K}^0_{s-1} \hat{K}^0_{s-1} \rangle \tag{2.3.15}$$

since $\hat{K}_{s-1}$ and $\hat{K}^0_{s-1}$ are orthogonal, due to orthogonality of the spin s primaries ($\langle J_s J_{s'} \rangle \sim \delta_{ss'}$). Then, the correct singlet anomalous dimension is obtained by simply subtracting from (2.3.13) the contribution of the two-point function $\langle \hat{K}^{(0)}_{s-1} \hat{K}^{(0)}_{s-1} \rangle$. By this procedure, we get the final result

$$\gamma_s = 2\gamma_\phi \frac{1}{(d/2+s-2)(d/2+s-1)} \left[(s-1)(d+s-2) - \frac{\Gamma(d+1)\Gamma(s+1)}{2(d-1)\Gamma(d+s-3)} \right], \tag{2.3.16}$$

in agreement with [45]. One may also check that setting $d = 4-\epsilon$, (2.3.16) and (2.3.13) precisely match the results (2.2.24), (2.2.25), (2.2.26) expanded to order $1/N$.

It is again interesting to mention the $s \to \infty$ behavior [39,40]. The expansion at large s yields

$$\gamma_s = 2\gamma_\phi - 2\gamma_\phi \frac{\Gamma(d+1)}{2(d-1)} \frac{1}{s^{d-2}} - 2\gamma_\phi \frac{d(d-2)}{4} \frac{1}{s^2} + ...,$$
$$\gamma_{s(ij)} = 2\gamma_\phi - 2\gamma_\phi \frac{d(d-2)}{4} \frac{1}{s^2} + ... \tag{2.3.17}$$

We see a tower of higher spin currents ($1/s^{d-2}$) in the singlet channel, as well as the σ ($1/s^2$) in both the singlet and the traceless parts. The higher-spin contribution vanishes for the symmetric traceless. The $1/s^2$ coming from σ is universal and can be calculated. Using (1.4.13) and plugging $O = \phi$, $O_{\tau_{min}} = \sigma$ such that $\Delta = d/2 - 1$, $\tau_{min} = 2$, $C_{O_{\tau_{min}} O_{\tau_{min}}}$ is the three-point function coefficient of $\phi\phi\sigma$, $C_{\phi\phi} = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}$, $C_{\sigma\sigma}$ is defined above. Putting all the factors together, the coefficient of $1/s^2$ is exactly reproduced.

2.4 Cubic models in $d = 6 - \epsilon$

let us consider the following model with $N+1$ scalars and $O(N)$ invariant cubic interactions

$$S = \int d^d x \Big(\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \sigma^3 \Big). \tag{2.4.1}$$

As argued in [30], in $d = 6 - \epsilon$ this model possesses IR stable, perturbatively unitary fixed points which provide a "UV completion" of the large N UV fixed points of the $O(N)$ model in $d > 4$. This proposal has passed various non-trivial checks [30,49,50]. These perturbative fixed points exist for $N > 1038(1 + O(\epsilon))$, and are expected to be unitary to all orders in ϵ and $1/N$ expansions. However, non-perturbative effects presumably render the vacuum metastable via instanton effects. In this section we just perform perturbative calculations, and in particular we obtain further non-trivial agreement with the large N expansion in $d > 4$. [7]

In this model, a novel feature in the calculation of the anomalous dimension of the higher spin operators is that the free theory contains two independent towers of conserved, $O(N)$ singlet, higher

[7] Calculations of anomalous dimensions of higher spin operators in a similar cubic model with an adjoint scalar and $\mathrm{tr}\,\phi^3$ interaction were carried out in [51,52].

spin currents:

$$
\begin{aligned}
\hat{J}_{s,\phi} &= \frac{1}{\sqrt{N}}(\hat{\partial}_1+\hat{\partial}_2)^s C_s^{3/2}\Big(\frac{\hat{\partial}_1-\hat{\partial}_2}{\hat{\partial}_1+\hat{\partial}_2}\Big)\phi^i(x_1)\phi^i(x_2)\Big|_{x_1,x_2\to x} \\
\hat{J}_{s,\sigma} &= (\hat{\partial}_1+\hat{\partial}_2)^s C_s^{3/2}\Big(\frac{\hat{\partial}_1-\hat{\partial}_2}{\hat{\partial}_1+\hat{\partial}_2}\Big)\sigma(x_1)\sigma(x_2)\Big|_{x_1,x_2\to x}
\end{aligned} \tag{2.4.2}
$$

where we have normalized $J_{s,\phi}$ so that both currents have $\langle J_s J_s\rangle \sim O(1)$. Once interactions are turned on, we expect these operators to mix non-trivially, and one should determine the appropriate eigenstates of the dilatation operator.

The equations of motion are

$$
\begin{aligned}
\partial^2\phi^i &= g_1\sigma\phi^i \\
\partial^2\sigma &= \frac{1}{2}(g_1\phi^i\phi^i+g_2\sigma^2)\,.
\end{aligned} \tag{2.4.3}
$$

It is evident that the equations of motion will induce the mixing between the currents, since $\langle\partial\cdot J_{s,\phi}\partial\cdot J_{s,\sigma}\rangle \neq 0$ due to the g_1-dependent interactions in (2.4.3). The descendant operators (in this case we find it more convenient to include the coupling constants into the definition of the K_{s-1}'s)

$$
\partial_\mu D_z^\mu \hat{J}_{s,\phi} = \hat{K}_{s-1,\phi}\,, \qquad \partial_\mu D_z^\mu \hat{J}_{s,\sigma} = \hat{K}_{s-1,\sigma} \tag{2.4.4}
$$

can be computed in a straightforward way by following similar steps as in the previous sections. Explicitly, they are given by

$$
\begin{aligned}
\hat{K}_{s-1,\phi} &= \frac{1}{\sqrt{N}}\Big(h_s(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2)+1\leftrightarrow 2\Big)\,g_1\phi^i(x_1)\phi^i(x_2)\sigma(x_3)\Big|_{x_{1,2,3}\to x} \\
\hat{K}_{s-1,\sigma} &= \Big(h_s(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2)+1\leftrightarrow 2\Big)\Big(\frac{g_1}{2}\phi^i(x_1)\sigma(x_2)\phi^i(x_3)+\frac{g_2}{2}\sigma(x_1)\sigma(x_2)\sigma(x_3)\Big)\Big|_{x_{1,2,3}\to x} \\
h_s(u,v) &= 6(u+v)^{s-1}\big[C_{s-1}^{5/2}\Big(\frac{u-v}{u+v}\Big)-\frac{5v}{u+v}C_{s-2}^{7/2}\Big(\frac{u-v}{u+v}\Big)\big]
\end{aligned} \tag{2.4.5}
$$

We can then use the general relation (1.4.11), suitably generalized to the present case with non-trivial mixing, to obtain the following anomalous dimension mixing matrix

$$
\begin{bmatrix}
\frac{g_1^2}{48\cdot 4\pi^3}(1-\frac{6}{(s+1)(s+2)}) & -\frac{g_1^2}{4\pi^3}\frac{1}{8(s+1)(s+2)}\sqrt{N} \\
-\frac{g_1^2}{4\pi^3}\frac{1}{8(s+1)(s+2)}\sqrt{N} & \frac{g_1^2}{4\pi^3}\frac{1}{2\cdot 48}N+\frac{g_2^2}{2\cdot 48\cdot 4\pi^3}(1-\frac{12}{(s+1)(s+2)})
\end{bmatrix} \tag{2.4.6}
$$

where the non-diagonal terms comes from the non-zero two-point function $\langle\hat{K}_{s-1,\phi}\hat{K}_{s-1,\sigma}\rangle$. From this mixing matrix, one can compute the two eigenvalues to leading order in ϵ and finite N, using the expression for the fixed point couplings given in [30]. The resulting finite N expressions are easy to get, but rather lengthy. At large N, using the expressions for the fixed point couplings [30]

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1+\frac{22}{N}+\frac{726}{N^2}-\frac{326180}{N^3}+\dots\right) \tag{2.4.7}$$

$$g_2^* = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1+\frac{162}{N}+\frac{68766}{N^2}+\frac{41224420}{N^3}+\dots\right) \tag{2.4.8}$$

one finds that the eigenvalues are given by

$$\begin{aligned}\gamma_1 &= \frac{2\epsilon}{N}\frac{(s-2)(s+5)(s^2+3s+8)}{(s+1)^2(s+2)^2}+O(1/N^2)\\ \gamma_2 &= \epsilon+\frac{16\epsilon}{N}\frac{5s^4+30s^3+38s^2-21s-25}{(s+1)^2(s+2)^2}+O(1/N^2)\end{aligned} \tag{2.4.9}$$

The higher order corrections can be obtained to any desired order, but for simplicity we have listed here only the leading order in $1/N$. We see that the γ_1 eigenvalue vanishes at $s=2$, and one can check that this is true for any N. This eigenvalue then corresponds to the tower of "single-trace" higher spin currents which include the stress-energy tensor. Indeed, one can explicitly verify that γ_1 matches the $1/N$ expansion result (2.3.16) expanded in $d=6-\epsilon$. The dimension corresponding to the second eigenvalue is

$$\Delta_2 = d-2+s+\gamma_2 = 4+s+\frac{16\epsilon}{N}\frac{5s^4+30s^3+38s^2-21s-25}{(s+1)^2(s+2)^2}+O(1/N^2) \tag{2.4.10}$$

which suggests that this should match the "double-trace" operator $\sigma\partial^s\sigma \sim \phi^2\partial^s\phi^2$ in the large N approach. Indeed, one can match (2.4.10) with the result given in [45] for the dimension of such composite operators of spin s. For $s=0$, this is the scalar operator σ^2 of the large N model, which has dimension $\Delta = 4-100\epsilon/N+\dots$ near $d=6$ and corresponds to a particular mixture of the mass operators $\phi^i\phi^i$ and σ^2 in the cubic model [30].

One can also study the spin s operators in the symmetric traceless and antisymmetric representations of $O(N)$, where no mixing occurs (since $\hat{J}_{s,\sigma}$ is a singlet of $O(N)$). Following similar steps to the previous sections, we obtain the result

$$\gamma_{s(ij)} = \gamma_{s[ij]} = \frac{(g_1^*)^2}{192\pi^3}\frac{(s-1)(s+4)}{(s+1)(s+2)} = \frac{2\epsilon}{N}(1+\frac{44}{N}+\dots)\frac{(s-1)(s+4)}{(s+1)(s+2)} \tag{2.4.11}$$

The order $1/N$ is seen to exactly match the large N result (2.3.13). Furthermore, we checked that the $1/N^2$ term also matches with the result obtained in [53] using large N methods for arbitrary d.[8]

Let us now study the large spin limit of these results. For the eigenvalues of the singlet mixing matrix, the large spin expansion can be written in a simple form, valid for finite N, in terms of the fixed point couplings

$$\begin{aligned}\gamma_1 &= \frac{(g_1^*)^2}{192\pi^3} - \frac{(g_1^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots, = 2\gamma_\phi - \frac{(g_1^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots \\ \gamma_2 &= \frac{(g_1^*)^2 N + (g_2^*)^2}{384\pi^3} - \frac{(g_2^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots = 2\gamma_\sigma - \frac{(g_2^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots\end{aligned} \tag{2.4.12}$$

where we have used the known expression for the one-loop anomalous dimensions of ϕ and σ in the cubic model [30]. The leading terms are precisely consistent with the expected large spin limit. The subleading $1/s^2$ contributions are clearly coming from the exchange of the σ field, which has $\Delta_\sigma = \tau = 2$. We can check explicitly the prediction of the formula (1.4.13) for the coefficients of the $1/s^2$ terms. For the γ_1 eigenvalue, we should take that $O = \phi, O_{\tau_{min}} = \sigma$. The two-point function coefficients are $C_{\phi\phi}$=$C_{\sigma\sigma} = \frac{1}{4\pi^3}$. The three point function coefficient is given at the lowest order by a diagram with one $g_1\phi^i\phi^i$ vertex in the middle. The diagram is given by the integral:

$$\frac{g_1}{(4\pi^3)^3}\int \frac{d^6x_0}{x_{10}^4 x_{20}^4 x_{30}^4} = \frac{g_1}{(4\pi^3)^3}\frac{\pi^3}{x_{12}^2 x_{23}^2 x_{31}^2} \tag{2.4.13}$$

Combining the factors we get $c_{\tau_{min}} = 2\frac{(g_1^*)^2}{(4^3\pi^6)^2}(4\pi^3)^3 = \frac{(g_1^*)^2}{32\pi^3}$. Overall, we then get

$$\tau_{s,\phi} = d - 2 + \frac{(g_1^*)^2}{192\pi^3} - \frac{(g_1^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots = 2\tau_\phi - \frac{(g_1^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots \tag{2.4.14}$$

since $\tau_\phi = \Delta_\phi = d/2 - 1 + \frac{(g_1^*)^2}{384\pi^3}$ to the leading order. The same applies for the eigenvalue γ_2, corresponding to $\tau_{s,\sigma}$, where we get:

$$\tau_{s,\sigma} = d - 2 + \frac{(g_1^*)^2 N + (g_2^*)^2}{384\pi^3} - \frac{(g_2^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots = 2\tau_\sigma - \frac{(g_2^*)^2}{32\pi^3}\frac{1}{s^2} + \ldots \tag{2.4.15}$$

where $\tau_\sigma = \Delta_\sigma = d/2 - 1 + \frac{(g_1^*)^2 N + (g_2^*)^2}{768\pi^3}$ is the leading order dimension (and thus twist) of σ. The coefficient of $1/s^2$ is reproduced the three-point function $\langle\sigma\sigma\sigma\rangle$.

The cubic model in $d = 6-\epsilon$ also admits non-unitary fixed points which are of interest in statistical mechanics. The simplest case is the $N = 0$ model, which just consists of a single scalar field σ with

[8] As far as we know, the $1/N^2$ term in the large N expansion of the anomalous dimensions of the *singlet* higher spin operators has not been obtained in the literature.

cubic interaction $g_2/6\sigma^3$. This model has a non-unitary fixed point at

$$(g_2^*)^2 = -\frac{128\pi^3}{3}\epsilon + O(\epsilon^2) \tag{2.4.16}$$

As pointed out by Fisher [54], this theory is related to the Lee-Yang edge singularity of the Ising model. For $d = 2$ ($\epsilon = 4$), the fixed point corresponds to the non-unitary minimal model $M(2,5)$. Using the result (2.4.6) for $g_1 = 0, N = 0$, we can deduce the dimension of the higher spin operators $\sim \sigma\partial^s\sigma$ in the Fisher model to be

$$\gamma_s = 2\gamma_\sigma \left(1 - \frac{12}{(s+1)(s+2)}\right) = -\frac{\epsilon}{9}\left(1 - \frac{12}{(s+1)(s+2)}\right) + O(\epsilon^2)\,. \tag{2.4.17}$$

where we have used $\gamma_\sigma = \frac{(g_2^*)^2}{768\pi^3}$ at one loop.

Another interesting non-unitary model is obtained at the formal value $N = -2$. In this case the model is equivalent to a theory of a complex anticommuting scalar θ and a commuting scalar σ [55]

$$S = \int d^dx \left(\partial_\mu\theta\partial^\mu\bar{\theta} + \frac{1}{2}\left(\partial_\mu\sigma\right)^2 + g_1\sigma\theta\bar{\theta} + \frac{1}{6}g_2\sigma^3\right)\,. \tag{2.4.18}$$

with $Sp(2)$ global symmetry. The IR stable fixed point occurs at [55]

$$g_2^* = 2g_1^*\,, \qquad g_1^* = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}\left(1 + O(\epsilon)\right)\ , \tag{2.4.19}$$

where the first equality holds to all orders in perturbation theory. For such a relation between couplings, one can verify that the model has an enhanced "supersymmetry" $OSp(1|2)$ which implies that the dimension of θ and σ are equal. It turns out that this $OSp(1|2)$ invariant fixed point is equivalent to the $q \to 0$ limit of the q-state Potts model [56]. The dimension of the $Sp(2)$ invariant higher spin currents at the fixed point can be obtained from (2.4.6) setting $N = -2$ and $g_2 = 2g_1$. This yields the two eigenvalues

$$\begin{aligned}\gamma_1 &= \frac{(g_1^*)^2}{192\pi^3}\frac{(s-2)(s+5)}{(s+1)(s+2)} = -\frac{\epsilon(s-2)(s+5)}{15(s+1)(s+2)}\\ \gamma_2 &= \frac{(g_1^*)^2}{192\pi^3}\frac{(s(s+3)-16)}{(s+1)(s+2)} = -\frac{\epsilon(s(s+3)-16)}{15(s+1)(s+2)}\end{aligned} \tag{2.4.20}$$

We see that the first eigenvalue corresponds to the tower which includes the stress tensor of the theory, since it vanishes at $s=2$ (this eigenvalue corresponds to an $OSp(1|2)$ singlet). In the large spin limit one gets

$$\begin{aligned}\gamma_1 &= -\frac{\epsilon}{15}+\frac{4\epsilon}{5s^2}+\dots\\ \gamma_2 &= -\frac{\epsilon}{15}+\frac{6\epsilon}{5s^2}+\dots\,.\end{aligned} \tag{2.4.21}$$

The equality of the leading terms is a consequence of $\Delta_\theta = \Delta_\phi$, as follows from the $OSp(1|2)$ symmetry. One may also obtain the dimension of the non-singlet currents, which are the same as in (2.4.11), with g_1 given in (2.4.19).

2.5 Nonlinear sigma model

It is well established that the critical behavior of the $O(N)$ ϕ^4 model can be related to the critical nonlinear sigma model, see e.g. [57] for a review. One of the ways to understand this relation is via the $1/N$ expansion, which provides an explicit "interpolation" between the UV fixed points of the sigma model in $d=2+\epsilon$ and the IR fixed points of the ϕ^4 model in $d=4-\epsilon$. In this section, we calculate the anomalous dimensions of the higher-spin currents at the critical point of the sigma model in $d=2+\epsilon$, at finite N. As far as we know, this result has not been obtained elsewhere.

We start with the action with an auxiliary field inserted to resolve the sphere constraint on the ϕ^i field, $\phi^i\phi^i = 1/g^2$

$$S=\int d^dx\Big(\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i+\sigma(\phi^i\phi^i-\frac{1}{g^2})\Big). \tag{2.5.1}$$

To develop perturbation theory, one may resolve the constraint by introducing a set of $N-1$ independent fields. A convenient parametrization is

$$\begin{aligned}\phi^a &= \varphi^a, \qquad a=1,\dots,N-1;\\ \phi^N &= \frac{1}{g}\sqrt{1-g^2\varphi^a\varphi^a}=\frac{1}{g}-\frac{g}{2}\varphi^a\varphi^a+O(g^3).\end{aligned} \tag{2.5.2}$$

In terms of the φ^a fields, the action is

$$S=\int d^dx\Big(\frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a+\frac{g^2}{2}\frac{(\varphi^a\partial_\mu\varphi^a)^2}{1-g^2\varphi^a\varphi^a}\Big) \tag{2.5.3}$$

To leading order in perturbation theory and in $d = 2+\epsilon$, the coupling constant has the beta function

$$\beta = \frac{\epsilon}{2}g - (N-2)\frac{g^3}{4\pi} \tag{2.5.4}$$

and there is a UV fixed point at [58, 59]

$$g_*^2 = \frac{2\pi\epsilon}{N-2} \tag{2.5.5}$$

The factor of $N-2$ is due to the fact that the $O(2)$ model is conformal and has a trivial beta function in $d = 2$. Consequently, the perturbative UV fixed point in $d = 2+\epsilon$ only exists for $N > 2$.

Before moving onward to the higher spin operators, we will calculate the anomalous dimension of the φ field (or equivalently ϕ^i in the action (2.5.1)) using the classical equations of motion. To leading order in g, they are given by[9]

$$\partial^2\varphi^a = -g^2\varphi^a\partial_\mu\varphi^b\partial_\mu\varphi^b + O(g^4)\,. \tag{2.5.6}$$

In full analogy with the discussion in $d = 4-\epsilon$, we can apply the equations of motion (2.5.6) to the two-point function of φ field, obtaining

$$g^4x^4\,\frac{\langle\varphi^a(\partial_\mu\varphi^c)^2(x)\varphi^b(\partial_\mu\varphi^d)^2(0)\rangle}{\langle\varphi^a(x)\varphi^b(0)\rangle} = 4\gamma_\phi(\gamma_\phi+1)(d-2+2\gamma_\phi)(d+2\gamma_\phi)\,. \tag{2.5.7}$$

To specialize to the expansion in $d-2=\epsilon$ we need to mention several important points: γ will be of the order ϵ and not ϵ^2, unlike in $d = 4-\epsilon$. This means that the third term in the right-hand side is of the order ϵ as well. Second, the bare propagator is

$$\langle\varphi^a(x)\varphi^b(0)\rangle = \frac{\Gamma(\frac{\epsilon}{2})}{4\pi\pi^{\epsilon/2}(x^2)^{\epsilon}}, \tag{2.5.8}$$

and applying derivatives to it produces powers of ϵ. They will combine with $\Gamma(\epsilon/2) = \frac{2}{\epsilon} + O(1)$. Since $g_*^4 \sim \epsilon^2$ at the critical point, only terms of order ϵ^2 are needed from the two-point function in the numerator, since we have three $\Gamma's$ on top and one on bottom, which amounts to $1/\epsilon^2$. Having said that, the relevant term of the two-point function is easy to calculate. The first and the second

[9] These may be also obtained starting from the equations of motion coming from (2.5.1), which are $\partial^2\phi^i = -g^2\phi^i\partial_\mu\phi^j\partial_\mu\phi^j$, and resolving the constraint by (2.5.2).

derivatives of the propagator are:

$$\partial_\mu \frac{1}{(x^2)^{\epsilon/2}} = -\epsilon \frac{x_\mu}{(x^2)^{\epsilon/2}}, \tag{2.5.9}$$

$$\partial_\nu \partial_\mu \frac{1}{(x^2)^{\epsilon/2}} = \frac{-\epsilon}{(x^2)^{\epsilon/2+1}} \big(\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\big) + \epsilon^2 \frac{x_\mu x_\nu}{(x^2)^{\epsilon/2+2}}. \tag{2.5.10}$$

It is evident then that the only way to get $O(\epsilon^2)$ is to contract φ^a and φ^b which would be undifferentiated, and the other φ's accordingly so that the ϵ from (2.5.10) is picked up two times. Overall one gets for the left hand side of (2.5.7)

$$g_*^4 (N-1) \frac{2 \cdot 2 \cdot \epsilon^2}{(4\pi)^2} \frac{2^2}{\epsilon^2} = (N-1) \frac{4\epsilon^2}{(N-2)^2} \,. \tag{2.5.11}$$

The right-hand side of (2.5.7) yields $8\gamma_\phi(\epsilon + 2\gamma_\phi)$ to leading order in ϵ. Solving the resulting quadratic equation for γ_ϕ gives

$$\gamma_\phi = \frac{\epsilon}{2(N-2)}, \tag{2.5.12}$$

which is the well-known result [58, 59].

Let us now move to the higher spin operators, restricting to the case of the $O(N)$ singlets. The form of the higher spin currents is most easily written in terms of the constrained fields $\phi^i, i = 1, \ldots, N$ appearing in the action (2.5.7). In terms of these fields, they take the same form (2.1.7)

$$\hat{J}_s = (\hat{\partial}_1 + \hat{\partial}_2)^s C_s^{-1/2} \Big(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2} \Big) \phi^i(x_1) \phi^i(x_2) \Big|_{x_1, x_2 \to x}. \tag{2.5.13}$$

where we have set $d = 2$ since we will only perform a leading order calculation.

It turns out that due to the properties of $C_s^{-1/2}(x)$, in the currents (2.5.13) all terms have both $\phi^i(x_1)$ and $\phi^i(x_2)$ differentiated at least once, so that after resolving the constraint (2.5.2), we have in terms of φ^a

$$\hat{J}_s = (\hat{\partial}_1 + \hat{\partial}_2)^s C_s^{-1/2} \Big(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2} \Big) \big(\varphi^a(x_1) \varphi^a(x_2) + \frac{g^2}{4} \varphi^a \varphi^a(x_1) \varphi^b \varphi^b(x_2) \big) \Big|_{x_1, x_2 \to x} + O(g^4) \tag{2.5.14}$$

One may check, for instance, that for $s = 2$ this yields the correct stress tensor coming from (2.5.3). The reason that we have to keep the term of order g^2 is that, when we compute the descendant by $\partial_\mu D_z^\mu \hat{J}_s$, both terms in (2.5.14) yield a contribution of order g^2 (because the first term is a conserved current at $g = 0$, but the second is not). Using the general equation (2.1.17), we have (recall that s is even)

$$\partial_\mu D_z^\mu \hat{J}_s = \left(h_s(\hat{\partial}_1, \hat{\partial}_2)\partial_1^2 + h_s(\hat{\partial}_2, \hat{\partial}_1)\partial_2^2\right)\left(\varphi^a(x_1)\varphi^a(x_2) + \frac{g^2}{4}\varphi^a\varphi^a(x_1)\varphi^b\varphi^b(x_2)\right)\Big|_{x_1,x_2\to x}. \quad (2.5.15)$$

When acting with ∂^2 on the first term, we use the equation of motion (2.5.6). When acting on the second term, on the other hand, we can actually use the free equation of motion $\partial^2\varphi^a = 0$ to this order, so that $\partial_1^2\varphi^a\varphi^a(x_1) = 2\partial_\mu\varphi^a\partial^\mu\varphi^a$. The final result for the descendant to order g^2 can then be written in the form

$$\begin{aligned}
&\partial_\mu D_z^\mu \hat{J}_s = g^2\hat{K}_{s-1} \\
&\hat{K}_{s-1} = -\left(h_s(\hat{\partial}_1 + \hat{\partial}_3 + \hat{\partial}_4, \hat{\partial}_2) + h_s(\hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4, \hat{\partial}_1) - h_s(\hat{\partial}_4 + \hat{\partial}_3, \hat{\partial}_1 + \hat{\partial}_2)\right) \times \\
&\qquad\qquad \times \partial_{3\mu}\partial_{4\mu}\varphi^a(x_1)\varphi^a(x_2)\varphi^b(x_3)\varphi^b(x_4)\Big|_{x_{1,2,3,4}\to x} \\
&h_s(u,v) = 2v(u+v)^{s-2}C_{s-2}^{3/2}\left(\frac{u-v}{u+v}\right)
\end{aligned} \quad (2.5.16)$$

Note that all $O(N)$ indices here run from 1 to $N-1$. The rest of the calculation is almost exactly the same as in the $d = 4-\epsilon$ case. Computing the descendant two-point function, using (1.4.11) for $d = 2$ and the current two-point function (2.1.14) (with $N \to N-1$), we find the result

$$\gamma_s = \frac{g_*^4}{4\pi^2}(N-2)\left(\frac{1}{s} - \frac{1}{2} + H_{s-2}\right) = \frac{\epsilon^2}{N-2}\left(\frac{1}{s} - \frac{1}{2} + H_{s-2}\right) \quad (2.5.17)$$

where $H_k = \sum_{n=1}^k 1/n$ is the harmonic number. The $1/N$ expansion of this result precisely matches the expansion of (2.3.16) in $d = 2+\epsilon$. In the large spin limit, we see the logarithmic behavior (since $H_k \sim \log(k)$ at large k)

$$\gamma_s = \frac{\epsilon^2}{N-2}\left(\log(s) + \gamma - \frac{1}{2} - \frac{1}{2s} + O(1/s^2)\right). \quad (2.5.18)$$

Also, we note that the leading order in γ_s is ϵ^2, although the leading order anomalous dimension of the ϕ field is ϵ (2.5.12). This may seem to contradict the expected $s \to \infty$ behavior. The simple resolution of this "paradox" is suggested by looking at the large N result (2.3.17) for the singlet currents, expanded near $d = 2 + \epsilon$:

$$\gamma_s = \frac{\epsilon}{N} - \frac{\epsilon}{N}\frac{\Gamma(3+\epsilon)}{2(1+\epsilon)}\frac{1}{s^{\epsilon}} + ... = \frac{\epsilon^2}{N}\log(s) + ... \tag{2.5.19}$$

We see that for the singlets the $2\gamma_\phi$ term is canceled by the expansion of the second term, coming from the higher-spin current tower, and the $\log(s)$ is exactly what one gets from expanding the harmonic number H_{s-2}. As for the non-singlet operators, from (2.3.17) one gets $\gamma_{s(ij)} = \frac{\epsilon}{N} + O(\frac{1}{s^2})$, and it is evident that the leading order is indeed $2\gamma_\phi$ as expected [39, 40, 46–48]. Thus, a finite N calculation of the anomalous dimensions of the non-singlet operators should yield a result starting at order ϵ, unlike (2.5.17). We leave the more detailed discussion of the non-singlet currents for future work.

2.6 Some $d = 3$ estimates

For the $O(N)$ models with $N \geq 3$, we can combine the information from the $d = 4 - \epsilon$ and $d = 2 + \epsilon$ expansions to obtain some estimates for the anomalous dimensions of the singlet higher spin currents in $d = 3$. The simplest way to do this is to use a "two-sided" Padé approximant. For any given physical quantity assumed to be a continuous function of dimension d, we can construct the Padé approximant

$$\text{Padé}_{[m,n]}(d) = \frac{A_0 + A_1(4-d) + A_2(4-d)^2 + \ldots + A_m(4-d)^m}{1 + B_1(4-d) + B_2(4-d)^2 + \ldots + B_n(4-d)^n}\,, \tag{2.6.1}$$

where the coefficients are fixed by matching the known perturbative expansions in $d = 4 - \epsilon$ and $d = 2 + \epsilon$. Rather than performing this procedure on $\gamma_s(d)$ itself, guided by the expected large spin behavior [39, 40, 46–48], we find it more convenient to consider the quantity

$$f_s(d) = \gamma_s(d) - 2\gamma_\phi(d) \tag{2.6.2}$$

From the results (2.2.24), (2.5.17), we can obtain the ϵ expansion of this quantity to order ϵ^2. Further information in $d = 4-\epsilon$ can be obtained using the result of [51], who derived the anomalous dimensions of the higher spin operators in the $O(N)$ theory to order ϵ^3.[10] For the singlet currents, it reads

$$\gamma_s = \frac{(N+2)\lambda^2\left(s^2+s-6\right)}{128\pi^4 s(s+1)}$$

[10] For $N = 1$, the result is known to order ϵ^4 [60].

$$-\quad \frac{(N+2)(N+8)\lambda^3\left(16s(s+1)H_s+s\left(s^3+2s^2-39s-16\right)+12\right)}{4096\pi^6 s^2(s+1)^2}+O(\lambda^4) \quad (2.6.3)$$

where H_s is the harmonic number. This vanishes at $s=2$, as expected. It is also interesting to check the large spin behavior, which yields (using the known result for γ_ϕ to order λ^3, see e.g. [61])

$$\gamma_s = 2\gamma_\phi - \frac{(N+2)\left(12\pi^2\lambda^2+(N+8)\lambda^3\left(\log(s)-\gamma-5/2\right)\right)}{256\pi^6 s^2}+O(1/s^3)\,. \quad (2.6.4)$$

We see that a logarithmic term arises at subleading order in the coupling constant, consistently with general expectations [39, 62]. Using the value of the critical coupling [61]

$$\lambda_* = \frac{8\pi^2\epsilon}{N+8}+\frac{24\pi^2(3N+14)\epsilon^2}{(N+8)^3}+\dots \quad (2.6.5)$$

we can obtain the ϵ expansion of γ_s around $d=4$ to order ϵ^3. Further using the ϵ expansions of γ_ϕ near $d=2$ and $d=4$, we can get the function $f(d)$ defined in (2.6.2) to the same order

$$\begin{aligned} f_s(4-\epsilon) &= -\frac{3\epsilon^2(N+2)}{s(s+1)(N+8)^2}+O(\epsilon^3) \\ f_s(2+\epsilon) &= \left(-\frac{\epsilon}{N-2}+\frac{(N-1)\epsilon^2}{(N-2)^2}\right)+\frac{\epsilon^2}{N-2}\left(\frac{1}{s}-\frac{1}{2}+H_{s-2}\right)+O(\epsilon^3) \end{aligned} \quad (2.6.6)$$

where for simplicity we did not write explicitly the $O(\epsilon^3)$ in $d=4-\epsilon$, it can be read off from (2.6.4). This allows to construct Padé approximants (2.6.1) with a maximum value $n+m=6$. Carrying out this procedure for general N, we find that Padé$_{[3,2]}$ (which only uses $f(4-\epsilon)$ to order ϵ^2) and Padé$_{[4,2]}$ appear to give the best agreement with the analytic large N result (2.3.16) over the full range $2\leq d\leq 4$, with Padé$_{[3,2]}$ in fact working slightly better. Using this approximant, we obtain a $d=3$ estimate for the function $f_s(d)$ in (2.6.2). To obtain the anomalous dimensions γ_s, we can then add back the contribution $2\gamma_\phi$ using the best available estimates that were collected in Table 2 of [63] for a few low values of N. The results of this procedure for $s=4,6,8,10$ and for several values of N are listed in the table below. For comparison, the large N formula (2.3.16) gives for $d=3$

$$\gamma_s = \frac{16(s-2)}{(2s-1)3\pi^2}\frac{1}{N}+O(1/N^2)\,. \quad (2.6.7)$$

Using this for $N=20$, one would get $\gamma_4=0.0077, \gamma_6=0.0098, \gamma_8=0.0108, \gamma_{10}=0.0114$. The results for $s=4$ given in Table 2.1 appear to be consistent with the ones given in [66].

N	3	4	5	6	10	20
$\gamma_{s=4}$ (Padé$_{[3,2]}$)	0.0261	0.0257	0.0208	0.0195	0.0158	0.0082
$\gamma_{s=6}$ (Padé$_{[3,2]}$)	0.0318	0.0310	0.0258	0.0240	0.0191	0.0100
$\gamma_{s=8}$ (Padé$_{[3,2]}$)	0.0342	0.0332	0.0278	0.0259	0.0206	0.0110
$\gamma_{s=10}$ (Padé$_{[3,2]}$)	0.0353	0.0343	0.0289	0.0269	0.0214	0.0115

Table 2.1: Padé estimates for the anomalous dimensions of the singlet currents with $s = 4, 6, 8$ in the 3d critical $O(N)$ models. The estimates are obtained by constructing a "two-sided" Padé approximant of the function (2.6.2) and adding at the end the contribution $2\gamma_\phi$ using the available results collected in [63]. For $N = 10, 20$, the value of γ_ϕ is obtained from the large N result known to order $1/N^3$ [64,65].

For $N = 1$ and $N = 2$, the nonlinear sigma model result cannot be used since there is no perturbative fixed point in $d = 2+\epsilon$ for these values of N. Simple Padé approximants of the $d = 4-\epsilon$ result appear to yield poles in $2 < d < 4$ in this case, so we will resort to the unresummed ϵ expansion to obtain some estimates. For $N = 1$, setting $\epsilon = 1$ in $f(4-\epsilon)$ expanded to order ϵ^3, and adding back the 3d value of $2\gamma_\phi^{N=1} = 0.0363$ [67–70], we obtain the following $d = 3$ estimates

$$\begin{aligned} \gamma_{s=4}^{N=1} &= 0.0240\,, \qquad \gamma_{s=6}^{N=1} = 0.0300 \\ \gamma_{s=8}^{N=1} &= 0.0324\,, \qquad \gamma_{s=10}^{N=1} = 0.0336\,. \end{aligned} \tag{2.6.8}$$

While we do not expect these to be high precision results, we observe that they appear to be quite close to the estimates derived in [62]. For the spin 4 operator, [71] obtained the slightly lower value $\gamma_4 = 0.0208(12)$. For $N = 2$, following a similar procedure and using $2\gamma_\phi^{N=2} = 0.0381$ [72], we obtain

$$\begin{aligned} \gamma_{s=4}^{N=2} &= 0.0252\,, \qquad \gamma_{s=6}^{N=2} = 0.0315 \\ \gamma_{s=8}^{N=2} &= 0.0340\,, \qquad \gamma_{s=10}^{N=2} = 0.0353\,. \end{aligned} \tag{2.6.9}$$

In all cases, we observe that the anomalous dimensions of the higher spin operators is rather small (similarly to what happens for the anomalous dimension of ϕ). From the results in Table 2.1, and (2.6.8),(2.6.9), we also notice some non-monotonic behavior as a function of N, with a maximum between $N = 3$ and $N = 4$. A qualitatively similar non-monotonic behavior can be observed in the sphere free energy [73, 74] and C_T [63]. It would be interesting to understand better the origin of this behavior and the relation between these quantities.

2.7 Appendix: Technical details on the computation of the descendant 2-point function

The calculation of the descendant two-point functions is carried out using the relation of Gegenbauer polynomials to hypergeometric functions (see appendix of [75]). We will illustrate the technique to calculate the function A_s (see eq. 2.2.20) as a function of s in terms of Gegenbauer integrals. The calculation of the other structures is similar. After introducing the Schwinger parametrization, the h_s function can be written as:

$$\begin{gathered} h_s(-2\alpha_1\hat{x} - 2\alpha_3\hat{x} - 2\alpha_4\hat{x}, -2\alpha_2\hat{x}) = (-1)^{s-1}(2\hat{x})^{s-1}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^s\tilde{h}_s(1-2\tilde{\alpha}_2); \\ \tilde{\alpha}_n = \frac{\alpha_n}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \qquad \tilde{h}_s(x) = C^{3/2}_{s-1}(x) - 3(1-x)C^{5/2}_{s-2}(x) \end{gathered} \tag{2.7.1}$$

and analogously for other arguments. The contraction is then compactly written as:

$$\frac{(-1)^{s-1}(2\hat{x})^{2s-2}}{(4\pi^2)^4}\int_0^\infty \prod_{n=1}^4 d\alpha_n(\sum_{n=1}^4 \alpha_n)^{2s-2}\exp(-x^2\sum_{n=1}^4\alpha_n)(\tilde{h}_s(1-2\tilde{\alpha}_2)+(-1)^s\tilde{h}_s(1-2\tilde{\alpha}_1))^2 \tag{2.7.2}$$

It is convenient to separate the integration other the sum $\sum_{n=1}^4 \alpha_n$ by introducing a delta function $\int_0^\infty dp\delta(\sum_{n=1}^4\alpha_n - p)$:

$$\begin{aligned} &\int_0^\infty dp p^{2s-2+3}\exp(-x^2p)\int_0^1\prod_{n=1}^4 d\tilde{\alpha}_n\delta(\sum_{n=1}^4\tilde{\alpha}_n - 1)(\tilde{h}_s(1-2\tilde{\alpha}_2)+(-1)^s\tilde{h}_s(1-2\tilde{\alpha}_1))^2 \\ &\qquad = \frac{(2s+1)!}{(x^2)^{2s+2}}\iint\limits_{0<1-\tilde{\alpha}_1-\tilde{\alpha}_2<1} d\tilde{\alpha}_1 d\tilde{\alpha}_2(1-\tilde{\alpha}_1-\tilde{\alpha}_2)(\tilde{h}_s(1-2\tilde{\alpha}_2)+(-1)^s\tilde{h}_s(1-2\tilde{\alpha}_1))^2 \end{aligned} \tag{2.7.3}$$

The goal now is to calculate the integral of functions $\tilde{h}$. First we study the integral of two $\tilde{h}$ with the same argument:

$$\int_0^1 d\tilde{\alpha}(1-\tilde{\alpha})^2(\tilde{h}_s(1-2\tilde{\alpha}))^2 \tag{2.7.4}$$

The idea now is to employ the Rodrigues formula for the Gegenbauer polynomial:

$$C^\nu_s(1-2\tilde{\alpha}) = \frac{4^s}{s!}\frac{\Gamma(s+\nu)\Gamma(s+2\nu)}{\Gamma(\nu)\Gamma(2s+2\nu)}(\tilde{\alpha}(1-\tilde{\alpha}))^{-\nu+1/2}\frac{d^s}{d\tilde{\alpha}^s}(\tilde{\alpha}(1-\tilde{\alpha}))^{s+\nu-1/2} \tag{2.7.5}$$

We again split the integral into two parts:

$$\int_0^1 d\tilde{\alpha}(1-\tilde{\alpha})^2\big(C^{3/2}_{s-1}(1-2\tilde{\alpha})-3(2\tilde{\alpha})C^{5/2}_{s-2}(1-2\tilde{\alpha})\big)\tilde{h}_s(1-2\tilde{\alpha}_1) \tag{2.7.6}$$

We now act on the second $\tilde{h}$ with the $C^{3/2}_{s-1}$ using the Rodrigues formula, integrating by parts $s-1$ times. The boundary terms vanish thanks to the power of $\tilde{\alpha}(1-\tilde{\alpha})$ under the derivative. The prefactor $(\tilde{\alpha}(1-\tilde{\alpha}))^{-1}$ combines with $(1-\tilde{\alpha})^2$ to $\frac{1}{\tilde{\alpha}}-1$. Then, since $\tilde{h}_s(1-2\tilde{\alpha})$ is a polynomial of degree $s-1$ in $\tilde{\alpha}$:

$$\tilde{h}_s(1-2\tilde{\alpha})=\sum_{k=0}^{s-1}c_k\tilde{\alpha}^k, \tag{2.7.7}$$

after $s-1$ integrations by parts only two terms survive the differentiation

$$\begin{aligned}(-1)^{s-1}\frac{d^{s-1}}{d\tilde{\alpha}^{s-1}}c_{s-1}\tilde{\alpha}^{s-1}&=(-1)^{s-1}(s-1)!c_{s-1};\\ (-1)^{s-1}\frac{d^{s-1}}{d\tilde{\alpha}^{s-1}}c_0\frac{1}{\tilde{\alpha}}&=\frac{(s-1)!}{\tilde{\alpha}^s}\end{aligned} \tag{2.7.8}$$

The remaining integrals are now of the beta-function type, for instance,

$$\begin{aligned}&\int_0^1 d\tilde{\alpha}c_{s-1}(\tilde{\alpha}(1-\tilde{\alpha}))^s\\ &\int_0^1 d\tilde{\alpha}c_0(1-\tilde{\alpha}))^s\end{aligned} \tag{2.7.9}$$

This is the main idea of the calculation, the rest is basically collecting all the coefficients and applying the same method to the other integrals which will appear (all of them will be of the same type though). For the sake of reference the coefficients of $\tilde{h}_s(1-2\tilde{\alpha})$ are obtained most easily by using the relation of Gegenbauer polynomials to the hypergeometric function:

$$C^\nu_s(1-2x)=\frac{(2\nu)_s}{s!}F\big(-s,s+2\nu;\nu+\frac{1}{2};x\big) \tag{2.7.10}$$

(this is the reason we used $1-2\tilde{\alpha}$ as the argument). After collecting all the factors, the overall answer for the integral in (2.7.3) is:

$$\frac{(s-1)s(s+1)(s+2)}{8(2s+1)} \tag{2.7.11}$$

Chapter 3

Anomalous dimensions in the fermionic O(N) and U(N) models

3.1 Introduction and Summary

The Gross-Neveu (GN) model [76]

$$\mathcal{L}_{\rm GN} = \bar{\psi}_i \partial\!\!\!/ \psi^i + \frac{g}{2}(\bar{\psi}_i \psi^i)^2 \tag{3.1.1}$$

is a classic example of quantum field theory of interacting fermions. Here ψ^i, $i = 1, \ldots, N_f$ denotes a collection of N_f Dirac fermions, so that the theory has a manifest $U(N_f)$ global symmetry. When studied as a function of dimension d, there is evidence that the GN model describes a unitary interacting CFT in $2 < d < 4$, which corresponds to a non-trivial UV fixed point of (3.1.1). Despite the fact that the quartic interaction is irrelevant, the model is formally renormalizable in the framework of the $1/N$ expansion [77,78], and this approach can be used to compute various physical quantities at the interacting fixed point as a function of d (with $d = 3$ being the physically interesting dimension), see [57] and references therein for a comprehensive review. Another approach to the Gross-Neveu CFT is the Wilson-Fisher ϵ-expansion: in $d = 2$ the four-fermi interaction is renormalizable, and working in $d = 2 + \epsilon$ one finds UV fixed points which are weakly coupled for small ϵ. Critical exponents at these fixed points can be computed by usual perturbation theory for finite N_f. In [79,80], it was suggested that the fermionic CFT in $2 < d < 4$ admits yet another perturbative description near $d = 4$, in terms of the Gross-Neveu-Yukawa theory

$$\mathcal{L}_{\rm GNY} = \frac{1}{2}(\partial_\mu \sigma)^2 + \bar{\psi}_i \partial\!\!\!/ \psi^i + g_1 \sigma \bar{\psi}_i \psi^i + \frac{1}{4!} g_2 \sigma^4 \ . \tag{3.1.2}$$

Working in $d = 4 - \epsilon$, one finds stable IR fixed points for any N_f, and there is considerable evidence that these fixed points correspond to the same CFT defined by the UV fixed point of the GN model. The information from the various perturbative approaches to the fermionic CFT can be used to obtain estimates for critical exponents and other physical quantities in the physical dimension $d = 3$, see for instance [30, 81–83].

When the interactions are turned off, the theory of N_f massless fermions defines a unitary CFT in any dimension d. Being a free CFT, it enjoys an exact higher-spin (HS) symmetry and corresponding exactly conserved currents of all spins which are constructed from fermion bilinears. In general d, the spectrum of these currents is more involved than that of the free scalar CFT. There are totally symmetric currents, but also currents in mixed-symmetry representations of $SO(d)$ that are obtained using the totally antisymmetric products $\gamma_{\nu_1 \dots \nu_k}$ of gamma matrices. Their explicit construction will be discussed in section 3.2 below. In the free CFT, all these currents are conserved; they have exact scaling dimension $\Delta = d - 2 + s$, where s is the spin, and belong to short representations of the conformal algebra. When interactions are turned on, the currents acquire anomalous dimensions and are no-longer exactly conserved (except for the stress tensor or spin 1 currents corresponding to the global symmetry):

$$\partial \cdot J_s \sim g K_{s-1} \, , \tag{3.1.3}$$

where g is a parameter that controls the HS symmetry breaking. In the large N expansion we have $g \sim 1/\sqrt{N}$, and in the ϵ-expansion g is a power of ϵ, so that the HS symmetry is weakly broken at large N or small ϵ. In the above equation, K_{s-1} denotes an operator of spin $s - 1$ and dimension $d - 1 + s + O(g)$: this is the primary operator of the unbroken theory ($g = 0$) which recombines with J_s to form a long multiplet in the interacting theory. As reviewed in section 3.3.1, the non-conservation equation (3.1.3) can be used to deduce the anomalous dimensions of the broken currents to leading order in g, by computing correlators in the unbroken theory [22, 23]. This method was applied recently in [33, 84] to the scalar $O(N)$ model in its large N and ϵ-expansions, and in [35] it was also used to extract the $1/N$ anomalous dimensions of HS currents in the bosonic and fermionic 3d Chern-Simons vector models of [85, 86]. In this chapter, we will apply the same method to the critical GN model, both in the large N expansion for any d, and in the ϵ-expansions near $d = 2$ and $d = 4$, and extract the anomalous dimensions of weakly broken currents to leading order. At large N, we reproduce known results obtained long ago by diagrammatic methods [87]. In $d = 2+\epsilon$ and $d = 4-\epsilon$, as far as we know, our results are new. In all cases, we find precise matching of ϵ-expansions and large N (including the recent $1/N^2$ results of [88]) in their overlapping regime of validity, which provides a nice cross-check of the various approaches to the interacting CFT.

In the context of the AdS/CFT correspondence, the free fermionic CFT$_d$ (restricted to its $U(N_f)$ singlet sector) should be holographically dual to the so-called "type B" higher-spin gravity theory in AdS$_{d+1}$, which includes towers of massless higher-spin gauge fields in one-to-one correspondence with the conserved currents in the boundary CFT$_d$.[1] In this context, the critical Gross-Neveu CFT can be thought of as a "double-trace" deformation of the free theory, and it follows from general arguments [31] that the AdS dual of the UV fixed point should be the same higher-spin gravity theory, with the choice of alternate boundary condition ($\Delta = 1$ instead of $\Delta = d-1$) on the bulk scalar field dual to the $\bar{\psi}\psi$ operator, in analogy with the original conjecture [29] in the case of the $O(N)$ model. With alternate boundary conditions in the bulk, the higher-spin fields are expected to acquire masses at loop level, corresponding to the fact that the anomalous dimensions start at $1/N$-order on the CFT side. The role of the Higgs field in the bulk [32] is played by a two-particle state with the appropriate quantum numbers, which should correspond to the operator appearing on the right-hand side of the non-conservation equation (3.1.3). In the large N CFT this operator is indeed of the double-trace type, and we will determine its explicit form in section 3.3.2. Schematically,

$$\partial \cdot J_s \sim \frac{1}{\sqrt{N}} \sum_{s'<s} \partial^n B_{(s',1)} \partial^m J_0 \,, \tag{3.1.4}$$

where $B_{(s,1)}$ denotes the mixed-symmetry current in the representation $[s, 1, 0 \ldots, 0]$ (dual to the corresponding mixed-symmetry field in the bulk), and J_0 denotes the scalar operator with $\Delta = 1 + O(1/N)$. In particular, this equation implies that the relevant bulk one-loop diagrams responsible for the anomalous dimensions involve the cubic coupling of a totally symmetric field, a mixed-symmetry field, and a scalar. It would be interesting to fix the form of these couplings directly in the bulk. Note

[1] Such type B theory is known at non-linear level only in the case of AdS$_4$ in the form of Vasiliev equations [26], see [89–91] for reviews with a focus on AdS/CFT applications. In general d, one can construct the spectrum and free equations of motion of the bulk theory, and in principle reconstruct interactions order by order in perturbation theory, but fully non-linear equations of motion of the Vasiliev type [26, 28] are not known.

that in $d = 3$ the mixed-symmetry fields are in fact related to the totally symmetric fields, due to $\gamma_{\mu\nu} = i\epsilon_{\mu\nu\rho}\gamma_\rho$, but this is not so in general d, and it would be interesting to study more generally the 3-point couplings involving mixed-symmetry fields in the bulk. Note also that (3.1.4) contains more information than just the anomalous dimensions of J_s: for instance, it implies that the 3-point functions $\langle J_s(x)B_{(s',1)}(y)J_0(z)\rangle$ break the J_s current conservation (for $s > s'$) already at leading order in N.

The methods we use to fix the HS anomalous dimensions, based on the idea of multiplet recombination, are closely related to the approach put forward in [24], see [52, 92–100] for subsequent related work. In this approach, the leading order anomalous dimensions of various composite operators in the ϵ-expansion of $O(N)$ or GN models were fixed using conformal symmetry and the required form of multiplet recombination (essentially dictated by the classical equations of motion) of the nearly free fields ϕ or ψ. In section 3.4.1, we apply a similar approach to fix the scaling dimensions of some scalar composites in the critical GN model at large N, as well as in the GNY model in $d = 4 - \epsilon$. We also show how similar methods can be used in the case of the large N expansion of the scalar $O(N)$ model. In particular, this appears to lead to a relatively simple derivation of the $1/N$ anomalous dimension of the scalar singlet operator (with $\Delta = 1 + O(1/N)$ in GN and $\Delta = 2 + O(1/N)$ in the $O(N)$ model) compared to the traditional diagrammatic expansion (see e.g. [65, 101, 102]).

In section 3.4.2, we move on to study a different type of operators with spin, namely the "double-trace" operators $\sim \sigma\partial^s\sigma$ built out of the scalar singlet $\sigma \sim \bar{\psi}\psi$. These operators have twist $2+O(1/N)$ at large N, and for general d they are not almost conserved currents.[2] We compute their anomalous dimensions in section 3.4.2 directly from Feynman diagrams in $1/N$ perturbation theory. The general d result is given in (3.4.34), and in $d = 3$ it reads

$$\Delta_{\sigma\partial^s\sigma} - s - 2\Delta_\sigma = \frac{32}{\pi^2(2s+1)}\frac{1}{N} + O(1/N^2)\,. \tag{3.1.5}$$

From the AdS point of view, the anomalous dimension defined by the right-hand side has the interpretation of the interaction energy associated to the two-particle state of two bulk scalar fields with orbital angular momentum s. Perhaps surprisingly, we find that this quantity is positive, corresponding to an effective repulsive interaction, for all spins in $2 < d < 4$.[3] In section 3.4.2 we compare this

[2] They become conserved in the $d \to 4$ limit, where they correspond to one of the two towers of exactly conserved HS operators in the GNY model (3.1.2) at the $d = 4$ trivial fixed point $g_1 = g_2 = 0$. The two towers non-trivially mix in $d = 4 - \epsilon$, as explained in section 3.3.4.

[3] This result is not in violation of Nachtmann's theorem [103], because in $d < 4$ the operators $\sim \sigma\partial^s\sigma$ are not the

result, as well as the one for the analogous operators (with $\sigma \sim \phi^2$) in the $O(N)$ model [45], to the analytic bootstrap analysis [39, 40] (see [41, 62, 104–111] for relevant related work) of the large spin expansion of the anomalous dimensions of double-trace-like operators of the form $O\partial^s O$. The OPE data needed for the bootstrap analysis is obtained in Appendix 3.5 from that of the free theories using the AdS/CFT dictionary for double-trace flows [29,31]. We find that the $\sigma\partial^s\sigma$ anomalous dimensions in the GN and $O(N)$ model can be exactly reproduced in the analytic bootstrap approach, provided one suitably regulates the sum over the exchange of the infinite tower of nearly conserved currents of all even spins. Even though the contribution of each nearly conserved even spin current to the $\sigma\partial^s\sigma$ anomalous dimension is negative, the regularized sum over the HS tower appears to yield a final positive result in the GN model in agreement with (3.1.5), and a vanishing result for the $d = 3$ $O(N)$ model, in agreement with [45] (see also [112]). More generally, the arbitrary d results can also be reproduced in the same way. As a consistency check of the regularized sum over spins, we also show that it correctly implies vanishing of the anomalous dimensions of the double-trace operators in the free fermionic and scalar CFT in any d. Finally, in section 3.4.2 we use the same analytic bootstrap approach to compute the anomalous dimensions of the same type of double-trace operators in the bosonic and fermionic vector models coupled to Chern-Simons gauge theory in $d = 3$ [85, 86], working to leading order in $1/N$ but exactly in the 't Hooft coupling λ. In the CS-fermion model, the anomalous dimensions vanish to the order $1/N$ for all λ, and in the CS-scalar model they are given by an expression that smoothly interpolates between the free scalar at $\lambda = 0$ and the critical GN model at $\lambda \to 1$, in agreement with the conjectured 3d bosonization duality [21, 113].

3.2 Free Fermions

Let us consider the free CFT of N_f massless Dirac fermions. For general d, the spectrum of bilinear primary operators is more complicated than that of the free scalar CFT. In addition to a tower of totally-symmetric conserved tensors $J_{\mu_1\cdots\mu_s}$, as in the free scalar theory, and the scalar operator $J_0 = \bar{\psi}\psi$ of dimension $\Delta = d-1$, we have towers of conserved tensors of mixed-symmetry $B_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_k}$ and a finite number of anti-symmetric tensors $B_{\nu_1\cdots\nu_k}$ that are not conserved currents, see e.g. [114, 115].

leading twists in the $\sigma\sigma$ OPE, due to the presence of the nearly conserved HS currents with twist $d - 2 + O(1/N)$.

3.2.1 Totally symmetric higher-spin currents

As in the previous chapter, we focus on studying the contracted form of the current:

$$\hat{J}_s(x,z) = J_{\mu_1 \dots \mu_s}(x) z^{\mu_1} \dots z^{\mu_s} \,. \tag{3.2.1}$$

where z_μ is again a constant null-vector. One may restore the explicit indices on the currents by acting with the differential operator (1.4.2) in z-space. The explicit form of the currents can be conveniently given as

$$(\hat{J}_s)^i{}_j = f_s(\hat{\partial}_1, \hat{\partial}_2) \bar{\psi}_j(x_1) \hat{\gamma} \psi^i(x_2) \Big|_{x_{1,2}=x} \,, \qquad s \geq 1 \,, \tag{3.2.2}$$

where $\hat{\partial}_{1,2} \equiv z \cdot \partial_{1,2}$, and $f_s(u,v)$ is a homogeneous function of total degree $s-1$. Here $i,j = 1, \dots, N_f$ are the flavor indices, and we can of course decompose $(J_s)^i{}_j$ into the $U(N_f)$ singlet part, and the adjoint (traceless) currents $(J_s^A)^i{}_j \sim \bar{\psi}_j \hat{\partial}^{s-1} \hat{\gamma} \psi^i - \frac{1}{N_f} \delta^i_j \bar{\psi} \hat{\partial}^{s-1} \hat{\gamma} \psi$. In the remaining of this section we will mostly omit flavor indices for simplicity.

Imposing the conservation condition $\partial_\mu D_z^\mu \hat{J}_s = 0$ and using the free Dirac equation one finds for $f_s(u,v)$:

$$\Big(\frac{d}{2} (\partial_u + \partial_v) + u \partial_u^2 + v \partial_v^2 \Big) f_s = 0 \,. \tag{3.2.3}$$

The solution is given by

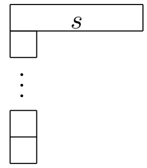
$$f_s = (\hat{\partial}_1 + \hat{\partial}_2)^{s-1} C_{s-1}^{d/2-1/2} \Big(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2} \Big) \,, \tag{3.2.4}$$

which takes the same form as the free scalar CFT (see (2.1.7) and also [33, 84]), up to the shifts $s \to s-1$, $d \to d+2$. Alternatively, we can obtain the same differential equation (3.2.3) by imposing that (3.2.2) is a conformal primary (see e.g. [37]). Of course, these operators have exact dimension $\Delta_s = d - 2 + s$ in the free CFT.

3.2.2 Mixed-symmetry currents

The currents constructed above are totally symmetric, corresponding to the representation $(s, 0, 0, \dots)$ of $SO(d)$. In general dimension d, there also exist conserved tensor primaries of the symmetry

$(s, 1, .., 1, 0, \ldots)$, corresponding to the Young diagram



(3.2.5)

All such mixed-symmetry currents can in principle be extracted from a simple generating formula [115]

$$\tilde{B}_{\nu_1 \ldots \nu_k}(x) = \bar{\psi}(x+y)\gamma_{\nu_1 \ldots \nu_k}\psi(x-y)\Big|_{y=0}, \tag{3.2.6}$$

where $\gamma_{\nu_1 \ldots \nu_k} \equiv \gamma_{[\nu_1} \ldots \gamma_{\nu_k]}$ is the totally anti-symmetrized product of γ-matrices. One can easily show that it generates conserved currents except for the case of the totally anti-symmetric primaries $(1, ..., 1)$

$$\bar{\psi}\gamma_{\nu_1 \ldots \nu_k}\psi\,, \qquad\qquad k > 1\,, \tag{3.2.7}$$

which are not conserved and should be AdS/CFT dual to anti-symmetric massive fields. However, the simple generating function (3.2.6), when expanded in y, does not give conformal primaries, but a mixture with descendants (the expansion of (3.2.6) does not produce irreducible tensors). To obtain the primary operators, let us look for the generating function

$$B_{\nu_1 \ldots \nu_k} = \sum_s \frac{1}{s!} B_{\mu_1 \ldots \mu_s, \nu_1 \ldots \nu_k} z^{\mu_1} \cdots z^{\mu_s} \tag{3.2.8}$$

where, as for the totally symmetric tensors, we use a null polarization vector to contract all the symmetric indices. The mixed-symmetry primaries have to obey a number of irreducibility conditions:

$$B_{(\mu_1 \ldots \mu_s, \mu_{s+1})\nu_1 \ldots \nu_k} = 0\,, \qquad\qquad \delta^{\rho\sigma} B_{\mu_1 \ldots \mu_{s-2}\rho\sigma, \nu_1 \ldots \nu_k} = 0\,, \tag{3.2.9}$$

$$\partial^\lambda B_{\mu_1 \ldots \mu_{s-2}\lambda[\nu_0, \nu_1 \ldots \nu_k]} = 0\,. \tag{3.2.10}$$

Here symmetrization over all μ indices and anti-symmetrization over all ν indices is implied, which is indicated by the brackets. The first condition imposes $(s, 1, \ldots, 1, 0, \ldots)$ symmetry; the second one tells that the tensor is traceless in all the indices provided the first condition is satisfied; the third one implies that the divergence projected onto the $(s-1, 1 \ldots, 1, 0, \ldots)$ symmetry vanishes (there are two independent divergences: $(s, 1, \ldots, 1, 0, \ldots)$ and $(s-1, 1, \ldots, 1, 0, \ldots)$ and only the latter is the primary that needs to be decoupled). The most general ansatz for the generating function reads:[4]

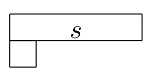
$$
\begin{aligned}
B_{\nu_1\dots\nu_k}(\hat{\partial}_i; z) = F_1\bar{\psi}(x_1)\gamma_{\nu_1\dots\nu_k\rho}z^\rho\psi(x_2) + F_2\bar{\psi}(x_1)\gamma_{[\nu_1\dots\nu_{k-1}}z_{\nu_k]}\psi(x_2)+ \\
+ F_3\partial_{[\nu_1}\bar{\psi}(x_1)\gamma_{\nu_2\dots\nu_{k-1}\rho}z^\rho z_{\nu_k]}\psi(x_2) + F_4\bar{\psi}(x_1)\gamma_{[\nu_1\dots\nu_{k-2}\rho}z^\rho z_{\nu_{k-1}}\partial_{\nu_k]}\psi(x_2)\,,
\end{aligned}
\tag{3.2.11}
$$

where again anti-symmetrization over ν indices is implied. Functions $F_{1,2,3,4}$ depend on $\hat{\partial}_{1,2}$. The usage of the null polarization vector z^μ takes away the traces in the μ indices. However, the trace with respect to one symmetric and one antisymmetric index $\delta^{\mu\nu}$ needs to be subtracted by hand. Altogether, the Young, the conservation and the tracelessness conditions, when expressed in terms of the generating function, give:

$$
z^\rho B_{\rho\nu_1\dots\nu_{k-1}} = 0\,, \qquad \partial_\mu D^\mu_z D^z_{[\nu_1}B_{\nu_2\dots\nu_{k+1}]} = 0\,, \qquad D^\mu_z B_{\mu\nu_1\dots\nu_{k-1}} = 0\,, \tag{3.2.12}
$$

where in the second expression the anti-symmetrization over all ν's is implied. The trace with respect to z and a free index μ has to be taken with the help of the Thomas derivative (1.4.2).

In the following we would like to compute the anomalous dimensions of the totally-symmetric higher-spin currents. The non-conservation operator of those, as will be shown below, contains no more than two gamma-matrices. Therefore it will only involve the simplest mixed-symmetry primaries with symmetry of the hook diagram



. (3.2.13)

A simplification occurs in this case and only two terms of (3.2.11) survive

$$
B_\mu = F_1\bar{\psi}(x_1)\gamma_{\mu\nu}z^\nu\psi(x_2) + F_2\bar{\psi}(x_1)z_\mu\psi(x_2)\,. \tag{3.2.14}
$$

The Young condition is trivial here and the conservation/tracelessness can be read from

$$
\partial_\mu D^\mu_z D_{z[\nu_1}B_{\nu_2]} = 0\,, \qquad D^\mu_z B_\mu = 0\,. \tag{3.2.15}
$$

[4]In principle, one can introduce auxiliary anti-commuting variables as to hide the ν indices and work out the super-symmetric Thomas derivative. Fortunately we will need only the simplest mixed-symmetry currents.

Solving these equations, we find that the result for the $(s,1)$ mixed-symmetry currents is

$$B_\mu(x,z) = F_1 \bar{\psi}(x_1)\gamma_{\mu\nu} z^\nu \psi(x_2) + F_2 \bar{\psi}(x_1) z_\mu \psi(x_2)\Big|_{x_{1,2}=x}\,, \tag{3.2.16}$$

$$F_1 = (\hat{\partial}_1 + \hat{\partial}_2)^{s-1} C_{s-1}^{d/2-1/2}(w)\,, \qquad F_2 = (\hat{\partial}_1 + \hat{\partial}_2)^{s-1} C_{s-2}^{d/2-1/2}(w)\,, \tag{3.2.17}$$

where $w = (\hat{\partial}_1 - \hat{\partial}_2)/(\hat{\partial}_1 + \hat{\partial}_2)$, and it is understood that $x_{1,2} \to x$ after taking all derivatives. Let us give few examples. The $s=1$ case is trivial — it is not a current:

$$B_\mu = \bar{\psi}\gamma_{\mu\nu}\psi z^\nu\,. \tag{3.2.18}$$

The simplest genuine mixed-symmetry current is $(2,1)$ (see also [115] for the index form):

$$B_\mu = (d-1)(\hat{\partial}_1 - \hat{\partial}_2)\bar{\psi}\gamma_{\mu\nu} z^\nu \psi + (\hat{\partial}_1 + \hat{\partial}_2) z_\mu \bar{\psi}\psi\,. \tag{3.2.19}$$

Note that while the divergence of the mixed-symmetry current $\partial^\mu B_{\mu\mu_2,\ldots\mu_{s-1}[\nu_0,\nu_1]}$ that has $(s-1,1)$ symmetry does vanish, but the divergence with respect to the ν index is not zero. It defines a descendant

$$\partial^\nu B_\nu = F_d(\hat{\partial}_1, \hat{\partial}_2)\bar{\psi}\psi\,, \tag{3.2.20}$$

$$F_d(u,v) = (v-u)(u+v)^{s-1} C_{s-1}^{d/2-1/2}\left(\frac{u-v}{u+v}\right) + (u+v)^s C_{s-2}^{d/2-1/2}\left(\frac{u-v}{u+v}\right)\,, \tag{3.2.21}$$

which will be shown below to naturally enter the non-conservation operator of totally symmetric currents in the interacting CFT.

3.2.3 Two-point functions

The two-point functions of the totally-symmetric currents can be computed as in [33, 84] by using the Schwinger representation for the two-point function $\langle\psi\bar{\psi}\rangle$, which in our conventions reads

$$\langle\psi^i(x_1)\bar{\psi}_j(x_2)\rangle = \delta^i_j \frac{C_{\psi\psi}\not{x}_{12}}{(x_{12}^2)^{\Delta_\psi+\frac{1}{2}}}\,, \qquad C_{\psi\psi} = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\,, \tag{3.2.22}$$

so that we can write

$$\langle\psi(x)\bar{\psi}(0)\rangle = -\frac{\Gamma\left(d/2-1\right)}{4\pi^{d/2}}\not{\partial}\frac{1}{(x^2)^{d/2-1}} = -\not{\partial}\int_0^\infty \frac{d\alpha}{4\pi^{d/2}}\alpha^{d/2-2}e^{-\alpha x^2}\,. \tag{3.2.23}$$

Using this, one finds after integration over the Schwinger parameters

$$\langle J_s(x,z)J_s(0,z)\rangle = C_{ss}\times\frac{(z\cdot x)^{2s}}{(x^2)^{d+2s-2}}\,, \tag{3.2.24}$$

$$C_{ss} = N\frac{\pi 2^{-2d+2s+1}\Gamma(d+s-2)\Gamma(d+2s-3)}{\pi^d\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(s)}\,, \tag{3.2.25}$$

where $N = N_f\mathrm{tr}\mathbf{1}$ is the total number of fermion components. The non-singlet currents have $\mathrm{tr}\mathbf{1}$ instead of N.

Also, we will need the two-point functions of the hook currents and their descendants (3.2.20), which are given by a two-by-two matrix:

$$\begin{pmatrix}\langle B^s_\nu\eta^\nu B^s_\mu\eta^\mu\rangle & \langle\partial^\nu B^s_\nu B^s_\mu\eta^\mu\rangle\\ \langle B^s_\nu\eta^\nu\partial^\mu B^s_\mu\rangle & \langle\partial^\nu B^s_\nu\partial^\mu B^s_\mu\rangle\end{pmatrix} = NC_s\times\frac{(z\cdot x)^{2s-2}}{(x^2)^{d+2s-2}}\times \tag{3.2.26}$$

$$\times\begin{pmatrix}\eta^2(z\cdot x)^2-2(z\cdot x)(\eta\cdot x)(z\cdot\eta)+\frac{(d+2s-4)}{2(d+s-3)}x^2(z\cdot\eta)^2 & (z\cdot x)(z\cdot\eta)\frac{s(d-2)}{d+s-3}\\ (z\cdot x)(z\cdot\eta)\frac{-s(d-2)}{d+s-3} & \frac{2(d-2)s(d+2s-2)}{d+s-3}\frac{(z\cdot x)^2}{x^2}\end{pmatrix}\,,$$

where we introduced an additional vector η^μ to hide the index ν away. The overall factor is the C_s, (3.2.25), from the two-point function of the symmetric currents.

3.2.4 Some OPE coefficients

It is in principle straightforward to work out 3-point (or higher) correlation functions by similar methods. As an example, the 3-point function of the totally symmetric currents and two fermions is found to be (omitting flavor indices for simplicity, and denoting by α,β the spinor indices)

$$\begin{aligned}\langle J_s(x_1,z)\psi^\alpha(x_2)\bar{\psi}_\beta(x_3)\rangle &= C^2_{\psi\psi}C_{s\psi\psi}\left(\not{x}_{12}z\cdot\gamma\not{x}_{13}\right)^\alpha_{\ \beta}\frac{\left(\frac{z\cdot x_{12}}{x^2_{12}}-\frac{z\cdot x_{13}}{x^2_{13}}\right)^{s-1}}{x^d_{12}x^d_{13}}\,,\\ C_{s\psi\psi} &= \frac{(-1)^s2^{s-1}\Gamma\left(\frac{d}{2}+s-1\right)\Gamma(d+s-2)}{\Gamma(d-1)\Gamma\left(\frac{d}{2}\right)\Gamma(s)}\,,\end{aligned} \tag{3.2.27}$$

and similarly one may derive the 3-point functions with mixed-symmetry operators. In the following, we will also need the 3-point function of J_s with two $\Delta = d-1$ scalar bilinears. A short calculation using the Schwinger representation and the generating function for J_s yields

$$\begin{aligned}
&\langle J_s(x_1,z)\bar\psi\psi(x_2)\bar\psi\psi(x_3)\rangle = C_{s00}\frac{\left(\frac{z_1\cdot x_{13}}{x_{13}^2}-\frac{z_1\cdot x_{12}}{x_{12}^2}\right)^s}{x_{12}^{d-2}x_{13}^{d-2}x_{23}^{d}}\,,\\
&C_{s00} = 2^{s-1}NC_{\psi\psi}^3(1+(-1)^s)\frac{\Gamma\left(s+\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma\left(d+s-2\right)}{\Gamma\left(d-1\right)\Gamma\left(s\right)}\,.
\end{aligned} \tag{3.2.28}$$

It is instructive to compare this result with the conformal block expansion of the 4-point function of the $\bar\psi\psi$ operator. An explicit calculation yields

$$\begin{aligned}
&\langle\bar\psi\psi(x_1)\bar\psi\psi(x_2)\bar\psi\psi(x_3)\bar\psi\psi(x_4)\rangle = N^2C_{\psi\psi}^4\frac{g(u,v)}{(x_{12}^2x_{34}^2)^{d-1}}\,,\qquad u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,,\\
&g(u,v) = 1+u^{d-1}+\left(\frac{u}{v}\right)^{d-1}+\frac{1}{N}\left(\frac{u}{v}\right)^{d/2}\left(u^{\frac{d}{2}}-u^{\frac{d}{2}-1}(1+v)+\frac{(1-v)(1-v^{\frac{d}{2}})}{u}-(1+v^{\frac{d}{2}})\right)
\end{aligned} \tag{3.2.29}$$

The function $g(u,v)$ has the conformal block expansion $g(u,v) = 1+\sum_{\tau,\ell}a_{\tau,\ell}g_{\tau,\ell}(u,v)$, with $\tau=\Delta-\ell$ the twist of the intermediate state, and $a_{\tau,\ell}$ are related to squares of the OPE coefficients. In the limit $u\to 0$, $g_{\tau,\ell}(u,v)$ reduces to the so-called collinear conformal blocks

$$g_{\tau,\ell}(u,v)\simeq u^{\tau/2}(-\tfrac{1}{2})^{\ell}(1-v)^{\ell}{}_2F_1(\frac{\tau}{2}+\ell,\frac{\tau}{2}+\ell,\tau+2\ell;1-v)\,. \tag{3.2.30}$$

The term of order $u^{(d-2)/2}$ in the small u expansion of (3.2.29) should be reproduced by the sum over the exchanged conserved currents J_s of all even spins, with $\tau = d-2$. Using the OPE coefficients in C_{s00}, and $a_s = C_{s00}^2/(C_{ss}N^2C_{\psi\psi}^4)$ (see the Appendix), we have verified that indeed

$$\sum_\ell a_\ell(-\tfrac{1}{2})^\ell(1-v)^\ell{}_2F_1(\frac{d-2}{2}+\ell,\frac{d-2}{2}+\ell,d-2+2\ell;1-v)=\frac{1}{N}\frac{1}{v^{d/2}}(1-v)(1-v^{d/2})\,. \tag{3.2.31}$$

3.3 Weakly broken currents in fermionic CFT

3.3.1 Generalities

Applying the general formula (1.4.11)derived in the introduction one may derive the following formula for the anomalous dimension, valid to leading order in the breaking parameter

$$\gamma_s = -\frac{1}{s(s+d/2-2)(s+d/2-1)(s+d-3)}\frac{(z\cdot x)^2\langle \hat{K}_{s-1}(x,z)\hat{K}_{s-1}(0,z)\rangle_0}{\langle \hat{J}_s(x,z)\hat{J}_s(0,z)\rangle_0}\,, \tag{3.3.1}$$

where the subscript '0' means that the correlators are computed in the "unbroken" theory. Although for simplicity we have omitted flavor indices, this formula applies in the same way for singlet and non-singlet currents. In the following we will denote by γ_s the anomalous dimension of the singlet currents, and γ_s^A the one of the non-singlets (adjoint).

To derive the explicit formula for the non-conservation in the various models we consider below, we act with an operator $\partial_\mu D_z^\mu$ on the currents (3.2.2), which gives terms proportional to the "descendant operators" $\partial^\mu\bar{\psi}\gamma_\mu$, $\not{\partial}\psi$ and $\partial^2\bar{\psi}$, $\partial^2\psi$, which are non-zero in the interacting fermion theory:

$$\begin{aligned}
&\partial_\mu D_z^\mu f_s(\hat{\partial}_1,\hat{\partial}_2)\bar{\psi}(x_1)\hat{\gamma}\psi(x_2) = &(3.3.2)\\
&\qquad = \Big[\not{\partial}_1 q_s(\hat{\partial}_1,\hat{\partial}_2) + \not{\partial}_2\tilde{q}_s(\hat{\partial}_1,\hat{\partial}_2) + \hat{\gamma}\partial_1^2 h_s(\hat{\partial}_1,\hat{\partial}_2) + \hat{\gamma}\partial_2^2\tilde{h}_s(\hat{\partial}_1,\hat{\partial}_2)\Big]\,\bar{\psi}(x_1)\psi(x_2)\,,\\
&q_s(u,v) = \big((\tfrac{d}{2}-1)f_s + v(\partial_v f_s - \partial_u f_s)\big)\,, \qquad \tilde{q}_s(u,v) = \big((\tfrac{d}{2}-1)f_s + u(\partial_u f_s - \partial_v f_s)\big)\,,\\
&h_s(u,v) = \big(\tfrac{d}{2}\partial_u f_s + \frac{u-v}{2}\partial_u^2 f_s + v\partial_{uv}f_s\big)\,, &(3.3.3)\\
&\tilde{h}_s(u,v) = \big(t\tfrac{d}{2}\partial_v f_s + \frac{v-u}{2}\partial_v^2 f_s + u\partial_{uv}f_s\big)
\end{aligned}$$

Explicitly carrying out the differentiation and using recurrence relations for Gegenbauer polynomials we may represent the functions introduced above as:

$$\begin{aligned}
&q_s(u,v) \equiv q_s^d(u,v) = (u+v)^{s-1}\big[(\tfrac{d}{2}-1)C_{s-1}^{d/2-1/2}\Big(\frac{u-v}{u+v}\Big) - \frac{2(d-1)v}{u+v}C_{s-2}^{d/2+1/2}\Big(\frac{u-v}{u+v}\Big)\big],\\
&\tilde{q}_s(u,v) = (-1)^{s-1}q_s^d(v,u),\\
&h_s(u,v) = (d-1)q_{s-1}^{d+2}(u,v)\,, &(3.3.4)\\
&\tilde{h}_s(u,v) = (d-1)(-1)^{s-1}q_{s-1}^{d+2}(v,u)\,.
\end{aligned}$$

3.3.2 Large N expansion

One begins with the action of the Gross-Neveu model

$$S = \int d^d x \left(\bar{\psi} \partial\!\!\!/ \psi + \frac{1}{2} (\bar{\psi}\psi)^2 \right) \tag{3.3.5}$$

and introduces the Hubbard-Stratanovich field σ as

$$S = \int d^d x \left(\bar{\psi} \partial\!\!\!/ \psi + \frac{\sigma}{\sqrt{N}} (\bar{\psi}\psi) - \frac{1}{2}\sigma^2 \right) . \tag{3.3.6}$$

The auxiliary field acquires an induced (non-local) kinetic term via fermion loops

$$S_\sigma = -\frac{1}{2} \int d^d x d^d y \sigma(x)\sigma(y) \langle \frac{1}{\sqrt{N}} \bar{\psi}\psi(x) \frac{1}{\sqrt{N}} \bar{\psi}\psi(y) \rangle_0 + O(1/N) \,, \tag{3.3.7}$$

where we have dropped the quadratic term in (3.3.6) as it does not contribute in the UV limit. Inverting the induced quadratic term, one finds the σ 2-point function to leading order in $1/N$ to be

$$\langle \sigma(x_1)\sigma(x_2) \rangle = \frac{C_{\sigma\sigma}}{x_{12}^2} \,, \qquad C_{\sigma\sigma} = -\frac{2(d-2)\Gamma(d-1)\sin\left(\frac{\pi d}{2}\right)}{\pi \Gamma\left(\frac{d}{2}\right)^2} \,, \tag{3.3.8}$$

so that $\sigma \sim \bar{\psi}\psi$ is a scalar primary with $\Delta = 1 + O(1/N)$ at the UV fixed point.

The anomalous dimensions of ψ and σ to leading order in the $1/N$ expansion are well-known [101, 102, 116, 117]:[5]

$$\Delta_\psi = \frac{d-1}{2} - \frac{1}{N} \frac{(d-2)\Gamma(d-1)\sin\left(\frac{\pi d}{2}\right)}{\pi d \Gamma\left(\frac{d}{2}\right)^2} \,, \tag{3.3.9}$$

$$\Delta_\sigma = 1 + \frac{1}{N} \frac{4\Gamma(d)\sin\left(\frac{\pi d}{2}\right)}{\pi d \Gamma\left(\frac{d}{2}\right)^2} . \tag{3.3.10}$$

Since ψ is a nearly free field at large N, the leading anomalous dimension of ψ can be readily obtained using the equations of motion

$$\partial\!\!\!/ \psi = -\frac{1}{\sqrt{N}} \psi\sigma \,, \qquad \partial^\mu \bar{\psi}\gamma_\mu = +\frac{1}{\sqrt{N}} \bar{\psi}\sigma, \tag{3.3.11}$$

in the spirit of [24], but with $1/N$ playing the role of the small parameter (a similar calculation in the

[5] Δ_ψ is known up to $1/N^3$ [101, 117] and Δ_σ up to $1/N^2$ [101, 116].

large N scalar CFT was carried out in [33, 84]). In the interacting theory, the fermion ψ must have the two-point function

$$\langle \psi^i \bar{\psi}_j \rangle = \delta^i_j \frac{C_{\psi\psi} \not{x}_{12}}{(x_{12}^2)^{\Delta_\psi + \frac{1}{2}}} \tag{3.3.12}$$

with $\Delta_\psi = (d-1)/2 + \gamma_\psi$. Applying the Dirac operators $\not{\partial}_1$ and $\not{\partial}_2$ on this two-point function, one gets

$$\not{\partial}_1 \not{\partial}_2 \langle \psi^i \bar{\psi}_j \rangle = -2\gamma_\psi (d + 2\gamma_\psi) \delta^i_j \frac{C_{\psi\psi} \not{x}_{12}}{(x_{12}^2)^{\Delta_\psi + \frac{3}{2}}} \,, \tag{3.3.13}$$

which can be compared with the insertion of (3.3.11)

$$-\frac{1}{N} \langle \psi^i \sigma \bar{\psi}_j \sigma \rangle = -\frac{1}{N} C_{\sigma\sigma} \delta^i_j \frac{C_{\psi\psi} \not{x}_{12}}{(x_{12}^2)^{\Delta_\psi + \frac{3}{2}}} \,. \tag{3.3.14}$$

This yields to the leading order in $1/N$:

$$\gamma_\psi = \frac{C_{\sigma\sigma}}{2dN} \,, \tag{3.3.15}$$

which, using (3.3.8), can be seen to be in full agreement with (3.3.9). Interestingly, the anomalous dimension of σ can also be reconstructed by using the equation of motion method, by considering the 3-point function $\langle \psi \bar{\psi} \sigma \rangle$. We will carry out this calculation in section 3.4.1 below, and proceed here with the analysis of the weakly broken higher-spin operators.

To find the non-conservation operator for the higher-spin currents, we need to plug the equations of motion (3.3.11) into the master formula for the non-conservation operator (3.3.2). As a result we find two type of terms: with two gamma-matrices and without gamma-matrices:

$$\hat{K}_{s-1} = \frac{1}{\sqrt{N}} \Big(k_1(\partial_i) \bar{\psi}(x_1) \psi(x_2) \sigma(x_3) + k_2(\partial_i) \bar{\psi}(x_1) \gamma_{\mu\nu} \psi(x_2) \partial_3^\mu z^\nu \sigma(x_3) \Big) \tag{3.3.16}$$

where

$$
\begin{aligned}
k_1 \equiv & \left[q_s^d(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2)+(-1)^s q_s^d(\hat{\partial}_2+\hat{\partial}_3,\hat{\partial}_1)\right]+ \\
& \qquad + (d-1)\hat{\partial}_3\left[q_{s-1}^{d+2}(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2)+(-1)^s q_{s-1}^{d+2}(\hat{\partial}_2+\hat{\partial}_3,\hat{\partial}_1)\right], \\
k_2 \equiv & \,(d-1)\left[q_{s-1}^{d+2}(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2)-(-1)^s q_{s-1}^{d+2}(\hat{\partial}_2+\hat{\partial}_3,\hat{\partial}_1)\right].
\end{aligned}
\tag{3.3.17}
$$

The non-conservation operator $\hat{K}_{s-1}$ must be a conformal primary of the unbroken theory[6] In particular it should be possible to decompose it as

$$
\hat{K}_{s-1} = \sum_{a,c}\mathcal{B}^s_{a,c}\hat{\partial}^a(B^{s_1}_\mu\partial_3^\mu)\hat{\partial}^c\sigma + \sum_{a,c}\mathcal{C}^s_{a,c}\hat{\partial}^a(\partial^\mu B^{s_1}_\mu)\hat{\partial}^c\sigma + \sum_{a,c}\mathcal{A}^s_{a,c}\hat{\partial}^a(\bar{\psi}\psi)\hat{\partial}^c\sigma\delta_{a+c,s-1}\,, \tag{3.3.18}
$$

where the summation range and the spin s_1 of the operators $B^{s_1}_\mu$ in the sums are fixed by the spin and conformal dimension counting: $s_1+a+c+1=s$ in the first two sums and $a+c+1=s$ in the last one.

First of all, we observe that we need the hook currents, i.e. the ones with $(s,1)$ symmetry, as these contain two gamma matrices while the usual totally-symmetric currents do not contribute at all. It is also important to take the descendant $\partial^\mu B_\mu$ into account since it does not vanish. The last terms involving the $\bar{\psi}\psi$-singlet appears only in the case of singlet currents of even spins. Note that at the large N UV fixed point, this term should be projected out due to the σ equation of motion, which is formally $\bar{\psi}\psi=0$ (σ replaces $\bar{\psi}\psi$ at the UV fixed point).

Taking notice that the γ-part of the non-conservation operator is exactly like in the large-N bosonic model [33, 84], we immediately find

$$
\mathcal{B}^s_{a,c} = -\frac{2(a+c+1)!(a+c-\nu-s+1)(a+2c+1-2(s+\nu))!}{a!c!(c+1)!(a+c-2(s+\nu))!}\,, \tag{3.3.19}
$$

which is assumed to vanish for $a+c$ even. The formula works both for even and odd spins. Note that everything depends on $s+\nu$ only ($\nu=(d-3)/2$). Analogously,

$$
\mathcal{C}^s_{a,c} = \frac{2(a+c+1)!(-a-c+\nu+s-1)(a+c-2(\nu+s)+1)(a+2c-2(\nu+s))!}{a!(c!)^2(a+c-s+1)(a+c-2(\nu+s))!}\,,
$$

[6]This can be seen by acting with the special conformal generators on the non-conservation equation, see for instance [86, 118].

which is assumed to vanish for $a+c$ even or $a+c>s-2$. The only new part is due to the $(\bar{\psi}\psi)$-terms

$$\mathcal{A}^s_{a,c}=\frac{\left(\nu+\frac{s}{2}\right)^2(2\nu+s-1)!}{(a+2\nu+1)!}\frac{4(-1)^a s\left((1-s)_a\right)^2}{\Gamma(a+1)\Gamma(s)}\,, \tag{3.3.20}$$

which is assumed to vanish for s odd or unless $a+c=s-1$. The fact that the decomposition (3.3.18) is possible is a check of the non-conservation operator (3.3.16).

Examples. Let us consider a few explicit low spin examples. The first nontrivial example is the spin-two singlet current, for which we find (omitting an overall factor)

$$K^{s=2}\sim(\hat{\partial}_1+\hat{\partial}_2+(1-d)\hat{\partial}_3)\bar{\psi}\psi\sigma=(\partial(\bar{\psi}\psi)\sigma+(1-d)(\bar{\psi}\psi)\partial\sigma)\,, \tag{3.3.21}$$

which is conserved upon projecting out $\bar{\psi}\psi$, the operator that is replaced by σ in the large-N treatment. The spin-three non-conservation contains an anti-symmetric tensor:

$$\begin{aligned}K^{s=3}&=2(\hat{\partial}_1-\hat{\partial}_2)(-\hat{\partial}_1-\hat{\partial}_2+(d+1)\partial_3)\bar{\psi}\psi\sigma+(2(\hat{\partial}_1+\hat{\partial}_2))-d\partial_3)\bar{\psi}\gamma_{ab}\psi z^a\partial^b\sigma\\&=\left[dB_{a,u}\partial^u\partial_a\sigma-2\partial_a B_{a,u}\partial^u\sigma-2(d+1)\partial^u B_{a,u}\partial_a\sigma+2\partial_a\partial^u B_{a,u}\sigma\right]z^a z^a\,.\end{aligned}$$

The spin-four non-conservation contains a genuine mixed-symmetry current:

$$\begin{aligned}K^{s=4}&=\left[5B_{aa,u}\partial^u\partial_a\sigma-2\partial_a B_{aa,u}\partial^u\sigma-6\partial^u B_{aa,u}\partial_a\sigma+\partial_a\partial^u B_{aa,u}\sigma\right]z^a z^a z^a\\&+\left[\frac{8}{3}(\bar{\psi}\psi)\partial_a\partial_a\partial_a\sigma-12\partial_a(\bar{\psi}\psi)\partial_a\partial_a\sigma+8\partial_a\partial_a(\bar{\psi}\psi)\partial_a\sigma-\frac{2}{3}\partial_a\partial_a\partial_a(\bar{\psi}\psi)\sigma\right]z^a z^a z^a\end{aligned}$$

where the formula is written in $d=3$ to simplify the coefficients and the last line displays the contribution of the $\bar{\psi}\psi$ operator that needs to be dropped for the singlet currents. As a result one sees the formula from [21, 35].

In principle, one can use the decomposition (3.3.18) to directly compute the two-point function in (1.4.11) and thus the anomalous dimensions. However, the sums which appear are quite involved to calculate, and in practice we find more convenient to compute $\langle K_{s-1}K_{s-1}\rangle$ directly in terms of the form (3.3.16). To do this, one can start with the Schwinger representation (3.2.23). Note that, using this representation, we can trade derivatives at point 0 in (1.4.11) to x-derivatives by flipping their signs. The action of the projected derivatives on the integral is trivial: $\hat{\partial}^n e^{-\alpha x^2}=(-2\alpha\hat{x})^n e^{-\alpha x^2}$ since

$\hat{\partial}\hat{x} = 0$. Owing to this, the differential operators in the descendant are replaced with a polynomial in α parameters. The derivatives which are not contracted with the null polarization vector require a bit more work, but after some manipulations it is not difficult to see that the calculation eventually reduces to evaluating some integrals over α parameters, which can be performed using the properties of Gegenbauer polynomials. Following this procedure to compute $\langle K_{s-1}K_{s-1}\rangle$, and using the master formula (1.4.11), we arrive at the following result

$$\gamma_s^A = 2\gamma_\psi \left(1 - \frac{(d-2)d}{4(s+\frac{d}{2}-2)(s+\frac{d}{2}-1)}\right) , \tag{3.3.22}$$

which is valid for the non-singlet currents of all spins (and for odd spin singlets, which coincide with odd spin non-singlets). To the best of our knowledge, this result was first obtained in [87] (using a standard Feynman diagram approach).

For singlet currents of even spins, the above result is not correct, because one has to subtract by hand the piece of the descendant (3.3.18) involving the scalar singlet $\bar{\psi}\psi$, as explained above. Subtracting from (3.3.22) the contribution of the last term in (3.3.18)

$$\sum_{a,b} \mathcal{A}_{a,s-1-n}\mathcal{A}_{b,s-1-b}(\hat{\partial})^{a+b}(-1)^b\langle\bar{\psi}\psi\bar{\psi}\psi\rangle(\hat{\partial})^{2(s-1)-a-b}(-1)^{s-1-b}\langle\sigma\sigma\rangle \tag{3.3.23}$$

we obtain the final result for even spin singlets:

$$\gamma_s = 2\gamma_\psi \left(1 - \frac{(d-2)d}{4(s+\frac{d}{2}-2)(s+\frac{d}{2}-1)} - \frac{\Gamma(d+1)\Gamma(s+1)}{2(d-1)(s+\frac{d}{2}-2)(s+\frac{d}{2}-1)\Gamma(d+s-3)}\right) , \tag{3.3.24}$$

where the last term is due to the subtraction of (3.3.23). This result is again in agreement with [87]. Interestingly, the structure of this result, as well as (3.3.22), is identical to the ones in the critical large N scalar model [33, 45, 84, 119], up to overall factor of γ_ψ replacing γ_ϕ. In particular, note that the large spin expansion of (3.3.24) reads

$$\gamma_s = 2\gamma_\psi \left(1 - \frac{(d-2)d}{4}\frac{1}{s^2} - \frac{\Gamma(d+1)}{2(d-1)}\frac{1}{s^{d-2}} + \ldots\right) . \tag{3.3.25}$$

The leading spin independent term agrees, of course, with the general expectation that $\Delta_s \sim 2\Delta_\psi + s$ at large spin. Naively, following the arguments in [39, 40, 48], one might have expected a term of order $1/s$ corresponding to the exchange of the scalar operator σ with $\Delta = 1 + O(1/N)$, however we see that such term is absent. Expanding at large spin the recent $1/N^2$ result in [88], we find that the anomalous dimensions include terms of order $1/s^{d-2}$, $1/s^2$, $\log(s)/s^{d-2}$ and $\log(s)/s^2$, but no terms of order $1/s$. It would be interesting to understand this by generalizing the analysis of [39–41], or the approach of [109, 110], to the case of 4-point functions of fermionic operators.

3.3.3 Gross-Neveu in $d = 2 + \epsilon$

The Gross-Neveu model in dimension d is defined by the action:

$$S = \int d^d x \left(\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2} g (\bar{\psi}\psi)^2 \right) \tag{3.3.26}$$

where $\bar{\psi}\psi \equiv \bar{\psi}^i \psi_i$ and the action enjoys $U(N_f)$-symmetry. The one-loop results for the β-function and the anomalous dimensions of the lowest lying operators ψ and $\bar{\psi}\psi$ are well known (see e.g. [57] for a review):

$$\beta = \epsilon g - (N-2)\frac{g^2}{2\pi} + \dots\,, \qquad g_* = \frac{2\pi}{N-2}\epsilon\,, \tag{3.3.27}$$

$$\gamma_\psi = \frac{N-1}{16\pi^2} g^2 = \frac{N-1}{4(N-2)^2}\epsilon^2\,, \qquad \Delta_\psi = \frac{d-1}{2} + \gamma_\psi\,, \tag{3.3.28}$$

$$\gamma_{\psi^2} = -\frac{1}{2\pi} g = -\frac{\epsilon}{N-2}\,, \qquad \Delta_\phi = d - 1 + \gamma_{\psi^2}\,. \tag{3.3.29}$$

Note that N here is $N_f \mathrm{tr}\mathbf{1}$ the total number of the field components. The equations of motion take the following form

$$\partial\!\!\!/\psi = -g\psi(\bar{\psi}\psi)\,, \qquad \partial^\mu \bar{\psi}\gamma_\mu = +g\bar{\psi}(\bar{\psi}\psi)\,. \tag{3.3.30}$$

Following similar methods as in [24], it is possible to use these equations to derive the above anomalous dimensions, as well as the dimension of higher order composite scalar operators [94, 95]. Here, we use a similar approach to fix the leading order anomalous dimensions of the weakly broken higher-spin currents.

The calculation of the divergence of the currents follows similar steps as in the previous sections. We find

$$\hat{K}_{s-1} = g\Big(k_1 (\partial_i)\bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4) + k_2(\partial_i)\bar{\psi}(x_1)\gamma_{\mu\nu}\psi(x_2)(\partial_3^\mu + \partial_4^\mu) z^\nu \bar{\psi}(x_3)\psi(x_4) \Big) \tag{3.3.31}$$

where

$$
\begin{aligned}
k_1 \equiv & \left[q_s^2(\hat{\partial}_1+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_2)+(-1)^s q_s^2(\hat{\partial}_2+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_1)\right] \\
& +\hat{\partial}_3\left[q_{s-1}^4(\hat{\partial}_1+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_2)+(-1)^s q_{s-1}^4(\hat{\partial}_2+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_1)\right], \\
k_2 \equiv & \left[q_{s-1}^4(\hat{\partial}_1+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_2)-(-1)^s q_{s-1}^4(\hat{\partial}_2+\hat{\partial}_3+\hat{\partial}_4,\hat{\partial}_1)\right].
\end{aligned}
\tag{3.3.32}
$$

To find the anomalous dimension according to the formula (1.4.11), we have to calculate a two-point function of four fermionic operators at points x and 0. There are three different ways to contract the spinor and $U(N_f)$ indices, pair by pair or "threading" through all 8 operators. The first diagram, obtained by contracting the first pair at point x with the corresponding one at point 0, gives a contribution

$$
\frac{N\epsilon^2}{2(N-2)^2}\,. \tag{3.3.33}
$$

The second diagram is obtained by contracting the first pair at point x with the second pair at point 0. It is non-zero only for even singlets and gives a contribution to γ_s

$$
-\frac{1+(-1)^s}{2}\frac{N\epsilon^2}{2(N-2)^2}\,. \tag{3.3.34}
$$

Finally, the third diagram is the one where spinor and $U(N_f)$ indices make a single loop threading all 8 fermions, and yields a contribution

$$
-\frac{1-(-1)^s}{2}\frac{\epsilon^2}{(N-2)^2}\,. \tag{3.3.35}
$$

Summing these results up we get for the non-singlets:

$$
\begin{aligned}
\gamma_s^A &= \frac{N\epsilon^2}{2(N-2)^2}\,, \qquad s \ \text{ even}\,, \\
\gamma_s^A &= \frac{(N-2)\epsilon^2}{2(N-2)^2} = \frac{\epsilon^2}{2(N-2)}\,, \qquad s>1 \ \text{ odd}\,,
\end{aligned}
\tag{3.3.36}
$$

and $\gamma_{s=1}^A = 0$ in accordance with the exact $U(N_f)$ symmetry. For the even spin singlets, we obtain[7]

$$
\gamma_s = O(\epsilon^3)\,, \qquad s \geq 2 \ \text{ even}\,. \tag{3.3.37}
$$

Note that, to this order, all these results are independent of the spin (except for the vanishing of $\gamma_{s=1}^A$). One can check that they match precisely the expansion of the $1/N$ values (3.3.22) and (3.3.24)

[7] The result for odd spin singlets is the same as for odd spin non-singlets.

near $d = 2 + \epsilon$. Including also the recent $1/N^2$ results in [88], we find that the $2 + \epsilon$ expansion of the available large N results is

$$\begin{aligned} \gamma^A_{s>1} &= \epsilon^2 \left(\frac{1}{2N} + \frac{3 + (-1)^s}{2N^2} \right) + \epsilon^3 \left(\frac{\frac{1}{s-s^2} - 1}{4N} + \frac{1 + \frac{2}{s-s^2} - 2(-1)^s(\psi(s) + \gamma)}{4N^2} \right) + O(\epsilon^4), \\ \gamma_s &= \epsilon^3 \left(\frac{4}{s} - \frac{4}{s-1} - 2 + 4(\psi(s) + \gamma) \right) \left(\frac{1}{8N} + \frac{3}{8N^2} + \dots \right) + O(\epsilon^4)\,, \qquad s \text{ even}\,, \end{aligned}$$

where $\psi(s)$ is the digamma function and γ the Euler-Mascheroni constant. Note that to leading order in ϵ these precisely agree with the results derived above. Note also that a non-trivial spin dependence, including terms of order $\sim \log(s)$ at large s, appears at the next-to-leading order in the ϵ-expansion.

3.3.4 Gross-Neveu-Yukawa in $d = 4 - \epsilon$

The Gross-Neveu-Yukawa (GNY) action is defined as:

$$S = \int d^d x \left(\bar{\psi}_i (\partial\!\!\!/ + g_1 \sigma) \psi^i + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_2}{24} \sigma^4 \right) \tag{3.3.38}$$

This model has a perturbative IR fixed point in $d = 4 - \epsilon$ which is in the same universality class as the UV fixed point of the GN model, and thus provides a different description of the same interacting fermionic CFT [57, 79, 80]. The one-loop beta functions yield the fixed point couplings

$$\begin{aligned} &(g_1^*)^2 = \frac{16\pi^2 \epsilon}{N+6}\,, \qquad g_2^* = 16\pi^2 R \epsilon\,, \\ &R = \frac{24N}{(N+6)[(N-6) + \sqrt{N^2 + 132N + 36}]}\,, \end{aligned} \tag{3.3.39}$$

and the leading anomalous dimensions of ψ and σ are

$$\gamma_\psi = \frac{(g_1^*)^2}{32\pi^2} = \frac{\epsilon}{2(N+6)}\,, \tag{3.3.40}$$

$$\gamma_\sigma = \frac{N (g_1^*)^2}{32\pi^2} = \frac{N\epsilon}{2(N+6)}\,. \tag{3.3.41}$$

The equations of motion are

$$\begin{aligned} \partial\!\!\!/ \psi^i &= -g_1 \sigma \psi^i\,, \\ \partial^2 \sigma &= g_1 \bar{\psi}_i \psi^i + \frac{g_2}{6} \sigma^3\,. \end{aligned} \tag{3.3.42}$$

From the abstract CFT point of view, these equations describe the multiplet recombination which makes ψ and σ into long representations of the conformal algebra with $\Delta_\psi > (d-1)/2$ and $\Delta_\sigma > d/2-1$, and can be used to fix the leading order anomalous dimensions. Since the ψ equation of motion formally coincides with the large-N equation (3.3.11), one arrives at the same relation (3.3.15), with $d=4$:

$$\gamma_\psi = \frac{1}{8} C_{\sigma\sigma} (g_1^*)^2 = \frac{(g_1^*)^2}{32\pi^2}\,, \tag{3.3.43}$$

where we have used $C_{\sigma\sigma} = \frac{1}{4\pi^2}$, since σ is now a canonically normalized (near) free field. As for the σ anomalous dimension, one can see that to leading order in the breaking parameter we can neglect the σ^3 term in (3.3.42), and by a calculation similar to the one in the scalar ϕ^4 theory [24, 33, 84], one finds (setting $d=4$ which is appropriate to leading order):

$$\gamma_\sigma = \frac{1}{32 C_{\sigma\sigma}} (g_1^*)^2 x_{12}^6 \langle \bar\psi\psi(x_1)\bar\psi\psi(x_2)\rangle_0\,, \tag{3.3.44}$$

which yields

$$\gamma_\sigma = \frac{N C_{\psi\psi}^2 (g_1^*)^2}{32 C_{\sigma\sigma}} = \frac{N (g_1^*)^2}{32\pi^2}\,, \tag{3.3.45}$$

in agreement with (3.3.40). The relation between g_1^* and ϵ can also be reconstructed without input from Feynman diagrams and beta functions by applying the equations of motion (3.3.42) to the 3-point function $\langle \psi\bar\psi\sigma\rangle$, as will be explained in section 3.4.1 below.

Let us now turn to the weakly broken higher-spin currents of the model (focusing on the totally symmetric operators only). By applying the general non conservation formula (3.3.2), the divergence of the fermion bilinear currents (flavor indices omitted)

$$\hat J_{s,\psi} = (\hat\partial_1 + \hat\partial_2)^{s-1} C_s^{3/2}\Big(\frac{\hat\partial_1 - \hat\partial_2}{\hat\partial_1 + \hat\partial_2}\Big) \bar\psi(x_1)\hat\gamma\psi(x_2)\Big|_{x_1,x_2\to x} \tag{3.3.46}$$

is found to be

$$\hat K_{s-1,\psi} = g_1\Big(k_1(\partial_i)\bar\psi(x_1)\psi(x_2)\sigma(x_3) + k_2(\partial_i)\bar\psi(x_1)\gamma_{\mu\nu}\psi(x_2)\partial_3^\mu z^\nu \sigma(x_3)\Big) \tag{3.3.47}$$

and is almost identical to the large-N limit, with k_1 and k_2 given now for $d=4$ by

$$
\begin{aligned}
k_1 \equiv & \big[q_s^4(\hat\partial_1+\hat\partial_3,\hat\partial_2)+(-1)^s q_s^4(\hat\partial_2+\hat\partial_3,\hat\partial_1)\big]+ \\
& + 3\hat\partial_3\big[q_{s-1}^6(\hat\partial_1+\hat\partial_3,\hat\partial_2)+(-1)^s q_{s-1}^6(\hat\partial_2+\hat\partial_3,\hat\partial_1)\big]\,, \\
k_2 \equiv & \, 3\big[q_{s-1}^6(\hat\partial_1+\hat\partial_3,\hat\partial_2)-(-1)^s q_{s-1}^6(\hat\partial_2+\hat\partial_3,\hat\partial_1)\big]\,.
\end{aligned}
\tag{3.3.48}
$$

For the non-singlets and odd spin singlets, the fermionic bilinear currents do not mix with other operators, so the calculation of the anomalous dimensions proceeds exactly as in the $1/N$ expansion, with $C_{\sigma\sigma}$ in the latter replaced now by $\frac{1}{4\pi^2}(g_1^*)^2$, and we find

$$
\gamma_s^A = \frac{(g_1^*)^2}{16\pi^2}\frac{(s-1)(s+2)}{s(s+1)} = 2\gamma_\psi\left(1-\frac{2}{s(s+1)}\right)\,. \tag{3.3.49}
$$

Note that this vanishes for $s=1$, as it should. Expanding this result at large N to the order $1/N^2$, we find agreement with both (3.3.22) and the result of [88]

$$
\gamma_s^A = \epsilon\left(\frac{1}{N}-\frac{6}{N^2}+\dots\right)\left(1-\frac{2}{s(s+1)}\right)+O(\epsilon^2)\,. \tag{3.3.50}
$$

In the case of even spin singlets, a novelty compared to the large N calculation is the appearance of mixing with the scalar bilinear higher-spin currents

$$
\hat J_{s,\sigma} = (\hat\partial_1+\hat\partial_2)^s C_s^{1/2}\Big(\frac{\hat\partial_1-\hat\partial_2}{\hat\partial_1+\hat\partial_2}\Big)\sigma(x_1)\sigma(x_2)\Big|_{x_1,x_2\to x}\,, \qquad s=2,4,6,\dots \tag{3.3.51}
$$

The divergence of these currents, which we denote $\hat K_{s-1,\sigma}$, has two pieces according to equations of motion. However, to lowest order in ϵ we may ignore the piece coming with g_2 since $g_1^*\sim\sqrt{\epsilon}$ whereas $g_2^*\sim\epsilon$. To the lowest order, the descendant is then:

$$
\hat K_{s-1,\sigma} = 2g_1 q_s^4(\hat\partial_1+\hat\partial_2,\hat\partial_3)\bar\psi_i(x_1)\psi^i(x_2)\sigma(x_3)\,. \tag{3.3.52}
$$

It is evident that this term induces the mixing between the $\hat J_{s,\psi}$ and $\hat J_{s,\sigma}$ since there are non-zero off-diagonal 2-point functions $\langle\hat K_{s-1,\sigma}\hat K_{s-1,\psi}\rangle$.[8] The calculation can be carried out using the Schwinger

[8] A similar mixing of two towers of nearly conserved higher-spin currents takes place in the cubic $O(N)$ scalar field theory in $d=6-\epsilon$ [33].

representation of the propagator, and using (1.4.11) we find the mixing matrix

$$\frac{(g_1^*)^2}{16\pi^2}\begin{bmatrix} \frac{(s-1)(s+2)}{s(s+1)} & \frac{-2\sqrt{N}}{\sqrt{s(s+1)}} \\ \frac{-2\sqrt{N}}{\sqrt{s(s+1)}} & N \end{bmatrix} . \tag{3.3.53}$$

This leads to the anomalous dimensions:

$$\gamma_s^{\pm} = \frac{g_1^2}{16\pi^2}\frac{-2+(N+1)s(1+s)\pm\sqrt{4+s(1+s)(-4+20N+(N-1)^2s+(N-1)^2s^2)}}{2s(1+s)} \tag{3.3.54}$$

The eigenvalue γ_s^- corresponds to the tower of weakly broken currents with twist near $d-2$ which is present in the large N treatment of the CFT. Indeed, one may check that $\gamma_{s=2}^- = 0$, indicating that the corresponding eigenstate is the conserved stress tensor of the CFT. The large N expansion of the anomalous dimensions reads

$$\gamma_s^- = \frac{\epsilon}{N}\frac{(s-2)(s+3)}{s(s+1)} - \frac{2\epsilon}{N^2}\frac{(s-2)(s+3)(3s^2+3s+2)}{s^2(s+1)^2} + \dots , \tag{3.3.55}$$

which again agrees with the expansion around $d = 4-\epsilon$ of the $1/N$ value (3.3.24) and $1/N^2$ results from [88].

The second eigenvalue γ_s^+ should instead correspond in the large N approach to the "double-trace" operators $\sigma\partial^s\sigma \sim (\bar{\psi}\psi)\partial^s(\bar{\psi}\psi)$. This is suggested by expanding (3.3.54) at large N, which yields

$$\Delta^+ = d-2+s+\gamma_s^+ = 2+s-2\frac{\epsilon}{N}\frac{3s^2+3s-2}{s(1+s)} + O(1/N^2) , \tag{3.3.56}$$

corresponding to an operator with twist $2+O(1/N)$. In section 3.4.2, we will compute the scaling dimensions of the $\sigma\partial^s\sigma$ operators in the large N theory as a function of d, and explicitly verify agreement with (3.3.56). This provides a non-trivial test of the identification of the IR fixed point of the GNY model with the UV fixed point of the GN model.

Let us also give the large spin expansion of the anomalous dimensions (3.3.54). Writing the result in terms of γ_ψ and γ_σ, we find

$$\begin{aligned} \gamma_s^- &= 2\gamma_\psi\left(1 - \frac{6N-2}{(N-1)s^2} + \frac{6N-2}{(N-1)s^3} + \dots\right) , \\ \gamma_s^+ &= 2\gamma_\sigma\left(1 + \frac{4}{(N-1)s^2} - \frac{4}{(N-1)s^3} + \dots\right) . \end{aligned} \tag{3.3.57}$$

Note the absence of $1/s$ terms, as observed above in the large N results.

The case $N = 1$ is special and deserves a separate comment. Since $N = N_f\,\mathrm{tr}1$, this corresponds to a formal analytic continuation which should yield a CFT with a single 2-component Majorana fermion in $d = 3$. It has been argued that the IR fixed point in this case displays emergent supersymmetry [83,120–124]. Indeed, we can see that for $N = 1$ the dimension of σ and ψ in (3.3.40) coincide. Further checks, including higher orders in the ϵ-expansion, can be found in [83]. Note that for $N = 1$ the square roots in (3.3.54) simplify, and we get the results

$$\gamma_s^- = 2\gamma_\psi\left(1 - \frac{2}{s}\right), \qquad \gamma_s^+ = 2\gamma_\psi\left(1 + \frac{2}{s+1}\right). \tag{3.3.58}$$

It would be interesting to study the half-integer higher-spin operators $\sim \sigma\partial^{s-1/2}\psi$, and check how the operators organize into supersymmetric multiplets.

3.4 Composite operators in fermionic and scalar CFTs

In this section we will calculate the dimensions of various composite operators in fermionic $U(N)$ theories and also in the bosonic $O(N)$ model. Mostly, we will be interested in the large-N results for operators built from the auxiliary field σ: σ itself, σ^k and $\sigma\partial^s\sigma$.

3.4.1 Some scalar operators

Let us consider the traceless (adjoint) scalar operator $(O^A)^i_j = \bar\psi_j\psi^i - \frac{\delta^i_j}{N}\bar\psi\psi$ in the large N critical fermion theory. Its leading order anomalous dimension can be fixed by starting with the 3-point function

$$\langle\psi^i(x_1)\bar\psi_j(x_2)(O^A)^l_k(x_3)\rangle = (\delta^l_j\delta^i_k - \frac{1}{N}\delta^i_j\delta^l_k)\frac{C_{\psi\psi O}\,\not{x}_{13}\not{x}_{23}}{x_{12}^{2\Delta_\psi-\Delta_O+2}x_{13}^{\Delta_O+1}x_{23}^{\Delta_O+1}}, \tag{3.4.1}$$

where the structure on the right-hand side is completely fixed by conformal symmetry, for general Δ_ψ and Δ_O. Acting with the Dirac operators $\not{\partial}_1\not{\partial}_2$ on the right-hand side, we find

$$(2\Delta_\psi + \Delta_O - d)(2\Delta_\psi - \Delta_O)(\delta^l_j\delta^i_k - \frac{1}{N}\delta^i_j\delta^l_k)C_{\psi\psi O}\frac{\not{x}_{13}\not{x}_{23}}{x_{12}^{2\Delta_\psi-\Delta_O}x_{13}^{\Delta_O+1}x_{23}^{\Delta_O+1}} + \dots, \tag{3.4.2}$$

where we have omitted terms proportional to the identity in the spinor space, which will not play a role in the following leading order calculations. After writing $\Delta_\psi = d/2 - 1/2 + \gamma_\psi$, $\Delta_O = d - 1 + \gamma_O$, and expanding to leading order in the anomalous dimensions, the prefactor above becomes

$$(d-2)(2\gamma_\psi - \gamma_O) \tag{3.4.3}$$

and hence this correlator can be used to fix γ_O in terms of γ_ψ. Indeed, inserting the equation of motion $\partial\!\!\!/\psi = -\frac{\sigma\psi}{\sqrt{N}}$ on the left-hand side of (3.4.1), we get, to leading order at large N

$$\begin{aligned} &-\frac{1}{N}\langle \psi^i \sigma(x_1)\bar{\psi}_j\sigma(x_2)(O^A)^l_k(x_3)\rangle = -\frac{1}{N}\langle\sigma(x_1)\sigma(x_2)\rangle\langle\psi^i(x_1)\bar{\psi}_j(x_2)(O^A)^l_k(x_3)\rangle = \\ &= -\frac{1}{N}(\delta^l_j\delta^i_k - \frac{1}{N}\delta^i_j\delta^l_k)\frac{C_{\sigma\sigma}C_{\psi\psi O}\, x\!\!\!/_{13}\, x\!\!\!/_{23}}{x_{12}^2 x_{13}^d x_{23}^d}\,. \end{aligned} \tag{3.4.4}$$

This gives:

$$\gamma_O = 2\gamma_\psi + \frac{C_{\sigma\sigma}}{N(d-2)} = \frac{4(d-1)}{d-2}\gamma_\psi\,. \tag{3.4.5}$$

In $d = 3$, this agrees with the result obtained in [124] via a standard Feynman diagram calculation. As far as we know, the general d result was not given elsewhere in the literature.

The same calculation can be applied in the case of the GNY model in $d = 4 - \epsilon$, where following the similar steps we arrive at

$$\gamma_O = 2\gamma_\psi + C_{\sigma\sigma}\frac{(g_1^*)^2}{d-2} = \frac{3\epsilon}{N+6}\,. \tag{3.4.6}$$

Analogously, we can fix the anomalous dimension of the scalar singlet σ starting from the 3-point function

$$\langle\psi^i(x_1)\bar{\psi}_j(x_2)\sigma(x_3)\rangle = \delta^i_j\frac{C_{\psi\psi\sigma}\, x\!\!\!/_{13}\, x\!\!\!/_{23}}{x_{12}^{2\Delta_\psi-\Delta_\sigma}x_{13}^{\Delta_\sigma+1}x_{23}^{\Delta_\sigma+1}}\,. \tag{3.4.7}$$

Direct application of two Dirac operators yields

$$(d-2)(2\gamma_\psi + \gamma_\sigma)\delta^i_j\frac{C_{\psi\psi\sigma}\, x\!\!\!/_{13}\, x\!\!\!/_{23}}{x_{12}^d x_{13}^2 x_{23}^2} + \ldots \tag{3.4.8}$$

where we used the formula (3.4.2) with the O replaced by σ, together with the expansion $\Delta_\sigma = 1+\gamma_\sigma$. Inserting equations of motion, one gets

$$-\frac{1}{N}\langle\psi^i\sigma(x_1)\bar{\psi}_j\sigma(x_2)\sigma(x_3)\rangle = -\frac{1}{N}\langle\sigma(x_1)\sigma(x_2)\rangle\langle\psi^i(x_1)\bar{\psi}_j(x_2)\sigma(x_3)\rangle + \ldots = -\frac{1}{N}\frac{C_{\sigma\sigma}C_{\psi\psi\sigma}}{x_{12}^d x_{13}^2 x_{23}^2} + \ldots\,, \tag{3.4.9}$$

where we used that the three-point function $\langle\sigma\sigma\sigma\rangle$ vanishes, and in the second step we have selected only the relevant contraction that gives the tensor structure in (3.4.8). The final result is then

$$\gamma_\sigma = -2\gamma_\psi - \frac{C_{\sigma\sigma}}{N(d-2)} = -4\gamma_\psi \frac{d-1}{d-2}\,, \tag{3.4.10}$$

which agrees with the result quoted earlier (3.3.9). It is interesting to repeat the above calculation in the GNY model. Using (3.4.2) and $\Delta_\sigma = d/2 - 1 + \gamma_\sigma$, to leading order in the anomalous dimensions we find the relation

$$2\left(2\gamma_\psi + \gamma_\sigma - \frac{\epsilon}{2}\right) = -C_{\sigma\sigma}(g_1^*)^2 = -\frac{1}{4\pi^2}(g_1^*)^2\,. \tag{3.4.11}$$

Combining this with (3.3.43) and (3.3.45) obtained from the ψ and σ 2-point functions, we see that we recover the correct fixed-point coupling given in (3.3.40).

It should be possible by similar methods to fix the scaling dimensions of higher order composites of σ and ψ. We will not pursue this in detail here, and just use a shortcut to obtain the result for a simple class of operators, namely the composites σ^k with twist $k + O(1/N)$ (see also [124] for this calculation in the case $d = 3$). We note that the diagrammatic expansion suggests that, to leading order in $1/N$, Δ_{σ^k} is at most a quadratic function of k. More precisely, there are two diagrams, which connect the σ legs pair by pair, so the k dependence for these is $k(k-1)$. There are also leg corrections, which go like k. So, overall, we should have:

$$\Delta_{\sigma^k} = k + \frac{Ak^2 + Bk}{N} + O(1/N^2)\,. \tag{3.4.12}$$

We can then use the known results for $k = 1$ and $k = 2$ [102] to get:

$$\Delta_{\sigma^k} = k + \frac{2k(d-1)(d(k-1)-2)}{d-2}\gamma_\psi\,. \tag{3.4.13}$$

Note that we may also write this result as

$$\Delta_{\sigma^k} - k\Delta_\sigma = \frac{k(k-1)}{2}\frac{4d(d-1)\gamma_\psi}{d-2} = \frac{k(k-1)}{2}\left(\gamma_{\sigma^2} - 2\gamma_\sigma\right)\,, \tag{3.4.14}$$

where the right-hand side has the interpretation of interaction energy for the multi-particle state in AdS which is dual to σ^k.

We may also apply the same method to the bosonic $O(N)$ model in the $1/N$ expansion, which can be developed using the action for the scalar field ϕ^i and the auxiliary field σ (2.3.2) used in Chapter

2.

Similarly to the fermionic calculation above, we can fix the anomalous dimension of the traceless scalar $O^{ij} = \phi^{(i}\phi^{j)} - \frac{\delta^{ij}}{N}\phi^2$. One can study the following three-point function

$$\langle \phi^i(x_1)\phi^j(x_2)O^{kl}(x_3)\rangle = (\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{N}\delta^{ij}\delta^{kl})\frac{C_{\phi\phi O}}{x_{12}^{2\Delta_\phi - \Delta_O} x_{13}^{\Delta_O} x_{23}^{\Delta_O}} \tag{3.4.15}$$

as constrained by the conformal symmetry. The dimensions of operators are power series in $1/N$: $\Delta_\phi = d/2 - 1 + \gamma_\phi$, $\Delta_O = d - 2 + \gamma_O$. One can then act on this three-point function with the Laplacians $\Box_1$ and $\Box_2$ at points 1 and 2 respectively. Explicit differentiation gives, to the leading order in $1/N$:

$$2(d-2)(d-4)(2\gamma_\phi - \gamma_O)(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{N}\delta^{ij}\delta^{kl})\frac{C_{\phi\phi O}}{x_{12}^4 x_{13}^{d-2} x_{23}^{d-2}}\,, \tag{3.4.16}$$

where we replaced the dimensions of the operators with the tree-level values where possible. Now, inserting the equation of motion (2.3.4) in the correlation function, we get

$$\begin{aligned} &\frac{1}{N}\langle \sigma\phi^i(x_1)\sigma\phi^j(x_2)O^{kl}(x_3)\rangle = \frac{1}{N}\langle\sigma(x_1)\sigma(x_2)\rangle\langle\phi^i(x_1)\phi^j(x_2)O^{kl}(x_3)\rangle \\ &= \frac{1}{N}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{N}\delta^{ij}\delta^{kl})\frac{C_{\sigma\sigma}C_{\phi\phi O}}{x_{12}^4 x_{13}^{d-2} x_{23}^{d-2}} \end{aligned} \tag{3.4.17}$$

and so to the lowest order in $1/N$ we obtain:

$$\gamma_O = 2\gamma_\phi - \frac{C_{\sigma\sigma}}{2N(d-2)(d-4)} = -\frac{8\gamma_\phi}{d-4} = -\frac{2^d\Gamma(\frac{d-1}{2})\sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{d}{2}+1)}\,. \tag{3.4.18}$$

Now the calculation for the σ operator is slightly more complicated although the general idea is the same. We begin with the three-point function:

$$\langle \phi^i(x_1)\phi^j(x_2)\sigma(x_3)\rangle = \delta^{ij}\frac{C_{\phi\phi\sigma}}{x_{12}^{2\Delta_\phi - \Delta_\sigma} x_{13}^{\Delta_\sigma} x_{23}^{\Delta_\sigma}}\,. \tag{3.4.19}$$

Expanding $\Delta_\sigma = 2 + \gamma_\sigma$, and acting with the operators $\Box_1$ and $\Box_2$ gives:

$$2\delta^{ij}(d-2)(d-4)(\gamma_\sigma + 2\gamma_\phi)\frac{C_{\phi\phi\sigma}}{x_{12}^d x_{13}^2 x_{23}^2} + \dots\,, \tag{3.4.20}$$

where we omitted two other terms which do not contain γ_σ, and have a different coordinate dependence.

Inserting the equations of motion and working to leading order at large N, we get:

$$\begin{aligned}
&\frac{1}{N}\langle\sigma\phi^i(x_1)\sigma\phi^j(x_2)\sigma(x_3)\rangle \\
&= \frac{1}{N}\left(\langle\sigma(x_1)\sigma(x_2)\rangle\langle\phi^i(x_1)\phi^j(x_2)\sigma(x_3)\rangle + \langle\phi^i(x_1)\phi^j(x_2)\rangle\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\rangle + \ldots\right) \\
&= \frac{1}{N}\frac{(C_{\sigma\sigma}C_{\phi\phi\sigma} + C_{\phi\phi}C_{\sigma\sigma\sigma})}{x_{12}^d x_{13}^2 x_{23}^2} + \ldots\,,
\end{aligned} \tag{3.4.21}$$

where we have only kept the relevant structures that match (3.4.20). Comparing the two results, we obtain the relation

$$\gamma_\sigma = -2\gamma_\phi + \left(C_{\sigma\sigma} + \frac{C_{\phi\phi}C_{\sigma\sigma\sigma}}{C_{\phi\phi\sigma}}\right)\frac{1}{2N(d-2)(d-4)}\,, \tag{3.4.22}$$

where $C_{\sigma\sigma\sigma}$ and $C_{\phi\phi\sigma}$ are the 3-point function coefficients, which both start at order $1/N$. Their ratio is known to be, to leading order at large N [125]

$$\frac{C_{\sigma\sigma\sigma}}{C_{\phi\phi\sigma}} = \frac{2(d-3)C_{\sigma\sigma}}{C_{\phi\phi}}\,, \tag{3.4.23}$$

so finally we find

$$\gamma_\sigma = -2\gamma_\phi + (C_{\sigma\sigma} + 2(d-3)C_{\sigma\sigma})\frac{1}{2N(d-2)(d-4)} = \frac{4(d-1)(d-2)}{d-4}\gamma_\phi\,, \tag{3.4.24}$$

which is the correct result [65].

The result for σ^k operators for arbitrary k was obtained in [45], and is again constrained to be a quadratic function of k to leading order in $1/N$. It reads

$$\Delta_{\sigma^k} = 2k + \frac{2k(d-1)((k-1)d^2 + d + 4 - 3kd)}{d-4}\gamma_\phi + O(1/N^2)\,. \tag{3.4.25}$$

We can also express this result in terms of “binding energies”

$$\Delta_{\sigma^k} - k\Delta_\sigma = \frac{k(k-1)}{2}\frac{4d(d-1)(d-3)\gamma_\phi}{4-d} = \frac{k(k-1)}{2}\left(\gamma_{\sigma^2} - 2\gamma_\sigma\right)\,. \tag{3.4.26}$$

Quite interestingly, these vanish in $d=3$, so that $\Delta_{\sigma^k} = 2k + k\gamma_\sigma = k\Delta_\sigma$, as pointed out in [112]. In other words, from a bulk point of view, the interaction energy of the k-particle states of the AdS scalar field appear to vanish to leading order in $1/N$ in the special case $d=3$. We will comment on this further below.

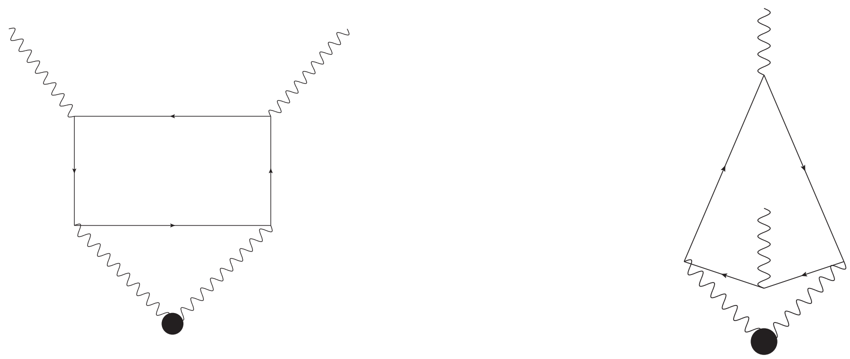


Figure 3.1: Diagrams contributing to the anomalous dimension of $\sigma\partial^s\sigma$ to order $1/N$.

3.4.2 Spinning double-trace operators

In this section we turn our attention to the spinning operators constructed from the σ fields, $\sigma\partial^s\sigma$ in the bosonic $O(N)$ model and the large-N Gross-Neveu model. The anomalous dimensions of $\sigma\partial^s\sigma$ and many other operators in the critical vector model were computed by Lang and Ruhl in [45]. Analogous results for the GN model are not available to the best of our knowledge. While an extensive discussion of the GN model can be found in many places, see e.g. [101, 116, 126], we will follow the conventions used in [88] and collect below only the necessary ingredients. The momentum space propagators for ψ and σ are:

$$D_\psi = \frac{\not{p}}{p^2}\,, \qquad D_\sigma = \frac{\tilde{C}_{\sigma\sigma}}{(p^2)^{d/2-1+\delta}}\,, \qquad \tilde{C}_{\sigma\sigma} = -\frac{2^d\pi^{d/2}\Gamma(d-1)}{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}\,, \tag{3.4.27}$$

where δ is a regulator and the only interaction vertex is $\frac{1}{\sqrt{N}}\bar{\psi}\psi\sigma$. Let us write the full conformal dimension of $O_s \sim \sigma\partial^s\sigma$ as

$$\Delta_{\sigma\partial^s\sigma} = 2\Delta_\sigma + s + \gamma_s = 2 + s + 2\gamma_\sigma + \gamma_s\,. \tag{3.4.28}$$

Note that in this and in the next subsection we adopt this definition of γ_s, which is naturally interpreted as an interaction energy from AdS point of view.

The anomalous dimensions can be extracted by computing the 3-point functions $\langle O_s\sigma\sigma\rangle$. This is most conveniently done in momentum space, and it is sufficient to set the momentum of the operator O_s to zero. There are two diagrams, given in figure 3.1, that contribute to the γ_s defined in (3.4.28) (the leg corrections only affect γ_σ), where the blob corresponds to the operator insertion. Up to $\tilde{C}_{\sigma\sigma}$

factors the integrals that correspond to these diagrams are, denoting by l the external momentum of the σ legs:

$$I_1 = \int \frac{d^dp}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \frac{(z\cdot q)^s tr[\not{p}(\not{p}-\not{q})\not{p}(\not{p}-\not{l})]}{(q^2)^{d-2+2\delta}(p-q)^2(p-l)^2(p^2)^2} \tag{3.4.29}$$

$$I_2 = \int \frac{d^dp}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \frac{(z\cdot q)^s}{(q^2)^{d-2+2\delta}} \frac{tr[\not{p}(\not{p}+\not{q})(\not{p}+\not{q}-\not{l})(\not{p}-\not{l})]}{p^2(p-l)^2(p+q)^2(p+q-l)^2}\,, \tag{3.4.30}$$

where, as usual, we conveniently take z to be null, $z^2 = 0$. The anomalous dimensions are related to the $1/\delta$ poles of these integrals. The second integral can be simplified by using

$$tr[\not{p}(\not{p}+\not{q})(\not{p}+\not{q}-\not{l})(\not{p}-\not{l})] = \frac{1}{2}\left[(p-l)^2(p+q)^2 + (p+q-l)^2p^2 - l^2q^2\right] \tag{3.4.31}$$

and dropping the last term as it does not contribute to the divergent part. The integrals are then standard and can be done with the help of

$$\int \frac{d^dq}{(2\pi)^d} \frac{(z\cdot q)^s}{q^{2a}(q-p)^{2b}} = \frac{\Gamma(d/2-b)\Gamma(a+b-d/2)\Gamma(-a+s+d/2)(z\cdot p)^s}{(4\pi)^{d/2}\Gamma(a)\Gamma(b)\Gamma(-a-b+s+d)(p^2)^{a+b-d/2}} \tag{3.4.32}$$

and similar formulae. The final results are:

$$\gamma_{s=0} = \frac{4(d-1)d}{d-2}\gamma_\psi\,, \tag{3.4.33}$$

$$\gamma_{s>0} = -2\gamma_\sigma \frac{2d\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\pi(d-1)(d-2)(d-4)} \frac{\Gamma\left(s+2-\frac{d}{2}\right)}{\Gamma\left(s+\frac{d}{2}\right)}\,. \tag{3.4.34}$$

Note that the $s = 0$ result is not related to the $s \to 0$ limit of the non-zero spin result. This is because the residue of the $1/\delta$-pole happens to be discontinuous at $s = 0$. As far as we know, the general spin result was not derived earlier in the literature. As a check, we note that using the known anomalous dimensions of σ and σ^2 [102]

$$\gamma_\sigma = -\frac{4(d-1)}{d-2}\gamma_\psi\,, \qquad \gamma_{\sigma^2} = 4(d-1)\gamma_\psi\,, \qquad \gamma_{\sigma^2} - 2\gamma_\sigma = \frac{4(d-1)d}{d-2}\gamma_\psi\,, \tag{3.4.35}$$

we find agreement between $\gamma_{s=0}$ and the last expression in the above equation. A non-trivial consistency check of the result (3.4.34) is the comparison with the $4-\epsilon$ expansion of the GNY model.

Indeed, expanding (3.4.34) in $d = 4 - \epsilon$, we find

$$(2\gamma_\sigma + \gamma_s)\Big|_{d=4-\epsilon} = -\frac{2\left(3s^2+3s-2\right)\epsilon}{s(s+1)} + O(\epsilon^4)\,, \tag{3.4.36}$$

which precisely agrees with (3.3.56).

From (3.4.34), we can extract the large spin behavior of the anomalous dimensions

$$\gamma_s = \frac{16\sin^2\left(\frac{\pi d}{2}\right)\Gamma(d-2)}{\pi^2(4-d)}\frac{1}{s^{d-2}} + \ldots\,, \tag{3.4.37}$$

suggesting that the leading term is due to the exchange of a tower of operators with twist $d-2$, i.e. the nearly conserved currents of the model at large N. We will comment further on this in the next subsection.

Comparison to analytic bootstrap approach

Let us discuss the relation of the results of the previous subsection to the general predictions [39–41] for the anomalous dimensions of double trace operators $O\partial^s O$, where O is a scalar primary of dimension Δ. One matches the anomalous dimension of the $O\partial^s O$ operators exchanged in the "s-channel" of the four-point function $\langle OOOO\rangle$ to the "t-channel" exchange of low-twist operators and their descendants. The expansion organizes itself as a function of conformal spin [105]:

$$J^2 = (\Delta + s + \gamma_s/2)(\Delta + s + \gamma_s/2 - 1)\,. \tag{3.4.38}$$

The contribution of the operator with spin l and twist τ exchanged in the t-channel is then:

$$\gamma_J(\tau, l) = -\frac{c_0(\tau,l)}{J^\tau}\left(1 + \sum_{k=1}^{\infty}\frac{c_k}{J^{2k}}\right)\,, \tag{3.4.39}$$

where the coefficient c_0 controlling the leading term in the large spin expansion is known from [39,40]:

$$c_0(\tau, l) = \frac{C^2_{OOO_\tau}}{C_{O_\tau O_\tau}C^2_{OO}}\frac{(-1)^l\Gamma(\tau+2l)\Gamma^2(\Delta)}{2^{l-1}\Gamma^2(\Delta-\frac{\tau}{2})\Gamma^2(l+\frac{\tau}{2})}\,. \tag{3.4.40}$$

In our case, $O = \sigma$ and $\Delta = 1 + O(1/N)$, and to leading order in $1/N$ we can study the exchange of the following operators in the t-channel: σ, the tower $\sigma\partial^l\sigma$ of twist $2 + O(1/N)$, and the fermionic higher-spin currents $\bar{\psi}\hat{\gamma}\hat{\partial}^{l-1}\psi$ of twist $d-2+O(1/N)$. Now the contribution of the σ operator vanishes since the three-point function $\langle\sigma\sigma\sigma\rangle = 0$ identically in the fermion model due to the parity symmetry. Further, we note that the $\sigma\partial^l\sigma$ would contribute to (3.4.40) only at the order $1/N^2$, since at tree level $\Delta = 1$, $\tau = 2$ and we would have hit $\Gamma^2(0)$ in the denominator, which gets resolved by expanding all the dimensions to the first order in $1/N$.

Hence, the exchange of the higher-spin currents of the form $\bar{\psi}\hat{\gamma}\hat{\partial}^{l-1}\psi$ should be what reproduces the answer for the anomalous dimensions (3.4.34). The relevant OPE coefficients that appear in (3.4.40) are explicitly computed in Appendix 3.5 as a function of the spin of the exchanged operator l and are of order $1/N$. This is sufficient to reproduce the leading large spin term in the anomalous dimensions. In [41], it has been shown how to sum the contributions other the descendants in (3.4.39) for the exchanged operator with a twist $d-2$, which is the case at hand, and obtain the finite spin dependence of the anomalous dimensions. The result is

$$\gamma_s(l) = -c_0(d-2,l)\frac{\Gamma(s+1)\Gamma(2\Delta+s-\frac{d}{2})}{\Gamma(s+\frac{d}{2})\Gamma(2\Delta+s-1)}\,. \tag{3.4.41}$$

Notice that in our case $\Delta = 1$ is the tree-level dimension of the σ operator, and we have to sum over the spin λ of the exchanged tower. Using (3.4.40) and the coefficient $a_{cr.F,s}$ from (3.5.9), the sum we need to perform is:

$$\sum_{l=2,4,6,\ldots}^{\infty}\frac{16(2l+d-3)\sin^2(\frac{d\pi}{2})\Gamma^2(d-2)\Gamma(l)}{\pi^2\Gamma(l+d-2)} = \frac{16\sin^2(\frac{d\pi}{2})\Gamma(d-2)}{(d-4)\pi^2}\,. \tag{3.4.42}$$

Note that this sum is power-like divergent for $d<4$, and we have evaluated it by analytic continuation in d, as usual in dimensional regularization. Putting it all together using (3.4.41), we arrive at the same result (3.4.34), in a slightly different form:

$$\gamma_s = -\frac{16\sin^2(\frac{d\pi}{2})\Gamma(d-2)\Gamma(s+2-\frac{d}{2})}{N(d-4)\pi^2\Gamma(s+\frac{d}{2})}\,. \tag{3.4.43}$$

In particular, in $d=3$ we obtain

$$\gamma_s = \frac{32}{\pi^2(2s+1)}\frac{1}{N} + O(1/N^2)\,. \tag{3.4.44}$$

Note that the anomalous dimensions are *positive* in $2<d<4$ for all spins. This may seem surprising, since each even spin field exchange contributes, according to (3.4.39)-(3.4.40), a negative amount to the anomalous dimensions (from AdS point of view, we expect attractive interactions due to each even spin l). However, the infinite sum in (3.4.42) is divergent, and its regularization appears to yield a positive final result. The fact that this procedure gives exact agreement with the direct diagrammatic calculation of the previous subsection is a good check that the regularization of the sum makes sense.

Let us also include here some results for these operators in the scalar theory. The $1/N$ result for γ_s has been obtained by Lang and Ruhl [45], and takes the form

$$\gamma_s = -2\gamma_\sigma \frac{d(d-3)}{(d-1)(d-2)(d-4)(s+1)(s+2)} \left(d^2 - 5d + 6 - \frac{2\Gamma(\frac{d}{2})\Gamma(s-\frac{d}{2}+4)}{\Gamma(4-\frac{d}{2})\Gamma(\frac{d}{2}+s)} \right) , \qquad (3.4.45)$$

with γ_σ given in (3.4.24). Note that in $d=3$ both of the spin-dependent contributions vanish, and the function (3.4.45) just reduces to $2\gamma_\sigma$, so that $\Delta_{\sigma\partial^s\sigma} = 2\Delta_\sigma + s$ in $d=3$. In other words, the "binding energy" of the corresponding two-particle state in the AdS_4 dual theory vanish, similarly to what observed earlier for the σ^k operators. In the range $2<d<6$ where the scalar CFT is unitary, we note that that the anomalous dimensions are positive in $3<d<4$, and negative in $2<d<3$ and $4<d<6$.[9] In general d, the result (3.4.45) has the large spin expansion

$$\gamma_s = -\frac{1}{s^2} \frac{2^d (d-3)^3 (d-2) \sin\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d-3}{2}\right)}{N\pi^{3/2}(d-4)\Gamma\left(\frac{d}{2}\right)} - \frac{1}{s^{d-2}} \frac{128(d-3)\sin^2\left(\frac{\pi d}{2}\right)\Gamma(d-2)}{N\pi^2(d-6)(d-4)^2} . \qquad (3.4.46)$$

Again, in light of the results of [39–41], we clearly see two different contributions in the t-channel accounting for this behavior: the exchange of σ operator with twist 2, producing the $1/s^2$, and the exchange of the higher spin tower $\phi\partial^l\phi$, accounting for the $1/s^{d-2}$. We can reproduce the full spin dependence of the contribution generated by the higher-spin tower by taking $\Delta = 2$ in (3.4.41) and using the bosonic OPE coefficients from (3.5.9):

$$\begin{aligned} \frac{\Gamma(s-\frac{d}{2}+4)}{(s+1)(s+2)\Gamma(s+\frac{d}{2})} \sum_{l=2,4,6,\ldots}^{\infty} \frac{64(2l+d-3)\sin^2(\frac{d\pi}{2})\Gamma^2(d-2)\Gamma(l+1)}{\pi^2(d-4)^2\Gamma(l+d-3)} = \\ = \frac{128(d-3)\sin^2(\frac{d\pi}{2})\Gamma(d-2)}{(d-4)^2(d-6)\pi^2} \frac{\Gamma(s-\frac{d}{2}+4)}{(s+1)(s+2)\Gamma(s+\frac{d}{2})} , \end{aligned} \qquad (3.4.47)$$

where we have again regularized the sum by analytic continuation. This is exactly equal to the last term in the expression (3.4.45). The other contribution in (3.4.45), which is responsible for the $1/s^2$

[9] The negativity in $4<d<6$ is consistent with Nachtmann theorem [103], since in this range the operators $\sim \sigma\partial^s\sigma$, with $\tau = 2 + O(1/N)$, are the minimal twists in the $\sigma\sigma$ OPE.

term in (3.4.46) at large spin, can be reproduced by using the known three-point function coefficients $\langle\sigma\sigma\sigma\rangle$ [125]:

$$\frac{C_{\sigma\sigma\sigma}^2}{C_{\sigma\sigma}^3}=\frac{1}{N}\frac{8(d-3)^2\Gamma(d-2)}{\Gamma(3-\frac{d}{2})\Gamma^3(\frac{d}{2}-1)}\,. \tag{3.4.48}$$

Plugging this into (3.4.40), we see that the resulting factor precisely agrees with the first term in (3.4.46). Remarkably, the full spin dependence can also be reconstructed in this case. This follows from analysis of [41], because for the special value $\Delta=\tau=2$ all higher order coefficients c_k in (3.4.39) vanish, and the finite spin answer is simply obtained from (3.4.46) by replacing $1/s^2$ with $1/J^2=1/(s+1)(s+2)$.

It is interesting to compare the above calculation to the case of the free fermionic and bosonic vector models. In the free CFT, we of course expect no anomalous dimensions for the double trace operators $\sim\bar\psi\psi\partial^s\bar\psi\psi$ and $\sim\phi^2\partial^s\phi^2$. But formally we can still apply the above analytic bootstrap results, and we would then expect to recover the vanishing of the anomalous dimensions from the sum over the infinite tower in the crossed channel. From a bulk point of view, this would correspond to computing the tree level 4-point functions of the bulk scalar, reading off the contribution to the binding energies of each diagram with a higher spin exchange (as well as the quartic scalar vertex), and summing up over all spins.[10] In the free fermion theory, using (3.4.40) and the OPE coefficients in 3.5.3, we encounter the sum (we omit the l-independent overall factors, and recall that the $\bar\psi\psi$ 3-point function is zero)

$$\sum_{l=2,4,6,\ldots}^{\infty}\frac{(2l+d-3)\Gamma(d+l-2)}{\Gamma(l)}=0\,, \tag{3.4.49}$$

and in the scalar theory

$$\sum_{l=0,2,4,6,\ldots}^{\infty}\frac{(2l+d-3)\Gamma(d+l-3)}{\Gamma(l+1)}=0\,. \tag{3.4.50}$$

Both of these sums, when regulated by analytic continuation in d, vanish. In $d=3$, one may also evaluate them by Riemann zeta-function regularization, see the next subsection. From the bulk perspective, the vanishing of the regularized sum over “binding energies” we encounter here is reminiscent of the vanishing of the regularized sum over one-loop vacuum energies found in [128, 129].

[10] Right the opposite was recently done in [127]: the quartic scalar self-interaction vertex was reconstructed in such a way that the full four-point function matches that of the free scalar CFT.

Chern-Simons vector models in $d = 3$

The analytic bootstrap results used in the previous subsection can be also readily applied to the bosonic and fermionic 3d Chern-Simons vector models of [85, 86]. We consider $U(N_c)$ Chern-Simons theory at level k coupled to a fundamental scalar or fermion, in the large N_c limit with $\lambda = N_c/k$ fixed, and use the approach described above to compute the anomalous dimensions of the spinning double trace operators $\sim \bar{\phi}\phi\partial^s\bar{\phi}\phi$ and $\sim \bar{\psi}\psi\partial^s\bar{\psi}\psi$.

Let us start with the scalar theory. The operators contributing in the t-channel, as before, will be the higher-spin tower (which still has twist $d-2+O(1/N_c)$, for any value of the 't Hooft coupling) and the scalar $\bar{\phi}\phi$. The latter has dimension $\Delta = d-2+O(1/N_c) = 1+O(1/N_c)$, so its contribution can be calculated exactly as a function of spin s using (3.4.41). Using the results of [21] which follow from the weakly broken higher-spin symmetries, the OPE coefficients we need still take the form (3.5.3), dressed with factors that depends on the parameters $\tilde{N}$ and $\tilde{\lambda}$ defined in [21], which can be fixed [113, 130] to be

$$\tilde{N} = 2N_c\frac{\sin(\pi\lambda)}{\pi\lambda}\,, \qquad \tilde{\lambda} = \tan(\frac{\pi\lambda}{2})\,. \tag{3.4.51}$$

Explicitly, using the results of [21], one finds that the contribution of the higher-spin tower acquires a factor $1/\tilde{N}$, and the scalar contribution comes with a factor $\frac{1}{\tilde{N}}\frac{1}{1+\tilde{\lambda}^2}$. Putting all the factors from (3.4.39), (3.4.40), (3.4.41), (3.5.3) together in $d = 3$, we get[11]

$$\gamma_{\bar{\phi}\phi\partial^s\bar{\phi}\phi} = -\frac{2}{2s+1}\left(\frac{1}{\tilde{N}}\sum_{l=2,4,6,\ldots}^{\infty}\frac{32}{\pi^2}+\frac{1}{\tilde{N}}\frac{1}{1+\tilde{\lambda}^2}\frac{16}{\pi^2}\right) = \frac{1}{\tilde{N}}\frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2}\frac{32}{\pi^2(2s+1)} + O(1/N_c^2)\,, \tag{3.4.52}$$

where we used the Riemann zeta function to regulate the sum (alternatively, one can regulate the sums dimensionally as in the previous section). Note that the answer goes to zero as $\tilde{\lambda} \to 0$, as expected since in that limit we recover the free scalar CFT, and for $\tilde{\lambda} \to \infty$ it goes into the 3d Gross-Neveu result (3.4.44), as it should be in accordance with the 3d bosonization duality.[12]

Let us now move to the fermionic theory. The calculation is almost the same, with the difference that the scalar contribution vanishes, since the three-point function of the scalar $\bar{\psi}\psi$ vanishes. As for the higher-spin tower (which acquires a factor $1/\tilde{N}$ as above), after plugging all the factors in $d = 3$,

[11] Keep in mind that we define $\gamma_{\bar{\phi}\phi\partial^s\bar{\phi}\phi} = \Delta_{\bar{\phi}\phi\partial^s\bar{\phi}\phi} - s - 2\Delta_{\bar{\phi}\phi}$. The scalar scaling dimension is $\Delta_{\bar{\phi}\phi} = 1+\gamma_{\bar{\phi}\phi}$, but $\gamma_{\bar{\phi}\phi} = f_0(\lambda)/N_c + O(1/N_c^2)$ is not currently known to all orders in λ.

[12] To be precise, at $\tilde{\lambda} \to \infty$ we get the result for the $U(k-N_c)$ GN model, and one should keep in mind that in (3.4.44) $N = N_c\mathrm{tr}1 = 2N_c$.

its contribution is proportional to the sum

$$\sum_{l=2,4,6,\dots}^{\infty} l^2 = 4\zeta(-2) = 0 \tag{3.4.53}$$

using the same zeta function regularization (equivalently, one gets the same result using the dimensionally continued sum in (3.4.49)), and hence we conclude that $\gamma_{\bar{\psi}\psi\partial^s\bar{\psi}\psi} = O(1/N_c^2)$ in the CS-fermion model. Note that this result is expected from the absence of anomalous dimensions in the free fermion theory, because apart from the overall factor of $1/\tilde{N}$ instead of $1/N$, the sum over the higher-spin tower is otherwise identical in the CS-fermion theory and free fermion theory, and has to vanish in the latter. This also explains the vanishing of the result (3.4.45) in the $d = 3$ critical scalar model: the scalar exchange contribution is absent in this model as well ($C_{\sigma\sigma\sigma} = 0$ in the 3d critical $O(N)$ model), and the higher-spin tower contribution is the same up to the overall function of $\tilde{\lambda}$; this is because the dimension of the external scalar is $\Delta = 2$ in both cases, and the relevant OPE coefficients then coincide, see eq. (3.5.10). In this sense, the vanishing of the σ binding energies in the 3d critical $O(N)$ model can be seen as a manifestation of the 3d bosonization duality.

3.5 Appendix: OPE Coefficients from AdS/CFT

AdS/CFT relates the OPE coefficients of the UV and IR duals, i.e. the duals of the same bulk theory for different choice of boundary conditions within the unitarity window [29, 31]. We can use this fact to compute some of the OPE coefficients in the critical $O(N)$ vector model and Gross-Neveu model using the OPE coefficients of the free scalar and free fermion, respectively.

The OPE coefficients we are interested in correspond to three-point function $\langle J_s J_0 J_0 \rangle$, where J_0 has scaling dimension $\Delta = 2$ for the critical model and $\Delta = 1$ for GN, and J_s are the totally symmetric (nearly) conserved currents with twist $\tau = d-2+O(1/N)$. We define the normalized OPE coefficients a_s as

$$a_s \equiv \frac{C_{s00}^2}{C_{ss}C_{00}^2}\,, \tag{3.5.1}$$

where we the coefficients on the right-hand side are defined in our conventions by

$$\langle J_0(x_1)J_0(x_2)\rangle = \frac{C_{00}}{x_{12}^{2\Delta}}\,, \qquad \langle \hat{J}_s(x_1,z_1)\hat{J}_s(x_2,z_2)\rangle = C_{ss}\frac{\left(z_1\cdot z_2 - \frac{2z_1\cdot x_{12}z_2\cdot x_{12}}{x_{12}^2}\right)^s}{x_{12}^{2\Delta_s}}\,,$$
$$\langle \hat{J}_s(x_1,z_1)J_0(x_2)J_0(x_3)\rangle = C_{s00}\frac{\left(\frac{z_1\cdot x_{13}}{x_{13}^2} - \frac{z_1\cdot x_{12}}{x_{12}^2}\right)^s}{x_{12}^{\tau}x_{13}^{\tau}x_{23}^{2\Delta-\tau}} \equiv C_{s00}\overline{\langle J_sO_\Delta O_\Delta\rangle}\,, \tag{3.5.2}$$

where z_1, z_2 are null vectors. The OPE coefficients for the free fermion theory were given in (3.2.28), and for the scalar can be found in [84, 127, 131, 132], and we obtain:

$$a_{B,s} = \frac{\sqrt{\pi}2^{-d-s+7}\Gamma\left(\frac{d}{2}+s-1\right)\Gamma(d+s-3)}{N\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(s+1)\Gamma\left(\frac{d-3}{2}+s\right)}\,, \tag{3.5.3}$$
$$a_{F,s} = \frac{\sqrt{\pi}(-)^s2^{-d-s+5}\Gamma\left(\frac{d}{2}+s-1\right)\Gamma(d+s-2)}{N\Gamma\left(\frac{d}{2}\right)^2\Gamma(s)\Gamma\left(\frac{d-3}{2}+s\right)}\,. \tag{3.5.4}$$

The three-point function of one higher-spin current J_s and two scalar operators of dimension Δ comes from the unique cubic interaction vertex in AdS

$$g_s\int \Phi_{\underline{m}(s)}\nabla^{\underline{m}(s)}\Phi_0\Phi_0 = g_s\tilde{b}_s \times \overline{\langle J_sO_\Delta O_\Delta\rangle}\,, \tag{3.5.5}$$
$$\tilde{b}_s = \frac{2^{-5+2s}\pi^{-d/2}(-3+d+2s)\Gamma\left[-1+\frac{d}{2}+s\right]^3\Gamma[-3+d+s]\Gamma[-1+s+\Delta]^2}{\Gamma[-2+d+2s]^2\Gamma[\Delta]^2}\,, \tag{3.5.6}$$

where g_s is the coupling constant of the bulk theory and b_s is a nontrivial factor produced by integrating the vertex on boundary-to-bulk propagators [133]. The coupling constant g_s can be chosen as to reproduce the OPE coefficients C_{s00} for free scalar $\Delta = d-2$ or free fermion $\Delta = d-1$. The results [84, 127] are

$$\text{boson}: \qquad g_s^B = \frac{1}{\sqrt{N}}\frac{\pi^{\frac{d-3}{4}}2^{\frac{1}{2}(3d+s-1)}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d+s-3)}\sqrt{\frac{\Gamma\left(\frac{d-1}{2}+s\right)}{\Gamma(s+1)}}\,, \tag{3.5.7}$$
$$\text{fermion}: \qquad (g_s^F)^2 = (g_s^B)^2\frac{s}{(d+s-3)}\,. \tag{3.5.8}$$

Next, we can change the boundary conditions to $\Delta = 2$ and $\Delta = 1$, respectively, and recompute the bulk integral, i.e. b_s. The result should give the OPE coefficients for the critical models:

$$\begin{aligned} a_{cr.B,s} &= \frac{2^{d-s+1}\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(s+1)\Gamma\left(\frac{d}{2}+s-1\right)}{\sqrt{\pi}N\Gamma\left(\frac{d-3}{2}+s\right)\Gamma(d+s-3)}\,, \\ a_{cr.F,s} &= \frac{2^{d-s+1}\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(s)\Gamma\left(\frac{d}{2}+s-1\right)}{\sqrt{\pi}N\Gamma\left(\frac{d-3}{2}+s\right)\Gamma(d+s-2)}\,. \end{aligned} \tag{3.5.9}$$

The same result can, of course, be obtained on the CFT side by attaching two propagators of the σ-field to the three-point functions of the free theories. As explained in [134, 135], the procedure of attaching a σ line on the CFT side is in one-to-one correspondence with changing the boundary condition on the bulk scalar propagator, and hence one is essentially guaranteed to obtain the same result. Nevertheless, to double-check our results we have explicitly computed (3.5.9) directly on the CFT side, and obtained the same result.

Note that in $d = 3$, when the leading large N dimensions of the scalar operator coincide in free fermion/critical scalar and free scalar/critical fermion, we have

$$\begin{aligned} a_{F,s}|_{d=3} &= a_{cr.B,s}|_{d=3}\,, \\ a_{B,s}|_{d=3} &= a_{cr.F,s}|_{d=3}\,. \end{aligned} \tag{3.5.10}$$

Chapter 4

Large N Chern-Simons Vector Models

4.1 Introduction and Summary

Chern-Simons (CS) gauge theories coupled to massless matter fields lead to a large class of conformal field theories in three dimensions, with or without supersymmetry. A particularly interesting non-supersymmetric example is obtained by coupling a $U(N)$ (or $O(N)$) CS gauge theory to a fermion or scalar in the fundamental representation [85, 86]. The Chern-Simons coupling k is quantized and cannot run (up to a possible integer shift at one loop). Therefore, in the fermionic case it is sufficient to tune away the relevant mass term to obtain a conformal field theory (CFT) for any N and k [85].[1] In the scalar case, one has a classically marginal coupling ϕ^6 that can get generated along RG flow, but in the presence of CS interactions one can find zeroes of its beta function, at least for sufficiently large N [86]. One may also obtain "critical" versions of these models by adding quartic self-interactions for the fundamental matter fields. In the scalar case, this leads to an IR fixed point which is a generalization of the familiar critical $O(N)$ model. In the fermionic case, at least in the large N expansion, one finds UV fixed points which generalize the critical 3d Gross-Neveu model.

The CFTs described above may be viewed as generalizations of the well-known bosonic and fermionic vector models by the addition of CS interactions, and we may refer to them as "Chern-

[1] The level k has to be half-integer due to the parity anomaly [136–138].

Simons vector models". Their investigation was initially motivated by the study of the AdS/CFT duality between Vasiliev higher-spin theory in AdS_4 [26][2] and free/critical vectorial CFTs with scalar or fermionic fields [29, 139, 140]. Gauging the global symmetries of the vector model by means of the CS gauge theory leads to a natural way to implement the singlet constraint, which is necessary in the conjecture of [29]. Remarkably, it turns out that in the 't Hooft limit of large N with $\lambda = N/k$ fixed, the CS vector models admit an approximate higher-spin (HS) symmetry, similarly to their ungauged versions, in the sense that the currents j_s are approximately conserved and have small anomalous dimension at large N [85, 86]. The fact that the anomalous dimensions are generated through $1/N$ corrections implies that the holographic dual to the CS vector models should be a parity breaking version of Vasiliev HS gravity, where the HS fields are classically massless, and masses are generated via bulk loop diagrams. The bulk HS theory is characterized by a parity breaking phase θ_0, which is mapped to the CFT 't Hooft coupling λ. See e.g. [89, 91, 141] for reviews of this duality.

A variety of new techniques have been developed and applied recently to the study of bosonic and fermionic vector models [24, 33, 52, 84, 88, 92–98, 106, 119, 142, 143], and bootstrap methods have also been applied for studying operators with large spin, e.g. [39, 40, 62, 104, 105, 107]. Partially motivated by this body of works, we study the spectrum of $1/N$ scaling dimensions of single-trace, primary operators with $s \geq 1$ in Chern-Simons vector models.

As we review in section 4.2, the spectrum of single-trace primary operators in these models is very simple: it just consists of bilinears in the fundamental matter fields. These include a scalar operator ($\bar{\phi}\phi$ or $\bar{\psi}\psi$), and a tower of spinning operators j_s of all integer spins. Owing to the topological nature of the CS gauge field, the addition of the CS interactions does not lead to any new local operator on top of the bilinears. It follows, as will be reviewed in more detail below, that the non-conservation of the HS currents j_s must take the schematic form [85, 86, 118]

$$\partial \cdot j_s \sim \sum_{s_1, s_2} \frac{1}{N} f^{(3)}_{s,s_1,s_2}(\lambda) \partial^n j_{s_1} \partial^m j_{s_2} + \sum_{s_1, s_2, s_3} \frac{1}{N^2} f^{(4)}_{s,s_1,s_2,s_3}(\lambda) \partial^n j_{s_1} \partial^m j_{s_2} \partial^p j_{s_3} \,, \tag{4.1.1}$$

where the "double-trace" and "triple-trace" operators on the right-hand side correspond to products of the bilinears and their derivatives, and no "single-trace" operator can appear, since there are none in the spectrum with the correct quantum numbers. The weakly broken HS symmetries corresponding to (4.1.1) can be used to constrain all planar 2-point and 3-point functions of the single-trace operators

[2] See for instance [27, 90, 91] for a review of the $4d$ Vasiliev equations.

in terms of two parameters [118].[3] The non-conservation equation (4.1.1) also encodes the anomalous dimensions of the weakly broken currents: schematically, $\gamma_s = \Delta_s - s - 1 \sim \langle \partial \cdot j_s | \partial \cdot j_s \rangle / \langle j_s | j_s \rangle$. Because the right-hand side of (4.1.1) contains no single-trace operators, it follows that the anomalous dimensions vanish at planar level, and the leading term is of order $1/N$:

$$\Delta_s = s + 1 + \frac{\gamma^{(1)}(s,\lambda)}{N} + \frac{\gamma^{(2)}(s,\lambda)}{N^2} + \dots . \tag{4.1.2}$$

In this chapter, we compute the term of order $1/N$ in the anomalous dimensions (4.1.2) for all $s \geq 1$ operators, in both fermionic and bosonic CS vector models, and to all orders in λ. As described in section 4.3.1, using the slightly broken higher-spin symmetry, one can show that the anomalous dimensions, or equivalently the twists $\tau_s = \Delta_s - s$, of the HS operators in the bosonic and fermionic CS-vector models must take the form:

$$\tau_s - 1 = \frac{1}{\tilde{N}} \left(a_s \frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2} + b_s \frac{\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2} \right) + O(\frac{1}{N^2}) \,, \tag{4.1.3}$$

where $\tilde{N}$ and $\tilde{\lambda}$ are the parameters introduced in the analysis of [118], as reviewed in section 4.2 and 4.3 below. The spin-dependent coefficients a_s and b_s can be determined by computing the 2-point function of the operator appearing in the non-conservation equation (4.1.1), neglecting the triple-trace term which does not affect the anomalous dimensions to this order. In section 4.3 we constrain the divergence of the HS currents using conformal invariance alone, up to some spin-dependent numerical coefficients, and in section 4.4 we use the classical equations of motion to calculate the divergence explicitly and fully fix the structure of the double-trace part of (4.1.1). A priori, the values of a_s and b_s may be different for the fermionic and bosonic theory. However, in our calculations below, we find that they are identical for both theories, and, in the case of $U(N)$ gauge group, they are given by

$$a_s = \begin{cases} \frac{16}{3\pi^2}\frac{s-2}{2s-1} \,, & \text{for even } s \,, \\ \frac{32}{3\pi^2}\frac{s^2-1}{4s^2-1} \,, & \text{for odd } s \,, \end{cases} \tag{4.1.4}$$

$$b_s = \begin{cases} \frac{2}{3\pi^2}\left(3g(s) + \frac{-38s^4+24s^3+34s^2-24s-32}{4s^4-5s^2+1}\right) \,, & \text{for even } s \,, \\ \frac{2}{3\pi^2}\left(3g(s) + \frac{20-38s^2}{4s^2-1}\right) \,, & \text{for odd } s \,, \end{cases} \tag{4.1.5}$$

[3] In the case of regular CS-scalar theory or critical CS-fermion theory, there is an additional marginal parameter corresponding to sextic couplings.

with

$$g(s) = \sum_{n=1}^{s} \frac{1}{n - 1/2} = \gamma - \psi(s) + 2\psi(2s) = H_{s-1/2} + 2\log(2)\,, \tag{4.1.6}$$

where $\psi(x)$ is the digamma function, and H_n the Harmonic number. In section 4.4, we also present the results for Chern-Simons theories based on $O(N)$ gauge group, which give slightly different coefficients that are reported in eq. (4.3.35). As a consistency check, note that the anomalous dimensions vanish for $s = 1$ and $s = 2$, as expected.

While the functions $\tilde{N}$ and $\tilde{\lambda}$ are not fixed by the weakly broken HS symmetry analysis, they can be fixed by an explicit calculation of 2-point and 3-point functions, and they were found to be [113,130]

$$\tilde{N} = 2N \frac{\sin(\pi\lambda)}{\pi\lambda}\,, \qquad \tilde{\lambda} = \tan(\frac{\pi\lambda}{2})\,, \tag{4.1.7}$$

in both CS-scalar and CS-fermion theories, in terms of the respective N and λ. Using these into (4.1.3), the anomalous dimensions take the form

$$\tau_s - 1 = \frac{\pi\lambda}{2N\sin(\pi\lambda)} \left(a_s \sin^2(\frac{\pi\lambda}{2}) + \frac{b_s}{4}\sin^2(\pi\lambda) \right)\,. \tag{4.1.8}$$

As an independent check of this result, in section 4.5 we also perform a direct Feynman diagram calculation in the CS-fermion model, from which we find the same values of the a_s and b_s coefficients.

Note that due to the harmonic sum in (4.1.5), we have $b_s \simeq \frac{2}{\pi^2}\log s$ for $s \gg 1$, while a_s is constant at large s, and so the large spin behavior of the anomalous dimensions is

$$\tau_s - 1 \simeq \frac{1}{\tilde{N}} \frac{\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2} \frac{2}{\pi^2} \log s = \frac{\lambda \sin(\pi\lambda)}{4\pi N} \log s\,. \tag{4.1.9}$$

This logarithmic behavior is a hallmark of gauge theory, and is expected from general arguments [48], see also the recent bootstrap analysis in [62]. The coefficient $f(\lambda) = \frac{\lambda\sin(\pi\lambda)}{4\pi N}$ of $\log s$ may be interpreted as the "cusp anomalous dimension" of the model; it would be interesting to see if it can be reproduced by computing the expectation value of a Wilson loop with a light-like cusp.

The result (4.1.8) applies to the "regular" CS-fermion and CS-scalar models. In the critical models, a calculation using the classical equations of motion, extended to all orders in λ by using the results of [118], yields

$$\tau_s^{\text{crit.}} - 1 = \frac{1}{\tilde{N}}\left(a_s \frac{1}{1+\tilde{\lambda}^2} + b_s \frac{\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2}\right) = \frac{\pi\lambda}{2N\sin(\pi\lambda)}\left(a_s \cos^2(\frac{\pi\lambda}{2}) + \frac{b_s}{4}\sin^2(\pi\lambda)\right) , \quad (4.1.10)$$

for both the critical CS-scalar and critical CS-fermion theory. In particular, at $\lambda = 0$, we recover the anomalous dimensions in the usual (Wilson-Fisher) critical $O(N)$ model [33, 45, 84, 142] and critical GN model [87, 119], which happen to coincide in $3d$

$$\gamma_s^{\text{W.F.}} = \gamma_s^{\text{GN}} = \frac{1}{2N} a_s = \begin{cases} \frac{8}{3N\pi^2}\frac{s-2}{2s-1} , & \text{for even } s , \\ \frac{16}{3N\pi^2}\frac{s^2-1}{4s^2-1} , & \text{for odd } s . \end{cases} \quad (4.1.11)$$

Note that the same anomalous dimensions arise in the strong coupling limit, $\lambda \to 1$ ($\tilde{\lambda} \to \infty$) of the regular CS-fermion and CS-scalar result (4.1.8). More precisely, in this limit one finds

$$\tau_s - 1 \overset{\lambda\to 1}{\simeq} \frac{1}{2(k-N)} a_s , \quad (4.1.12)$$

which are the anomalous dimensions in the $U(k-N)$ critical Wilson-Fisher or Gross-Neveu model. This is a manifestation of the "3d bosonization" duality [85, 113, 118] which conjecturally relates the critical/regular CS-scalar theory to the regular/critical CS-fermion theory. The precise form of the duality was spelled out in [113, 144], and reads[4]

$$U(N)_{k-1/2} \ \ \text{CS} - \text{Fermion} \quad \Leftrightarrow \quad U(|k|-N)_{-k} \ \ \text{Critical CS} - \text{Scalar} , \quad (4.1.13)$$

and a similar duality relating the regular CS-scalar to the critical CS-fermion.[5] So far we have assumed that k is the CS level defined in the dimensional reduction scheme [146], where no one-loop renormalization of the level occurs. To write the duality in a more familiar form, it is useful to express it in terms of $\kappa = k - \text{sign}(k)N$; this is the definition of the CS-level that arises when the theory is regularized with a Yang-Mills term in the UV.[6] In terms of this, the duality reads

$$U(N)_{\kappa-1/2} \ \ \text{CS} - \text{Fermion} \quad \Leftrightarrow \quad U(|\kappa|)_{-N} \ \ \text{Critical CS} - \text{Scalar} , \quad (4.1.14)$$

[4]Versions of this duality map involving the $SU(N)$ gauge group were also recently proposed in [144], where the mapping of baryon and monopole operators was discussed (see also [145]).

[5]In this case, the duality at large N also entails a mapping [130] between the additional marginal couplings $g_6(\bar{\psi}\psi)^3$ and $\lambda_6(\phi^*\phi)^3$ in these models.

[6]This definition of κ agrees with the level of the WZW theory dual to the CS theory.

and it can be recognized as a generalization of level-rank duality in pure CS theory [147–149]. Several non-trivial tests of the duality have been obtained in the large N 't Hooft limit [113, 130, 150–160]. If we denote by N_b and λ_b the rank and coupling in the critical CS-scalar theory, and by N_f, λ_f the ones in the CS-fermion theory, in the large N limit (where we can neglect the half-integer shift of the level on the fermionic side), the duality implies the map

$$\lambda_b = \lambda_f - \mathrm{sign}(\lambda_f)\,, \qquad \frac{N_b}{|\lambda_b|} = \frac{N_f}{|\lambda_f|}\,, \tag{4.1.15}$$

or equivalently $\tilde{N}_b = \tilde{N}_f$, $\tilde{\lambda}_b^2 = 1/\tilde{\lambda}_f^2$. Comparing (4.1.8) and (4.1.10), we see that the anomalous dimensions are indeed mapped into each other under the duality. Furthermore, by our explicit calculation using the classical equations of motion in section 4.4, we will verify that the non-conservation equations (4.1.1) in the dual theories correctly map into each other, including the normalization factors.

The "3d bosonization" (4.1.13) may also be regarded as a non-supersymmetric version of the supersymmetric dualities [161, 162], which are well established at finite N and k. Therefore, it is plausible that the bose/fermi duality (4.1.13) holds away from the large N limit. For small N and k, (4.1.13) and related dualities may have interesting applications in condensed matter physics, see for example [163–168] for recent closely related work. While exact results at finite N and k are hard to obtain, it would be interesting to see if the subleading terms in the large N expansion of the anomalous dimensions (or other quantities such as the thermal free energy) may be also computed for finite λ, and whether they agree with the duality. Note that the half-integer shift in the CS-fermion level can play a non-trivial role in this case. As a first step towards determining subleading corrections at large N, in section 4.4 we use the classical equations of motion method to fix the terms of order λ^2/N^2 in the anomalous dimensions of the CS-scalar and CS-fermion models. In particular, this result gives the term of order $1/k^2$ in the scaling dimensions of the spin-s operators in the $U(1)_k$ CS theory coupled to a fundamental fermion.[7]

Besides encoding the anomalous dimensions of the HS operators, the current non-conservation equation (4.1.1) can also be used to completely fix (including the overall normalization) the parity

[7] In one version of the dualities put forward in [144], see also [165], the $U(1)_{-1/2}$ CS-fermion theory is related to the critical $O(2)$ model without CS gauge field. Our result for the anomalous dimensions γ_s in the $U(1)_k$ theory to order $1/k^2$ shows logarithmic behavior at large s. On the other hand, we do not expect logarithmic growth in the critical $O(2)$ model. It is plausible that the $\log s$ behavior disappears in the strongly coupled ($k = -1/2$) theory, but it would be interesting to understand this better.

odd structure in the planar 3-point functions of $\langle j_{s_1} j_{s_2} j_{s_3} \rangle$ when the triangular inequality is violated, i.e. $s_3 > s_1 + s_2$; this is the case where the 3-point function breaks the j_{s_3} current conservation. In section 4.6, we use our results from the classical divergence calculation to determine explicitly all such parity odd 3-point functions. In particular, we derive some recursion relations that can be used to obtain the explicit form of the 3-point functions for general spins. The parity-odd three-point functions are further analyzed in Appendix 4.8, with some examples listed for low spins in Appendix 4.9.

An interesting open problem that we do not address in this chapter is the calculation of the scaling dimension of the scalar operators $\bar{\phi}\phi$ or $\bar{\psi}\psi$. It is possible to argue [85, 86] based on the structure of the HS breaking equations (where the scalar operators can appear on the right-hand side) that they must have dimensions $\Delta = 1 + O(1/N)$ or $\Delta = 2 + O(1/N)$, but it is not obvious if the weakly broken HS symmetry can be used to determine the order-$1/N$ correction for finite λ. A direct all-orders diagrammatic calculation may in principle be possible, but it appears to require a currently unavailable ingredient: the ladder diagram of [154, 156] for general off-shell external momenta.

Another interesting direction would be to extend the results of this chapter to various other related CS-matter theories. As an example, $U(N) \times U(M)$ Chern-Simons theories coupled to bi-fundamental matter, also possess a weakly broken HS symmetry when $M/N \ll 1$ [141, 169, 170], and the methods used in this chapter should be applicable to this class of models. As the non-supersymmetric theories have two independent Chern-Simons levels, the $1/N$ anomalous dimensions here appear to depend on two independent parameters, so it would be interesting to see how these parameters relate to the general analysis of [118] (which, in its present form, applies to theories with even spin currents only). It may be also interesting to consider general CS-vector models [153] with fundamental boson and fermions on the same side (including in particular the supersymmetric theories as a special case).

Perhaps the most interesting extension of this work would be to calculate the anomalous dimensions of higher spin operators in the $\mathcal{N} = 6$ ABJ theory [171], in the regime $M \ll N$, which has been conjectured to be dual to a particular limit of type IIA string theory. Our results here do not directly carry over to this case because of the additional matter fields and the presence of the Chern-Simons coupling for the second gauge field, but we expect a similar analysis to be possible in principle. We hope to return to this in future work.

As mentioned earlier, the weakly broken HS operators should correspond in the dual AdS_4 theory

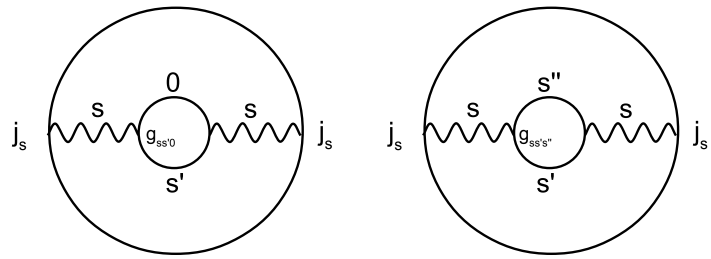


Figure 4.1: The one-loop bulk diagrams that are expected to reproduce the $1/N$ term in the anomalous dimensions of the HS currents at the boundary.

to classically massless HS gauge fields that acquire masses via loop corrections, through a HS analogue of the Higgs mechanism [32].[8] It would be interesting to see if the result for the anomalous dimensions (4.1.8) can be reproduced by a one-loop calculation in the parity breaking higher-spin theory, corresponding schematically to the diagrams depicted in figure 4.1. Note that the coupling constant in the bulk is fixed by the duality to be $1/G_N \sim \tilde{N} = 2N\frac{\sin(\pi\lambda)}{\pi\lambda}$, and the parity breaking 3-point couplings are expected to depend on the bulk parameter θ_0 as $g^{\rm odd}_{ss'0} \sim \sin\theta_0$ and $g^{\rm odd}_{ss's''} \sim \sin(2\theta_0)$ (see e.g. [89]), and we also have $g^{\rm even-A}_{ss's''} \sim \cos^2(\theta_0)$, $g^{\rm even-B}_{ss's''} \sim \sin^2(\theta_0)$. Therefore we see that if $\theta_0 = \pi\lambda/2$, which is required for agreement of the tree-level 3-point functions, the bulk one-loop diagrams would yield the expected coupling dependence we found in (4.1.8). It remains to be seen if the spin-dependent coefficients can be reproduced from the AdS calculation.

4.2 The Chern-Simons vector models

The action for the $U(N)$ Chern-Simons theory at level k coupled to a massless fundamental scalar field is given in our conventions by

$$S = \frac{ik}{4\pi} S_{\rm CS} + \int d^3x \left(D_\mu\bar{\phi} D^\mu\phi + \frac{\lambda_6}{N^2}(\bar{\phi}\phi)^3 \right) , \tag{4.2.1}$$

where

$$S_{\rm CS} = \int d^3x \epsilon^{\mu\nu\rho} {\rm Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) . \tag{4.2.2}$$

[8]The role of the Higgs field is played in this case by a multi-particle state in the bulk which is dual to the operator appearing on the right-hand side of (4.1.1).

We work in Euclidean signature throughout the chapter, and use the conventions $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$, $D_\mu\bar\phi = \partial_\mu\bar\phi + i\bar\phi A_\mu$, with $A_\mu = A_\mu^a T^a$, where T^a are the generators of $U(N)$ in the fundamental representation. One can show that in the large N limit with $\lambda = N/k$ and λ_6 fixed, the classically marginal coupling λ_6 is in fact exactly marginal. Hence, in the large N limit the model (4.2.1) defines a CFT (provided the scalar mass is suitably tuned to zero) labeled by two marginal parameters λ, λ_6 [9]. The value of λ_6 does not affect the anomalous dimensions of the higher-spin operators to the order $1/N$ we consider, and hence we will neglect this coupling in the following.

One may define another bosonic CFT, sometimes referred to as the critical bosonic theory, by adding to (4.2.1) a quartic interaction $\frac{\lambda_4}{N}(\bar\phi\phi)^2$ and flowing to the infrared. Rewriting the quartic coupling with the aid of a Hubbard-Stratonovich auxiliary field σ_b, the action of the IR CFT may be written as

$$S_{\rm crit} = \frac{ik}{4\pi}S_{\rm CS} + \int d^3x \left(D_\mu\bar\phi D^\mu\phi + \frac{1}{N}\sigma_b\bar\phi\phi\right) , \tag{4.2.3}$$

where the quadratic term in σ_b was dropped, which is appropriate in the IR limit. The factor of $1/N$ was introduced so that the 2-point function of σ_b scales like N. Note that the ϕ^6 term can be dropped since this coupling becomes irrelevant in the IR. This model defines a generalization of the Wilson-Fisher CFT by the addition of the Chern-Simons gauge coupling.

The action of a fundamental massless fermion coupled to the $U(N)$ CS gauge field at level k is given by

$$S = \frac{ik}{4\pi}S_{\rm CS} + \int d^3x \bar\psi \not{D}\psi\,, \tag{4.2.4}$$

where we define $\not{D} = \gamma^\mu D_\mu$ and $D_\mu\psi = \partial_\mu\psi - iA_\mu\psi$. Note that the level k should be half-integer due to the parity anomaly, however this condition will not be important for the large N computations we will perform below. The action (4.2.4) defines a CFT labeled by N and $\lambda = N/k$, provided the fermion mass term is tuned to zero.

Analogously to the scalar case, one may add to the model (4.2.4) a quartic self-interaction $\frac{g_4}{N}(\bar\psi\psi)^2$. Such theory is expected to have, at least in the large N limit, a non-trivial UV fixed point which is a generalization of the critical 3d Gross-Neveu model. The action describing the UV CFT can be taken to be

$$S_{\rm crit} = \frac{ik}{4\pi}S_{\rm CS} + \int d^3x \left(\bar\psi\not{D}\psi + \frac{1}{N}\sigma_f\bar\psi\psi\right) , \tag{4.2.5}$$

[9] Away from the $N\to\infty$ limit, $\beta_{\lambda_6}(\lambda_6,\lambda)\neq 0$, but one finds fixed points with $\lambda_6 = \lambda_6^*(\lambda)$ [86].

where σ_f is the auxiliary Hubbard-Stratonovich field, and the quadratic term was dropped as appropriate in the UV limit. At large N, the model also possesses an exactly marginal coupling $g_6(\bar{\psi}\psi)^3 \sim g_6\sigma_f^3$. This extra coupling (which is mapped under the bose-fermi duality to the λ_6 coupling in the CS-boson theory) does not affect the quantities we will compute in this chapter, and we will neglect it below.

4.2.1 The "single-trace" operators

Free theories Let us first review the spectrum of "single-trace" operators in the free bosonic and fermionic $U(N)$ vector models. In the scalar model, the spectrum consists of a scalar operator

$$j_0 = \bar{\phi}\phi \tag{4.2.6}$$

with scaling dimension $\Delta = 1$, and a tower of exactly conserved currents $j_s \sim \bar{\phi}\partial^s\phi$ of all integer spins. To give the explicit form of these currents, as usual we introduce an auxiliary null vector z^μ, $z^\mu z_\mu = 0$, and define the index-free operators

$$j_s(x,z) = j_{\mu_1\cdots\mu_s} z^{\mu_1}\cdots z^{\mu_s}\,. \tag{4.2.7}$$

A generating function $J_b(x,z) = \sum_{s=0}^{\infty} j_s(x,z)$ of the higher-spin operators in the scalar theory is given by [172]

$$\begin{aligned} J_{\rm b} &= \bar{\phi}(x) f_{\rm b}(z\cdot\overleftarrow{\partial}, z\cdot\overrightarrow{\partial})\phi(x) = f_{\rm b}(\hat{\partial}_1,\hat{\partial}_2)\bar{\phi}(x_1)\phi(x_2)|_{x_1,x_2\to x}\,, \\ f_{\rm b}(u,v) &= e^{u-v}\cos(2\sqrt{uv})\,. \end{aligned} \tag{4.2.8}$$

In the first line, we have introduced a bilocal notation which will be useful below, and we defined the shorthand $\hat{\partial} \equiv z\cdot\partial$. One may restore the explicit indices on the currents by acting with the differential operator (1.4.2) in z-space. For instance, to compute the divergence of the current j_s, one can evaluate $\partial^\mu D^z_\mu j_s(x,z) \propto \partial^\mu j_{\mu\mu_2\cdots\mu_s} z^{\mu_2}\cdots z^{\mu_s}$. Using the free equation of motion $\partial^2\phi = 0$, one can explicitly check that the currents in (4.2.8) are conserved. Indeed, the condition $\partial^\mu D^z_\mu J_{\rm b} = 0$ turns into the differential equation

$$\left(\frac{1}{2}(\partial_u + \partial_v) + u\partial_u^2 + v\partial_v^2\right) f_{\rm b}(u,v) = 0\,, \tag{4.2.9}$$

which is seen to be satisfied by the generating function given above. Expanding (4.2.8) in powers of z, one may also deduce the following explicit expression for the currents

$$j_s^{\mathrm{b}}(x,z) = \sum_{k=0}^{s} \frac{(-1)^{k+s}}{s!} \begin{pmatrix} 2s \\ 2k \end{pmatrix} \hat{\partial}_1^k \hat{\partial}_2^{s-k} \bar{\phi}(x_1)\phi(x_2)|_{x_1,x_2\to x} \,. \tag{4.2.10}$$

Using the free scalar propagator

$$\langle \bar{\phi}(x)\phi(0)\rangle = \frac{1}{4\pi|x|}\,, \tag{4.2.11}$$

it is straightforward to derive the 2-point function normalization of the higher-spin operators in the free scalar theory. One finds

$$\begin{aligned} \langle j_s^{\mathrm{b}}(x,z) j_s^{\mathrm{b}}(0,z)\rangle &= N n_s \frac{(z\cdot x)^{2s}}{(x^2)^{2s+1}}\,, \\ n_s &= \frac{2^{4s-5}\Gamma\left(s+\frac{1}{2}\right)}{\pi^{5/2} s!}\,. \end{aligned} \tag{4.2.12}$$

Similarly, for the $\Delta = 1$ scalar we have

$$\langle j_0(x) j_0(0)\rangle = \frac{N}{16\pi^2 x^2} \equiv \frac{N n_0}{x^2}\,. \tag{4.2.13}$$

In the free fermionic $U(N)$ vector model, the single-trace operators consist of the parity odd scalar

$$\tilde{j}_0 = \bar{\psi}\psi \tag{4.2.14}$$

with $\Delta = 2$, and the conserved higher-spin currents $j_s \sim \bar{\psi}\gamma\partial^{s-1}\psi$, given explicitly by the generating function [85]

$$\begin{aligned} J_{\mathrm{f}} &= \bar{\psi}(x) z\cdot\gamma f_{\mathrm{f}}(z\cdot \overleftarrow{\partial}, z\cdot\overrightarrow{\partial})\psi(x) = f_{\mathrm{f}}(\hat{\partial}_1,\hat{\partial}_2)\bar{\psi}(x_1)\hat{\gamma}\psi(x_2)|_{x_1,x_2\to x}\,, \\ f_{\mathrm{f}}(u,v) &= e^{u-v}\frac{\sin(2\sqrt{uv})}{2\sqrt{uv}}\,. \end{aligned} \tag{4.2.15}$$

To check that these currents are conserved when ψ obeys the free equation of motion, one can verify that $\partial^\mu D_\mu^z J_{\mathrm{f}}(x,z) = 0$. This yields

$$\left(\frac{3}{2}(\partial_u + \partial_v) + u\partial_u^2 + v\partial_v^2\right) f_{\mathrm{f}}(u,v) = 0\,, \tag{4.2.16}$$

which is satisfied by the generating function in (4.2.15). By expanding in powers of z, one can also derive the following explicit form

$$j_s^{\rm f}(x,z) = \sum_{k=0}^{s-1} \frac{(-1)^{k+s+1}}{2s!} \begin{pmatrix} 2s \\ 2k+1 \end{pmatrix} \hat{\partial}_1^k \hat{\partial}_2^{s-k-1} \bar{\psi}(x_1)\hat{\gamma}\psi(x_2)|_{x_1,x_2\to x} \,. \tag{4.2.17}$$

Using the free fermion propagator

$$\langle \psi(x)\bar{\psi}(0)\rangle = \frac{1}{4\pi}\frac{\not{x}}{x^3} \,, \tag{4.2.18}$$

one finds that the currents in the free fermion theory (4.2.15), (4.2.17) have exactly the same 2-point normalization as the scalar ones

$$\langle j_s^{\rm f}(x,z) j_s^{\rm f}(0,z)\rangle = N n_s \frac{(z\cdot x)^{2s}}{(x^2)^{2s+1}} \,, \qquad n_s = \frac{2^{4s-5}\Gamma\left(s+\frac{1}{2}\right)}{\pi^{5/2} s!} \,. \tag{4.2.19}$$

For the parity odd scalar operator, one finds

$$\langle \tilde{j}_0(x)\tilde{j}_0(0)\rangle = \frac{N}{8\pi^2 x^4} \equiv \frac{N\tilde{n}_0}{x^4} \,. \tag{4.2.20}$$

In the calculations below, we will sometimes find it convenient to introduce explicit light-cone coordinates, with metric

$$ds^2 = 2dx^+dx^- + dx_3^2 \,. \tag{4.2.21}$$

When we do this, we will take the auxiliary null vector to be $z^\mu = \delta^\mu_-$, and so $j_s(x,z) = j^s_{--\dots-}$, $\hat{\partial} = \partial_-$ and $z\cdot x = x_-$.

Interacting theories When the Chern-Simons coupling is turned on, the higher-spin operators defined above should be made gauge invariant by replacing derivatives with covariant derivatives. The currents in the bosonic theories are then

$$\begin{aligned} J_{\rm b} &= \bar{\phi}(x) f_{\rm b}(\overleftarrow{\hat{D}}, \overrightarrow{\hat{D}})\phi(x) = \sum_{s=0}^{\infty} j_s^{\rm b}(x,z) \,, \\ j_s^{\rm b}(x,z) &= \sum_{k=0}^{s} \frac{(-1)^{k+s}}{s!} \begin{pmatrix} 2s \\ 2k \end{pmatrix} \hat{D}_1^k \hat{D}_2^{s-k} \bar{\phi}(x_1)\phi(x_2)|_{x_1,x_2\to x} \,, \end{aligned} \tag{4.2.22}$$

where $\hat{D} = z^\mu D_\mu$, and recall $\hat{D}\phi = \hat{\partial}\phi - i\hat{A}\phi$, $\hat{D}\bar{\phi} = \hat{\partial}\bar{\phi} + i\bar{\phi}\hat{A}$. Similarly, in the fermionic theories one has

$$
\begin{aligned}
J_{\mathrm{f}} &= \bar{\psi}(x)\hat{\gamma} f_{\mathrm{f}}(\overleftarrow{\hat{D}}, \overrightarrow{\hat{D}})\psi(x) = \sum_{s=1}^{\infty} j_s^{\mathrm{f}}(x,z)\,, \\
j_s^{\mathrm{f}}(x,z) &= \sum_{k=0}^{s} \frac{(-1)^{k+s+1}}{2s!} \begin{pmatrix} 2s \\ 2k+1 \end{pmatrix} \hat{D}_1^k \hat{D}_2^{s-k} \bar{\psi}(x_1)\hat{\gamma}\psi(x_2)|_{x_1,x_2\to x}\,.
\end{aligned}
\tag{4.2.23}
$$

Note that contracting with the null vector z^μ automatically projects the currents onto their symmetric traceless part. The higher-spin operators above, together with the scalars $j_0 = \bar{\phi}\phi$ and $\tilde{j}_0 = \bar{\psi}\psi$, exhaust the single-trace spectrum in the interacting theories as well [85,86]. Note that the CS equation of motion $\frac{k}{4\pi}\epsilon^{\mu\nu\rho}(F_{\nu\rho})^i{}_j = (J^\mu)^i{}_j$, where $(J_\mu)^i{}_j$ is the $U(N)$ current, implies that naive single-trace operators obtained by inserting factors of the field strength inside matter bilinears are in fact multi-trace.

In the interacting theory the higher-spin currents are no longer conserved, however the breaking is small at large N and implies that anomalous dimensions are generated starting at order $1/N$. The 2-point and 3-point functions of the bilinear operators can be fixed in the planar limit and to all orders in λ by using the weakly broken higher-spin symmetry [118] and explicit computations for low spins [113, 130].

In the CS-boson theory, one finds for the exact planar 2-point functions [113]

$$
\begin{aligned}
\langle j_s^{\mathrm{b}}(x,z) j_s^{\mathrm{b}}(0,z)\rangle &= N\frac{\sin(\pi\lambda)}{\pi\lambda} n_s \frac{(z\cdot x)^{2s}}{(x^2)^{2s+1}}\,, \\
\langle j_0(x) j_0(0)\rangle &= \frac{2N\tan(\frac{\pi\lambda}{2})}{\pi\lambda}\frac{n_0}{x^2}\,.
\end{aligned}
\tag{4.2.24}
$$

In terms of the parameters $\tilde{N}$ and $\tilde{\lambda}$ introduced in the analysis of [118], these read[10]

$$
\begin{aligned}
\langle j_s^{\mathrm{b}}(x,z) j_s^{\mathrm{b}}(0,z)\rangle &= \tilde{N}\langle j_s(x,z) j_s(0,z)\rangle_{\mathrm{sc}}\,, \\
\langle j_0(x) j_0(0)\rangle &= \tilde{N}(1+\tilde{\lambda}^2)\langle j_0(x) j_0(0)\rangle_{\mathrm{sc}}\,,
\end{aligned}
\tag{4.2.25}
$$

where the correlators on the right-hand side refer to the theory of a single real free scalar, and we used [113]

$$
\tilde{N} = 2N\frac{\sin(\pi\lambda)}{\pi\lambda}\,, \qquad \tilde{\lambda} = \tan(\frac{\pi\lambda}{2})\,. \tag{4.2.26}
$$

[10] Note that the scalar operator $j_0 = \bar{\phi}\phi$ has a different normalization from the one chosen in [118]. They are related by $j_0^{\mathrm{MZ}} = j_0/(1+\tilde{\lambda}^2)$. Similarly, in the fermionic theory we define $\tilde{j}_0 = \bar{\psi}\psi$, $\tilde{j}_0^{\mathrm{MZ}} = \tilde{j}_0/(1+\tilde{\lambda}^2)$.

The 3-point functions of operators of non-zero spin are fixed to be

$$\langle j^{\rm b}_{s_1} j^{\rm b}_{s_2} j^{\rm b}_{s_3}\rangle = \tilde N \left[\frac{1}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm sc} + \frac{\tilde\lambda^2}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm fer} + \frac{\tilde\lambda}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm odd}\right] , \qquad (4.2.27)$$

where the suffix 'sc' and 'fer' refer to the correlators in the (real) free scalar and free fermion theories, and the 'odd' term is a structure which breaks parity. It also breaks current conservation when s_1, s_2, s_3 do not satisfy the triangular inequality, as will be explained below. When one of the operators is the scalar $j_0 = \bar\phi\phi$, the 3-point functions read

$$\langle j^{\rm b}_{s_1} j^{\rm b}_{s_2} j_0\rangle = \tilde N \left[\langle j_{s_1} j_{s_2} j_0\rangle_{\rm sc} + \tilde\lambda \langle j_{s_1} j_{s_2} j_0\rangle_{\rm odd}\right] . \qquad (4.2.28)$$

Here $\langle j_{s_1} j_{s_2} j_0\rangle_{\rm odd}$ is a parity odd tensor structure that breaks the spin s_1 current conservation when $s_1 > s_2$. Similarly one can write down the expression for correlators involving two or three scalar operators: these are completely fixed by conformal invariance up to the overall constant, and do not play a role in the analysis of the higher-spin anomalous dimensions to order $1/N$.

In the CS-fermion theory (4.2.4), one finds the analogous results [130]

$$\begin{aligned} \langle j^{\rm f}_s(x,z) j^{\rm f}_s(0,z)\rangle &= \tilde N \frac{n_s (z\cdot x)^{2s}}{2(x^2)^{2s+1}} = \tilde N \langle j_s(x,z) j_s(0,z)\rangle_{\rm fer} , \\ \langle \tilde j_0(x) \tilde j_0(0)\rangle &= \tilde N (1+\tilde\lambda^2)\frac{\tilde n_0}{2x^4} = \tilde N(1+\tilde\lambda^2)\langle \tilde j_0(x)\tilde j_0(0)\rangle_{\rm fer} , \end{aligned} \qquad (4.2.29)$$

where the subscript 'fer' indicates correlators in the free theory of a single real fermion. The parameters $\tilde N$ and $\tilde\lambda$ are given in terms of N, λ by the same expressions as in (4.2.26). The 3-point functions are

$$\langle j^{\rm f}_{s_1} j^{\rm f}_{s_2} j^{\rm f}_{s_3}\rangle = \tilde N \left[\frac{1}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm fer} + \frac{\tilde\lambda^2}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm sc} + \frac{\tilde\lambda}{1+\tilde\lambda^2}\langle j_{s_1} j_{s_2} j_{s_3}\rangle_{\rm odd}\right] , \qquad (4.2.30)$$

and

$$\langle j^{\rm f}_{s_1} j^{\rm f}_{s_2} \tilde j_0\rangle = \tilde N \left[\langle j_{s_1} j_{s_2} \tilde j_0\rangle_{\rm fer} + \tilde\lambda \langle j_{s_1} j_{s_2} \tilde j_0\rangle_{\rm odd}\right] . \qquad (4.2.31)$$

The 'odd' structure in the above equation breaks current conservation on j_{s_1} when $s_1 > s_2$. It breaks parity, but note that since $\tilde j_0 = \bar\psi\psi$ is parity odd, this tensor structure is actually parity even.

Let us now discuss the critical models defined by (4.2.3) and (4.2.5). In the scalar theory, the auxiliary field σ_b replaces the scalar operator $\bar\phi\phi$, and in the IR it behaves as a scalar operator with scaling dimension $\Delta = 2 + O(1/N)$. To leading order at large N, its two-point function is essentially the inverse of the $\bar{\phi}\phi$ 2-point function (in momentum space), and reads

$$\langle\sigma_b(x)\sigma_b(0)\rangle = N\frac{4\pi\lambda}{\tan(\frac{\pi\lambda}{2})}\frac{1}{\pi^2x^4}\,. \tag{4.2.32}$$

Note that this result is valid to all orders in λ. Defining the operator $\tilde{j}_0^{\text{crit.bos.}} = \sigma_b/(4\pi\lambda)$, one finds

$$\langle\tilde{j}_0^{\text{crit.bos.}}(x)\tilde{j}_0^{\text{crit.bos.}}(0)\rangle = \frac{N}{4\pi\lambda\tan(\frac{\pi\lambda}{2})}\frac{1}{\pi^2x^4} = \tilde{N}(1+\frac{1}{\tilde{\lambda}^2})\frac{\tilde{n}_0}{2x^4}\,. \tag{4.2.33}$$

We see that this 2-point function precisely matches the $\tilde{j}_0$ 2-point function in the fermionic theory, eq. (4.2.29), under the duality map (4.1.15). To leading order at large N,the 2-point and 3-point functions involving operators with spin are unchanged in the critical theory compared to the CS-boson theory, and the agreement with the duality follows by comparing (4.2.27) and (4.2.30). The 3-point functions involving one (or more) scalars σ_b can be obtained from the ones in the CS-boson theory by attaching a σ_b line to the every scalar operator $\bar{\phi}\phi$, using the vertex in (4.2.3). In terms of $\tilde{j}_0^{\text{crit.bos.}} = \sigma_b/(4\pi\lambda)$, the corresponding 3-point functions are related to those of the CS-fermion theory (4.2.31) by the duality map (4.1.15). Note that the tensor structure $\langle j_{s_1}j_{s_2}\tilde{j}_0\rangle_{\text{odd}}$ in (4.2.31) corresponds to the correlators of the critical $O(N)$ model (Wilson-Fisher), which is recovered in the $\tilde{\lambda}_f\to\infty$ limit of the CS-fermion model (or $\tilde{\lambda}_b\to 0$ limit of the critical CS-boson model).

The discussion of the critical fermion model (4.2.5) goes similarly. The auxiliary field σ_f becomes a scalar primary with dimension $\Delta = 1+O(1/N)$ in the UV, and its 2-point function can be computed to be

$$\langle\sigma_f(x)\sigma_f(0)\rangle = N\frac{2\pi\lambda}{\tan(\frac{\pi\lambda}{2})}\frac{1}{\pi^2x^2}\,. \tag{4.2.34}$$

The duality with the CS-boson model can be verified by defining the operator $j_0^{\text{crit.fer.}} = \sigma_f/(4\pi\lambda)$, which has the 2-point function

$$\langle j_0^{\text{crit.fer.}}(x)j_0^{\text{crit.fer.}}(0)\rangle = \frac{N}{8\pi\lambda\tan(\frac{\pi\lambda}{2})}\frac{1}{\pi^2x^2} = \tilde{N}(1+\frac{1}{\tilde{\lambda}^2})\frac{n_0}{2x^2}\,. \tag{4.2.35}$$

This matches the CS-boson 2-point function (4.2.25) under the duality map (4.1.15). Similarly, the 3-point functions involving a scalar can be seen to map to those of the CS-boson theory. The tensor structure in (4.2.28) which breaks current conservation corresponds to the correlators of the critical Gross-Neveu model, which is recovered in the limit $\tilde{\lambda}_f = 0$ of the critical CS-fermion (or $\tilde{\lambda}_b\to\infty$ in the CS-boson model). Note that in this case there is an additional marginal parameter g_6 on both sides of the duality, as discussed earlier, and a corresponding duality map [130]. We will neglect this coupling throughout the chapter.

4.3 Analysis based on Slightly Broken Higher-Spin Symmetry

The theories we study have a tower of single-trace primary spin-s operators j_s which have scaling dimension $\Delta = s+1+O(1/N)$ and are nearly conserved currents [85,86,118]. Following the terminology introduced in [118], we call "quasi-boson" theory the CFT whose single trace spectrum include, in addition to the spin-s operators, a scalar j_0 with $\Delta = 1 + O(1/N)$; and "quasi-fermion" theory the CFT with a scalar $\tilde{j}_0$ of dimension $\Delta = 2 + O(1/N)$. The "regular" CS-boson theory or critical CS-fermion theory fall in the quasi-boson class, while the regular CS-fermion or critical CS-boson fall in the quasi-fermion class.

In [118], the quasi-bosonic and quasi-fermionic theories are defined in terms of two parameters: $\tilde{\lambda}$ and $\tilde{N}$. (In the quasi-bosonic theory there is an additional parameter $\tilde{\lambda}_6$ which we ignore here.) The parameter $\tilde{N}$ can be defined via the normalization of the spin 2 operator (the stress-tensor) two-point function, while $\tilde{\lambda}$ is defined via the spin 4 anomalous conservation relation:

$$\partial \cdot j_4 \sim \frac{\tilde{\lambda}}{\tilde{N}} \left(\partial_- \tilde{j}_0^{\mathrm{MZ}} j_2 - \frac{2}{5} \tilde{j}_0^{\mathrm{MZ}} \partial_- j_2 \right) \tag{4.3.1}$$

in the quasi-fermion case, and similarly in the quasi-boson case. Here $\sim$ denotes equality up to a $\tilde{\lambda}$-independent numerical coefficient, and j_0^{MZ} denotes the scalar in the normalizations used in [118], which differ from ours by $\tilde{j}_0^{\mathrm{MZ}} = \tilde{j}_0/(1+\tilde{\lambda}^2)$. With $\tilde{\lambda}$ so defined, [118] derive expressions for all two-point functions and three-point functions of single-trace primary operators j_s.

4.3.1 General form of current non-conservation

To derive the general expression for anomalous dimensions of spin s currents, we need an expression for the divergence of j_s. As argued in [85, 86, 118] the divergence of j_s (for $s > 0$) takes the following form:

$$\partial \cdot j_s \equiv \sum_{s_1,s_2} \partial \cdot j_s\Big|_{s_1,s_2} + \sum_{s_1,s_2,s_3} \partial \cdot j_s\Big|_{s_1,s_2,s_3} \tag{4.3.2}$$

$$= \sum_{s_1,s_2} \left(C_{s_1,s_2,s}(\tilde{\lambda}) \frac{1}{\tilde{N}} [j_{s_1}][j_{s_2}] \right) + \sum_{s_1,s_2,s_3} \left(C_{s_1,s_2,s_3,s}(\tilde{\lambda}) \frac{1}{\tilde{N}^2} [j_{s_1}][j_{s_2}][j_{s_3}] \right) , \tag{4.3.3}$$

where $[j_s]$ denotes j_s or any of its conformal descendants, and $C_{s_1,s_2,s}$ and $C_{s_1,s_2,s_3,s}$ are numerical coefficients that depend on s_1, s_2 (and s_3) and also $\tilde{\lambda}$. The "double-trace" operator $[j_{s_1}][j_{s_2}]$ appearing on the right-hand side can be fixed by conformal symmetry up to the overall normalization that can be absorbed in $C_{s_1,s_2,s}$, as we will work out explicitly below. Similarly, one could fix the structure of the "triple-trace" term. However, it is easy to see that this term does not affect the anomalous dimension of j_s to order $1/N$ or the planar 3-point functions, and we will ignore it below.

We can fix the $\tilde{\lambda}$-dependence of $C_{s_1,s_2,s}(\tilde{\lambda})$ by calculating the correlation function of both sides of equation (4.3.3) with j_{s_1} and j_{s_2}. To leading order at large N, the resulting correlator factorizes and we find

$$\langle j_{s_1} j_{s_2} \partial \cdot j_s \rangle \sim \frac{1}{\tilde{N}} C_{s_1,s_2,s}(\tilde{\lambda}) \langle j_{s_1} j_{s_1} \rangle \langle j_{s_2} j_{s_2} \rangle \, . \tag{4.3.4}$$

On the other hand, from the results of [118], we have, see (4.2.27), (4.2.28) and (4.2.31):

$$\begin{aligned} &\langle j_{s_1} j_{s_2} \partial \cdot j_s \rangle \sim \tilde{N} \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} \, , \\ &\langle j_{s_1} j_0 \partial \cdot j_s \rangle \sim \tilde{N} \tilde{\lambda} \, , \\ &\langle j_{s_1} \tilde{j}_0 \partial \cdot j_s \rangle \sim \tilde{N} \tilde{\lambda} \, , \end{aligned} \tag{4.3.5}$$

where $\sim$ means equality up to $\tilde{N}$- and $\tilde{\lambda}$-independent numerical coefficients, and this result follows from the fact that the current non-conservation can only arise from the parity violating terms in the 3-point functions (4.2.27), (4.2.28) and (4.2.31). We also know that, see eq. (4.2.25) and (4.2.29): [11]

$$\begin{aligned} &\langle j_{s_1} j_{s_1} \rangle \sim \tilde{N} \, , \qquad s_1 \neq 0 \, , \\ &\langle j_0 j_0 \rangle \sim \tilde{N}(1+\tilde{\lambda}^2) \, , \qquad \langle \tilde{j}_0 \tilde{j}_0 \rangle \sim \tilde{N}(1+\tilde{\lambda}^2) \, . \end{aligned} \tag{4.3.6}$$

[11] Recall that our normalization of j_0 and $\tilde{j}_0$ differ from the one used in [118], where $\langle j_0 j_0 \rangle, \langle \tilde{j}_0 \tilde{j}_0 \rangle \sim \tilde{N}(1+\tilde{\lambda}^2)^{-1}$.

Putting everything together, we find

$$C_{s_1,s_2,s}(\tilde{\lambda}) \sim \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} = \left(\tilde{\lambda} + \frac{1}{\tilde{\lambda}}\right)^{-1} , \tag{4.3.7}$$

which is valid both for $s_1, s_2 \neq 0$ and for the case when either one of s_1 or s_2 is zero (in the case $s_1 = s_2 = 0$, we have $C_{0,0,s} = 0$). This $\tilde{\lambda}$-dependence holds both in the quasi-boson and quasi-fermion theories.

Via radial quantization (or equivalently directly using conformal invariance in flat space), the form (4.3.3) for the divergence of j_s implies that the twist, $\tau_s = \Delta_s - s$ of j_s takes the form, to the leading order in $1/\tilde{N}$

$$\tau_s - 1 = \sum_{s_1 \neq 0} \left(C_{s_1,0,s}(\tilde{\lambda})\right)^2 \alpha_{s_1,0,s} \frac{n_{s_1} n_0 (1+\tilde{\lambda}^2)}{\tilde{N} n_s} + \sum_{s_1,s_2 \neq 0} \left(C_{s_1,s_2,s}(\tilde{\lambda})\right)^2 \alpha_{s_1,s_2,s} \frac{n_{s_1} n_{s_2}}{\tilde{N} n_s} \tag{4.3.8}$$

in the quasi-boson theory, and a similar expression in the quasi-fermion theory, with n_0 replaced by $\tilde{n}_0$. Here $\alpha_{s_1,s_2,s}$ and $\alpha_{s_1,0,s}$ are numerical coefficients that depend on the explicit form of the "double-trace" primaries on the right-hand side of (4.3.3), and $n_s, n_0, \tilde{n}_0$ are the 2-point normalization coefficients defined in (4.2.12), (4.2.13) and (4.2.20). Note that the triple-trace component of the RHS of equation (4.3.3) does not contribute to the anomalous dimension at the order $\frac{1}{\tilde{N}}$.

From (4.3.8), we see that to order $1/\tilde{N}$ the twists take the form:

$$\tau_s - 1 = \frac{1}{\tilde{N}} \left(a_s^{(F/B)} \frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2} + b_s^{(F/B)} \frac{\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2} \right) , \tag{4.3.9}$$

where the value of a_s and b_s depends on the spin s only. A priori, the values of a_s and b_s may be different for the quasi-Fermionic theory and the quasi-Bosonic theory, hence the superscripts F and B. Assuming the uniqueness of the parity violating terms in the 3-point functions of non-zero spin operators, one expects from the analysis of [118] that $b_s^B = b_s^F$. We will verify this explicitly from the calculations in section 4.4. Note that the result $b_s^B = b_s^F$ is in fact necessary for the bose/fermi duality to work; this is because the calculation of b_s, or equivalently of $C_{s_1,s_2,s}$ with $s_1, s_2 \neq 0$, is identical in the regular CS-boson and critical CS-boson (the planar 3-point functions of non-zero spin operators are unaffected by the Legendre transform), and similarly in regular CS-fermion and critical CS-fermion. We will also find by our explicit calculations that $a_s^B = a_s^F$; this result appears to be more surprising, as it is not required by the bose/fermi duality.

4.3.2 Constraining the divergence of j_s

The divergence of j_s must be a conformal primary to leading order in $1/N$. A straightforward argument for this is given in Appendix A of [118]. Another simple way of seeing this is based on conformal representation theory [85] – at leading order in $1/N$, the primary operator j_s has twist 1, and therefore heads a short representation $(\Delta, s) = (s+1, s)$ of the conformal group. When $1/N$ corrections are included, the primary j_s acquires an anomalous dimension and now heads a long representation of the conformal group. To transform a short representation $(s+1, s)$ to a long representation, we require additional states, which must transform amongst themselves as a long representation $(s+2, s-1)$ to leading order in $1/N$. This long representation is headed by a primary operator, which is the divergence of j_s.

We denote the contribution of double-trace operators involving j_{s_1} and j_{s_2} to the RHS of (4.3.3) by

$$\partial \cdot j_s \Big|_{s_1,s_2} = C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]\,, \tag{4.3.10}$$

where for convenience we have absorbed the factor of $1/\tilde{N}$ in (4.3.3) into $C_{s_1,s_2,s}$. Below we explicitly determine the unique allowed combination of descendants of j_{s_1} and j_{s_2} represented by $[j_{s_1}][j_{s_2}]$ on the LHS of (4.3.10) up to a single overall constant, C_{s_1,s_2,s_3} by demanding that $\partial \cdot j_s\Big|_{s_1,s_2}$ is annihilated by the generator of special conformal transformations K_μ to leading order in $1/N$.

For simplicity, in this subsection we assume the null polarization vector z^μ always to be δ^μ_-, so $j_s(x,z) = (j_s)_{---\dots} = j_s^{+++\cdots}$. We also use $(j_s)_\mu$ and $(j_s)_{\mu\nu}$ to denote $(j_s)_{\mu---\dots}$ and $(j_s)_{\mu\nu---\dots}$ respectively.

s_1 and s_2 nonzero

Let us first consider the case when both spins are nonzero: $s_i \neq 0$.

The scaling dimension of the LHS of Equation (4.3.10) is $\Delta = s+2$. We match the scaling dimension in the RHS of (4.3.10) by including $p = s+2-(s_1+1+s_2+1) = s-s_1-s_2$ derivatives in $[j_{s_1}][j_{s_2}]$. A general expression with p derivatives acting on j_{s_1} and j_{s_2}, is:

$$\sum_{n=0}^{p} c_n \partial^{\mu_1}\dots\partial^{\mu_n} j_{s_1}^{\alpha_1\dots\alpha_{s_1}} \partial^{\nu_1}\dots\partial^{\nu_{n-p}} j_{s_2}^{\beta_1\dots\beta_{s_2}}\,. \tag{4.3.11}$$

Here, we wrote all free indices explicitly. This expression is symmetric with respect to permutations

$\alpha_i \leftrightarrow \alpha_j$, $\beta_i \leftrightarrow \beta_j$, $\mu_i \leftrightarrow \mu_j$, and $\nu_i \leftrightarrow \nu_j$.

We must now contract each term in expression (4.3.11) with dimensionless tensors. These can come from the following lists:

$$\text{List 1:} \qquad \eta_{-\alpha_i},\ \eta_{-\beta_i},\ \eta_{-\mu_i},\ \eta_{-\nu_i} \tag{4.3.12}$$

$$\text{List 2:} \qquad \epsilon_{\alpha_i\beta_j-},\ \epsilon_{\alpha_i\mu_j-},\ \epsilon_{\mu_i\beta_j-},\ \epsilon_{\alpha_i\nu_j-}, \epsilon_{\nu_i\beta_j-},\ \epsilon_{\mu_i\nu_j-} \tag{4.3.13}$$

$$\text{List 3:} \qquad \eta^{\mu_i\mu_j},\ \eta^{\nu_i\nu_j}, \ldots \tag{4.3.14}$$

$$\text{List 4:} \qquad \epsilon^{\alpha_i\beta_j\mu_k},\ \epsilon^{\alpha_i\beta_j\nu_k}, \ldots \tag{4.3.15}$$

Let us contract equation (4.8.23) with n_1 tensors from List 1, n_2 tensors from List 2, n_3 tensors from List 3 and n_4 tensors from list 4.

Because the total spin of $[j_{s_1}][j_{s_2}]$ must be $s-1$, we require $n_1+n_2 = s-1$. (Recall that we take all free indices in $\partial \cdot j_s$ to be in the $-$ direction, so the spin is simply the number of lower $-$ indices.) The total number of free indices in (4.3.11) is $p + s_1 + s_2 = s$; each tensor from List 1 removes one free index, each tensor from List 2 or 3 removes two free indices, and each tensor from List 4 removes 3 indices, so we also require $n_1 + 2n_2 + 2n_3 + 3n_4 = s$. This implies $n_2 + 2n_3 + 3n_4 = 1$, which then implies $n_2 = 1$, $n_1 = s-2$ (and $n_3 = n_4 = 0$).

Hence we require $s-2$ tensors from List 1 and 1 tensor from List 2. Choosing the tensor from List 2 automatically fixes which tensors from List 1 we need to use. Note that the resulting operators always involve the ϵ-tensor, illustrating the fact that the breaking of current conservation in 3-point functions arises from parity violating terms.

Contracting each of the six tensors in List 2 with equation (4.3.11) yields:

$$\begin{aligned} \partial \cdot j_s\Big|_{s_1,s_2} = \sum_{n=0}^{p} \epsilon_{\mu\nu-}\Big(& a_n \partial_-^n j_{s_1}^\mu \partial_-^{p-n} j_{s_2}^\nu + b_n \partial_-^{n-1}\partial^\nu j_{s_1}^\mu \partial_-^{p-n} j_{s_2} + \\ & + c_n \partial_-^{n-1}\partial^\mu j_{s_1} \partial_-^{p-n} j_{s_2}^\nu + d_n \partial_-^n j_{s_1}^\mu \partial_-^{p-n-1}\partial^\nu j_{s_2} \\ & + e_n \partial_-^n j_{s_1} \partial_\mu \partial_-^{p-n-1} j_{s_2}^\nu + f_n \partial_-^{n-1}\partial^\mu j_{s_1} \partial_-^{p-n-1}\partial^\nu j_{s_2}\Big) . \end{aligned} \tag{4.3.16}$$

However the six types of terms in (4.3.16) are not linearly independent, as one can check by explicitly writing out the sums over μ and ν. We can choose a basis of three linearly independent terms and write the most general form for $[j_{s_1}][j_{s_2}]$ with correct scaling dimension and spin as:

$$\partial \cdot j_s \Big|_{s_1,s_2} = \sum_{n=0}^{p} \epsilon_{\mu\nu-} \left(a_n \partial_-^n j_{s_1}^\mu \partial_-^{p-n} j_{s_2}^\nu + b_n \partial_-^{n-1} \partial^\nu j_{s_1}^\mu \partial_-^{p-n} j_{s_2} + e_n \partial_-^n j_{s_1} \partial^\mu \partial_-^{p-n-1} j_{s_2}^\nu \right) , \quad (4.3.17)$$

where $b_0 = 0$ and $e_p = 0$. We also must require that, when we interchange the spins $s_1 \leftrightarrow s_2$, $b_n \leftrightarrow -e_{p-n}$ and $a_n \leftrightarrow -a_{p-n}$.

Next we apply the constraint that the expression be a conformal primary. Acting on this expression with K_3 and K_+, as illustrated in Appendix 4.7, we are able to determine a_n, b_n and e_n up to one undetermined constant $C_{s_1,s_2,s}$.

$$\begin{aligned} a_n &= C_{s_1,s_2,s} \frac{(-1)^{n+1} \left(s_1(n-s+s_1-s_2) + s_2(n+2s_1)(-1)^{s+s_1+s_2}\right)}{(s-s_1-s_2)(s+s_1+s_2)} \binom{s-s_1-s_2}{n} \binom{s+s_1+s_2}{n+2s_1}, \\ b_n &= C_{s_1,s_2,s}(-1)^n \binom{s-s_1-s_2-1}{n-1} \binom{s+s_1+s_2}{n+2s_1}, \\ e_n &= C_{s_1,s_2,s}(-1)^{s-s_1-s_2+n+1} \binom{s-s_1-s_2-1}{n} \binom{s+s_1+s_2}{n+2s_1}. \end{aligned} \quad (4.3.18)$$

This formula is also valid if $s_1 = s_2$.

$s_2 = 0$, Quasi-Fermionic

Let us next consider the contribution to the non-conservation equation from j_{s_1} and $\tilde{j}_0$ in the quasi-fermionic theory.

In this case, $[j_{s_1}][\tilde{j}_0]$ requires $p = s - s_1 - 1$ derivatives, $s - 1$ tensors from List 1, and no tensors from the other lists. Hence, we have the following expression:

$$\partial \cdot j_s \Big|_{s_1,0} = \sum_{m=0}^{p} c_m \partial_-^m j_{s_1} \partial_-^{p-m} \tilde{j}_0 \, . \quad (4.3.19)$$

Now we apply the constraint that the expression be a conformal primary, as illustrated in appendix 4.7. We find that

$$c_m = \frac{-(m-p-1)(m-p-2)}{m(m+2s_1)} c_{m-1}, \quad (4.3.20)$$

which can be solved to give

$$c_m = (-1)^m \binom{s-s_1}{m} \binom{s+s_1-1}{m+2s_1} C_{s_1,\tilde{0},s}. \quad (4.3.21)$$

One can check that this form agrees with the various divergences calculated explicitly in [85], as well as those calculated in [33, 84].

$s_2 = 0$, Quasi-Bosonic

We now consider the contribution from j_{s_1} with $s_1 \neq 0$ and j_0 to the divergence of j_s in the quasi-bosonic theory, where the scalar primary has scaling dimension 1. Here, the analysis of section 4.3.2 applies again, but there are only three relevant tensors in List 2, of which only two are independent, yielding:

$$\partial \cdot j_s\Big|_{s_1,s_2} = \sum_{n=0}^{p} \epsilon_{\mu\nu-} \left(b_n \partial_-^{n-1} \partial^\nu j_{s_1}^\mu \partial_-^{p-n} j_0 + f_n \partial_-^n j_{s_1}^\mu \partial^\nu \partial_-^{p-n-1} j_0 \right) , \tag{4.3.22}$$

where $p = s - s_1$.

Requiring that the expression is annihilated by K_δ gives:

$$b_n = C_{s_1,0,s} (-1)^n \binom{s+s_1}{n+2s_1} \binom{s-s_1-1}{n-1} , \tag{4.3.23}$$

$$f_n = C_{s_1,0,s} (-1)^n \frac{s_1}{s+s_1} \binom{s+s_1}{n+2s_1} \binom{s-s_1-1}{n} , \tag{4.3.24}$$

with $f_p = 0$ and $b_0 = 0$. We checked that this matches the divergence of j_4 calculated explicitly in [86].

4.3.3 The anomalous dimensions

We can now use the explicit form of the non-conservation equation to determine the anomalous dimensions of the higher-spin operators to order $1/N$. Using the index-free notation in terms of the null polarization vector z, we can write the non-conservation equation as

$$\partial_\mu D_z^\mu j_s(x, z) = \mathcal{K}_{s-1}(x, z) , \tag{4.3.25}$$

where D_z^μ is the operator defined in (1.4.2). Recall that the two-point function a spin s primary operator of dimension Δ_s is fixed by conformal invariance to be (1.4.8):

$$\langle j_s(x_1, z_1) j_s(x_2, z_2) \rangle = \mathcal{N}_s \frac{\left(\frac{z_1 \cdot x_{12} z_2 \cdot x_{12}}{x_{12}^2} - \frac{1}{2} z_1 \cdot z_2 \right)^s}{(x_{12}^2)^{\Delta_s}} , \tag{4.3.26}$$

where z_1, z_2 are two polarization vectors. Writing $\Delta_s = s+1+\gamma_s$ and taking the divergence on x_1 and x_2 on both sides of this equation, one may derive the following formula for the anomalous dimension, valid to leading order in the breaking parameter [22, 23]

$$\gamma_s = -\frac{1}{s^2(s^2-\frac{1}{4})}\frac{(z\cdot x)^2\langle \mathcal{K}_{s-1}(x,z)\mathcal{K}_{s-1}(0,z)\rangle_0}{\langle j_s(x,z)j_s(0,z)\rangle_0}\,, \tag{4.3.27}$$

where the subscript '0' means that the correlators are computed in the "unbroken" theory (in our case, to leading order at large N).

Let us define

$$\begin{aligned}
&\mathcal{K}^{(a)}_{s-1} = \sum_{s_1} C_{s_1,0,s}[j_{s_1}][j_0]\,, \qquad \tilde{\mathcal{K}}^{(a)}_{s-1} = \sum_{s_1} C_{s_1,\tilde{0},s}[j_{s_1}][\tilde{j}_0]\,,\\
&\mathcal{K}^{(b)}_{s-1} = \sum_{s_1,s_2\neq 0} C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]\,,
\end{aligned} \tag{4.3.28}$$

so that in the quasi-boson theory we have $\partial\cdot j_s = \mathcal{K}^{(a)}_{s-1}+\mathcal{K}^{(b)}_{s-1}$, and in the quasi-fermion $\partial\cdot j_s = \tilde{\mathcal{K}}^{(a)}_{s-1}+\mathcal{K}^{(b)}_{s-1}$. Using the explicit form of these double-trace operators given in (4.3.22), (4.3.19) and (4.3.17), and computing their two-point functions using (4.2.25) and (4.2.29),[12] we find

$$\begin{aligned}
\frac{(z\cdot x)^2\langle \mathcal{K}^{(a)}_{s-1}\mathcal{K}^{(a)}_{s-1}\rangle_0}{\langle j_s j_s\rangle_0} &= \tilde{N}(1+\tilde{\lambda}^2)\sum_{s_1}\frac{s^3((s-1)!)^2}{128\pi^2(s^2-s_1^2)(s_1!)^2}(C_{s_1,0,s})^2\,,\\
\frac{(z\cdot x)^2\langle \tilde{\mathcal{K}}^{(a)}_{s-1}\tilde{\mathcal{K}}^{(a)}_{s-1}\rangle_0}{\langle j_s j_s\rangle_0} &= -\tilde{N}(1+\tilde{\lambda}^2)\sum_{s_1}\frac{(s!)^2(s-s_1)}{128\pi^2 s(s_1!)^2(s+s_1)}(C_{s_1,\tilde{0},s})^2\,,\\
\frac{(z\cdot x)^2\langle \mathcal{K}^{(b)}_{s-1}\mathcal{K}^{(b)}_{s-1}\rangle_0}{\langle j_s j_s\rangle_0} &= \tilde{N}\sum_{s_1,s_2\neq 0}\frac{s^3((s-1)!)^2(s-s_1-s_2-1)!(s+s_1+s_2-1)!}{64\pi^2(s_1!)^2(s_2!)^2(s+s_1-s_2)!(s-s_1+s_2)!}(C_{s_1,s_2,s})^2\,.
\end{aligned} \tag{4.3.29}$$

A direct calculation using the equations of motion, described in the next section, and the result (4.3.7), allow us to fix the undetermined "structure constants" to be

$$C^B_{s_1,s_2,s} = -C^F_{s_1,s_2,s} = \frac{1}{\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}\cdot\begin{cases}\frac{32i(s+s_1-s_2)!(s-s_1+s_2)!s_1!s_2!}{(s+s_1+s_2-1)!(s-s_1-s_2-1)!s!} & s_1+s_2 = s-2, s-4,\dots\,,\\ 0 & \text{otherwise}\end{cases} \tag{4.3.30}$$

$$C_{s_1,0,s} = \frac{1}{\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}\cdot\begin{cases}\frac{32i(s^2-s_1^2)s_1!}{s!} & s_1 = s-2, s-4,\dots\,,\\ 0 & \text{otherwise}\end{cases} \tag{4.3.31}$$

[12] To compute the two-point functions of currents with one "open" index, one may take derivatives of (1.4.8) with respect to the polarization vectors.

$$C_{s_1,\tilde{0},s} = \frac{1}{\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}\cdot\begin{cases}\frac{32(s+s_1)s_1!}{(s-1)!} & s_1 = s-2, s-4, \dots\,, \\ 0 & \text{otherwise}\end{cases} \tag{4.3.32}$$

Plugging these into (4.3.29) and using the formula (4.3.27), we find that the anomalous dimensions take the form (4.3.9), with $a_s^B = a_s^F = a_s$ and $b_s^B = b_s^F = b_s$ given by

$$a_s = \sum_{s_1=s-2,s-4,\dots} \frac{32(s^2-s_1^2)}{\pi^2 s\,(4s^2-1)} = \begin{cases}\frac{16(s-2)}{3\pi^2(2s-1)}\,, & s \ \text{ even}\,, \\ \frac{32(s^2-1)}{3\pi^2(4s^2-1)}\,, & s \ \text{ odd}\,,\end{cases} \tag{4.3.33}$$

and

$$\begin{aligned} b_s &= \sum_{s_1+s_2=s-2,s-4,\dots} \frac{64(s+s_1-s_2)!(s-s_1+s_2)!}{\pi^2 s\,(4s^2-1)\,(s-s_1-s_2-1)!(s+s_1+s_2-1)!} \\ &= \begin{cases}\frac{2}{3\pi^2}\left(3\sum_{n=1}^{s}\frac{1}{n-1/2}+\frac{-38s^4+24s^3+34s^2-24s-32}{(4s^2-1)(s^2-1)}\right)\,, & s \ \text{ even}\,, \\ \frac{2}{3\pi^2}\left(3\sum_{n=1}^{s}\frac{1}{n-1/2}+\frac{-38s^2+20}{4s^2-1}\right)\,, & s \ \text{ odd}\,.\end{cases} \end{aligned} \tag{4.3.34}$$

Let us note here that it is straightforward to adapt the above results to the case of $O(N)$ gauge group: one simply drops all the odd-spins from the sum. Doing this, we find

$$\begin{aligned} a_s^{O(N)} &= \frac{16(s-2)}{3\pi^2(2s-1)}\,, \\ b_s^{O(N)} &= \frac{4}{\pi^2}\left(-2\sum_{n=1}^{\frac{s}{2}-1}\frac{1}{n-\frac{1}{2}}+\frac{9}{4}\sum_{n=1}^{s-1}\frac{1}{n-\frac{1}{2}}-\frac{59s^4+18s^3-4s^2+54s+35}{6(4s^2-1)(s^2-1)}\right)\,, \end{aligned} \tag{4.3.35}$$

and the anomalous dimensions take the same form as in (4.3.9), with $\tilde{N}^{O(N)} = N(1+O(\lambda^2))$ and $\tilde{\lambda}^{O(N)} = \frac{\pi}{2}\lambda + O(\lambda^3)$. Note that $b_s^{O(N)}$ vanishes for $s=2$ and also for $s=4$, because in the $O(N)$ case the divergence of j_4 can only take the form (4.3.1).

4.4 Current non-conservation from classical equations of motion

4.4.1 CS-boson

The generating function $J_{\rm b}(x,z)$ of the higher-spin operators in the CS-boson theory was given in (4.2.22). Since we will be working to leading order in $1/k$, it will be sufficient to expand the generating function to linear order in the gauge field. We note that

$$
\begin{aligned}
(\hat{\partial} - i\hat{A})^n \phi &= \hat{\partial}^n \phi - i \sum_{k=0}^{n-1} \hat{\partial}^k \hat{A} \hat{\partial}^{n-1-k} \phi + O(A^2) \\
&= \hat{\partial}^n \phi - i \sum_{k=0}^{n-1} (\hat{\partial}_x + \hat{\partial}_y)^k \hat{\partial}_x^{n-1-k} \hat{A}(y)\phi(x)|_{y\to x} + O(A^2) \\
&= \hat{\partial}^n \phi - i \frac{(\hat{\partial}_x + \hat{\partial}_y)^n - \hat{\partial}_x^n}{\hat{\partial}_y} \hat{A}(y)\phi(x)|_{y\to x} + O(A^2)\,.
\end{aligned}
\tag{4.4.1}
$$

Since this expression involves the same power n everywhere, we can extend this formula to any function of $\hat{D}$ acting on ϕ:

$$
F(\hat{\partial} - i\hat{A})\phi = F(\hat{\partial})\phi - i \frac{F(\hat{\partial}_x + \hat{\partial}_y) - F(\hat{\partial}_x)}{\hat{\partial}_y} \hat{A}(y)\phi(x)|_{y\to x} + O(A^2)\,. \tag{4.4.2}
$$

A similar result applies when a power of the covariant derivative acts on $\bar{\phi}$. Therefore, to linear order in the gauge field, the generating function of the higher-spin operators is

$$
\begin{aligned}
&J_{\mathrm{b}} = f_{\mathrm{b}}(\hat{\partial}_1, \hat{\partial}_2)\bar{\phi}(x_1)\phi(x_2) + ig(\hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3)\bar{\phi}(x_1)\hat{A}(x_3)\phi(x_2)\,, \\
&g(u,v,w) = \frac{f_{\mathrm{b}}(u+w,v) - f_{\mathrm{b}}(u,v+w)}{w}\,, \qquad f_{\mathrm{b}}(u,v) = e^{u-v}\cos(2\sqrt{uv})\,,
\end{aligned}
\tag{4.4.3}
$$

where in the first line it is understood that after taking the derivatives all points are set to x.

To calculate the divergence of the spin s operators, we should evaluate

$$
\partial_\mu D_z^\mu J_{\mathrm{b}}(x,z) \equiv \partial \cdot J_{\mathrm{b}}\,. \tag{4.4.4}
$$

When the operator $\partial_\mu D_z^\mu$ acts on the A-independent piece of J_{b}, one gets [33, 84]

$$
\begin{aligned}
&\partial_\mu D_z^\mu f_{\mathrm{b}}(\hat{\partial}_1, \hat{\partial}_2)\bar{\phi}(x_1)\phi(x_2) = \left[h(\hat{\partial}_1, \hat{\partial}_2)\partial_1^2 + \tilde{h}(\hat{\partial}_1, \hat{\partial}_2)\partial_2^2\right]\bar{\phi}(x_1)\phi(x_2)\,, \\
&h(u,v) = (\frac{1}{2}\partial_u + \frac{u-v}{2}\partial_u^2 + v\partial_{uv})f(u,v)\,, \\
&\tilde{h}(u,v) = (\frac{1}{2}\partial_v + \frac{v-u}{2}\partial_v^2 + u\partial_{uv})f(u,v)\,.
\end{aligned}
\tag{4.4.5}
$$

In the interacting theory, the equations of motion to linear order in A are

$$\begin{aligned} \partial^2\phi &= i(\partial\cdot A)\phi + 2iA\cdot\partial\phi\,, \\ \partial^2\bar\phi &= -i\bar\phi(\partial\cdot A) - 2i(\partial\bar\phi)\cdot A\,. \end{aligned} \tag{4.4.6}$$

Using this into (4.4.5), and combining with the term that arises when $\partial^\mu D_z^\mu$ acts on the piece of $J_{\rm b}$ linear in A (where one can just use the free equation of motion $\partial^2\phi = 0$), one should find a gauge invariant result. We have explicitly checked that all the terms involving $\hat A$ indeed cancel out, and one is left with terms involving only the field strength $F = dA$. The final result takes the form

$$\partial\cdot J_{\rm b} = \Big[k_1(\hat\partial_1,\hat\partial_2,\hat\partial_3)\partial_1^\mu + k_2(\hat\partial_1,\hat\partial_2,\hat\partial_3)\partial_2^\mu + k_3(\hat\partial_1,\hat\partial_2,\hat\partial_3)\partial_3^\mu\Big]\,\bar\phi(x_1)(iF_{\mu\rho}(x_3)z^\rho)\phi(x_2)\,, \tag{4.4.7}$$

where

$$\begin{aligned} k_1(u,v,w) &= \frac{2}{w}h(u+w,v) - \frac{1}{w}\big(\frac{1}{2} - (v+w)\partial_u + v\partial_v + w\partial_w\big)g(u,v,w)\,, \\ k_2(u,v,w) &= -\frac{2}{w}\tilde h(u,v+w) - \frac{1}{w}\big(\frac{1}{2} + u\partial_u - (u+w)\partial_v + w\partial_w\big)g(u,v,w)\,, \\ k_3(u,v,w) &= \frac{1}{w}(h(u+w,v) - \tilde h(u,v+w)) - \frac{1}{w}\big(\frac{1}{2} + u\partial_u + v\partial_v - (u+v)\partial_w\big)g(u,v,w)\,. \end{aligned} \tag{4.4.8}$$

We can now use the equation of motion for A_μ, which reads (to linear order in A)

$$\frac{k}{4\pi}\epsilon^{\mu\nu\rho}(F_{\nu\rho})^i{}_j = (\partial^\mu\bar\phi_j)\phi^i - \bar\phi_j\partial^\mu\phi^i\,, \tag{4.4.9}$$

or equivalently

$$(F_{\mu\rho})^i{}_j = \frac{2\pi}{k}\epsilon_{\mu\rho\nu}\left((\partial^\nu\bar\phi_j)\phi^i - \bar\phi_j\partial^\nu\phi^i\right)\,. \tag{4.4.10}$$

After plugging this into (4.4.7), we get

$$\partial\cdot J_{\rm b} = -\frac{2\pi i}{k}\epsilon_{\mu\nu\rho}z^\rho\left[k_1\partial_1^\mu + k_2\partial_2^\mu + k_3(\partial_3^\mu + \partial_4^\mu)\right](\partial_3^\nu - \partial_4^\nu)\bar\phi_i(x_1)\phi^i(x_4)\bar\phi_j(x_3)\phi^j(x_2)\,. \tag{4.4.11}$$

Note that we had to point-split $\partial_3 \to \partial_3 + \partial_4$. To make contact with the analysis of the previous section, one should express this as a sum of double-trace primaries. Note that the scalar bilocals with derivatives acting on them can be expressed as linear combinations of the higher-spin operators and their derivatives. Doing this, one finds precisely the decomposition derived in the previous section

$$\partial \cdot J_{\rm b} = \mathcal{K}^{(a)} + \mathcal{K}^{(b)}\,, \tag{4.4.12}$$

where $\mathcal{K}^{(a)} \sim \sum_{s_1} C_{s_1,0,s}[j_{s_1}][j_0]$ and $\mathcal{K}^{(b)} \sim \sum_{s_1,s_2} C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]$ are the double-trace operators given respectively in (4.3.22) and (4.3.17), and the $C_{s_1,s_2,s}$ coefficients are determined to be

$$C_{s_1,s_2,s} = +\frac{2\pi i\lambda}{N}\frac{4(s+s_1-s_2)!(s-s_2+s_1)!}{(s+s_1+s_2-1)!(s-s_1-s_2-1)!}\frac{s_1!s_2!}{s!}\,, \qquad s_1+s_2 = s-2j\,,\ j>0\,, \tag{4.4.13}$$

and

$$C_{s_1,0,s} = 4\frac{2\pi i\lambda}{N}\frac{(s^2-s_1^2)s_1!}{s!}\,, \qquad s_1 = s-2j\,,\ j>0\,. \tag{4.4.14}$$

Extending these to all orders in λ by sending $\lambda \to \frac{2}{\pi}\tilde{\lambda}/(1+\tilde{\lambda}^2)$ and $N \to \tilde{N}/2$, one obtains the results quoted in (4.3.30) and (4.3.31).

Note that from the form (4.4.11) of the divergence, it is also straightforward to compute the anomalous dimensions by directly using Wick contractions of ϕ and the master formula (4.3.27). In fact, this allows to obtain the anomalous dimensions to order λ^2 and exactly in N. We find for $s = 1,2,3,\ldots$:

$$\gamma_s = \frac{\lambda^2}{N}\left\{0,0,\frac{32}{105}+\frac{8}{105N},\frac{12}{35}+\frac{4}{105N},\frac{1504}{3465}+\frac{24}{385N},\frac{4192}{9009}+\frac{32}{693N},\ldots\right\}+O(\lambda^3)\,. \tag{4.4.15}$$

This takes the form

$$\gamma_s = \frac{\pi^2\lambda^2}{2N}(a_s+b_s)+\frac{\lambda^2}{N^2}\gamma_s^{(2)}+O(\lambda^3)\,, \tag{4.4.16}$$

where a_s and b_s are given in (4.3.33) and (4.3.34), and the coefficients $\gamma_s^{(2)}$ at order λ^2/N^2 can be found to be:

$$\gamma_s^{(2)} = \begin{cases} 2\left(H_{s-\frac{5}{2}} - H_{\frac{s-3}{2}}\right) - \frac{4(s-2)\left(4s^2-4s+3\right)}{3(s-1)(2s-3)(2s-1)}\,, & s \ \text{even}\,,\\ 2\left(H_{s-\frac{3}{2}} - H_{\frac{s}{2}-1}\right) - \frac{2(s-1)\left(8s^2+8s+3\right)}{3s(2s-1)(2s+1)}\,, & s \ \text{odd}\,,\end{cases} \tag{4.4.17}$$

where H_n is the harmonic number. We note that the dimensions of even spin currents differ by a simple fraction from that of the odd spin ones. We also observe that, unlike the order $1/N$ term, these coefficients do not display logarithmic behavior at large spin.

Critical boson

Let us now study the critical boson theory obtained by adding the $(\bar{\phi}\phi)^2$ interaction and flowing to the IR. As reviewed earlier, the $1/N$ expansion of the CFT can be developed using the action (4.2.3). In the IR, σ_b becomes a scalar primary with $\Delta = 2 + O(1/N)$, and the σ_b equation of motion formally removes $\bar{\phi}\phi$ from the spectrum.

It is evident that the equations of motion and hence the divergence of the higher-spin currents will be modified due to the interaction with σ_b (the form of the currents themselves stay the same as in (4.2.22)). Working to linear order in the gauge field, the equations of motion are modified to

$$\begin{aligned} \partial^2\phi &= i(\partial\cdot A)\phi + 2iA\cdot\partial\phi + \frac{1}{N}\sigma_b\phi\,, \\ \partial^2\bar{\phi} &= -i\bar{\phi}(\partial\cdot A) - 2i(\partial\bar{\phi})\cdot A + \frac{1}{N}\sigma_b\bar{\phi}\,. \end{aligned} \tag{4.4.18}$$

Consequently, when computing the divergence of $J_{\rm b}$, the descendant acquires an additional term linear in σ, and to leading order in $1/N$ and $1/k$ is given by

$$\begin{aligned} \partial\cdot J_{\rm b} &= \mathcal{K}_{\rm reg.CS-bos.} + \mathcal{K}_{\rm crit.bos.}\,, \\ \mathcal{K}_{\rm crit.bos.} &= \frac{1}{N}(h(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2) + \tilde{h}(\hat{\partial}_1,\hat{\partial}_2+\hat{\partial}_3))\bar{\phi}(x_1)\phi(x_2)\sigma_b(x_3)\,, \end{aligned} \tag{4.4.19}$$

where $h(u,v)$ and $\tilde{h}(u,v)$ were defined in (4.4.5), and $\mathcal{K}_{\rm reg.CS-bos.}$ is the descendent computed in the previous section, given in (4.4.7). To get the final result for the divergence, one should still impose that $\bar{\phi}\phi = 0$ as a consequence of the equation of motion for σ_b. This means that we should drop the term $\mathcal{K}^{(a)} \sim C_{s_1,0,s}\sum_{s_1}[j_0][j_{s_1}]$ from $\mathcal{K}_{\rm reg.CS-bos.}$. Also writing $\mathcal{K}_{\rm crit.bos.}$ in terms of primaries and dropping all the $\bar{\phi}\phi$ terms, one finds the final result

$$\begin{aligned} &\partial\cdot J_{\rm b} = \tilde{\mathcal{K}}^{(a)} + \mathcal{K}^{(b)}\,, \\ &\tilde{\mathcal{K}}^{(a)} = \sum_{s_1=s-2,s-4,\ldots} C_{s_1,\tilde{0},s}\sum_{m=0}^{s-s_1-1}(-1)^m\binom{s-s_1}{m}\binom{s+s_1-1}{m+2s_1}\hat{\partial}^m j_{s_1}\hat{\partial}^{s-s_1-1-m}\sigma_b\,, \\ &C_{s_1,\tilde{0},s} = \frac{2(s+s_1)s_1!}{(s-1)!}\frac{1}{N}\,. \end{aligned} \tag{4.4.20}$$

where $\mathcal{K}^{(b)} \sim \sum_{s_1,s_2} C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]$ remains the same as in the regular CS-boson theory of the previous section. Note that $\tilde{\mathcal{K}}^{(a)}$ has precisely the form predicted by conformal symmetry for the quasi-fermion theory, eq. (4.3.19). Defining $\sigma_b = 4\pi\lambda\tilde{j}_0^{\rm crit.bos.}$, this result can be seen to be precisely related by the bose/fermi duality to the divergence in the CS-fermion theory, which we compute in the next section.

Extending the above result to all orders in $\tilde{\lambda}$ by using the arguments in section 4.3.1, one can deduce that the anomalous dimensions in the critical boson model coupled to Chern-Simons are

$$\gamma_s^{\text{crit.}} = \frac{1}{\tilde{N}}\left(\frac{1}{1+\tilde{\lambda}^2}a_s + \frac{\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2}b_s\right). \tag{4.4.21}$$

4.4.2 CS-fermion

The generating function of the higher-spin operators in the CS-fermion theory was given in (4.2.23). Linearizing it in the gauge field, as described in the boson case above, we find

$$\begin{aligned} J_{\rm f} &= f_{\rm f}(\hat{\partial}_1,\hat{\partial}_2)\bar{\psi}(x_1)\hat{\gamma}\psi(x_2) + ig(\hat{\partial}_1,\hat{\partial}_2,\hat{\partial}_3)\bar{\psi}(x_1)\hat{\gamma}\hat{A}(x_3)\psi(x_2)\,, \\ g(u,v,w) &= \frac{f_{\rm f}(u+w,v)-f_{\rm f}(u,v+w)}{w}\,, \qquad f_{\rm f}(u,v) = \frac{e^{u-v}\sin(2\sqrt{uv})}{2\sqrt{uv}}\,. \end{aligned} \tag{4.4.22}$$

The equations of motion to linear order in the gauge field are

$$\begin{aligned} \partial\!\!\!/\psi &= iA\!\!\!/\psi\,, \qquad \partial_\mu\bar{\psi}\gamma^\mu = -i\bar{\psi}A\!\!\!/\,, \\ \partial^2\psi &= \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\psi + i(\partial\cdot A)\psi + 2iA\cdot\partial\psi\,, \\ \partial^2\bar{\psi} &= \frac{i}{2}\bar{\psi}\gamma^{\mu\nu}F_{\mu\nu} - i\bar{\psi}\partial\cdot A - 2i(\partial^\mu\bar{\psi})A_\mu\,. \end{aligned} \tag{4.4.23}$$

We are now prepared to evaluate the divergence $\partial\cdot J_{\rm f}$. The calculation will consist of two terms essentially. The first one arises from acting with $\partial_\mu D_z^\mu$ on the A-independent part of (4.4.22), and which gives terms proportional to the "descendant operators" $\partial^\mu\bar{\psi}\gamma_\mu$, $\partial\!\!\!/\psi$ and $\partial^2\bar{\psi}$, $\partial^2\psi$, which are non-zero in the interacting fermion theory:

$$\begin{aligned} &\partial_\mu D_z^\mu f_{\rm f}(\hat{\partial}_1,\hat{\partial}_2)\bar{\psi}(x_1)\hat{\gamma}\psi(x_2) = \Big[\partial\!\!\!/_1 q(\hat{\partial}_1,\hat{\partial}_2) + \partial\!\!\!/_2\tilde{q}(\hat{\partial}_1,\hat{\partial}_2) + \hat{\gamma}\partial_1^2 h(\hat{\partial}_1,\hat{\partial}_2) + \hat{\gamma}\partial_2^2\tilde{h}(\hat{\partial}_1,\hat{\partial}_2)\Big]\bar{\psi}(x_1)\psi(x_2)\,, \\ &q(u,v) = \big(\frac{1}{2}f_{\rm f} + v(\partial_v f_{\rm f} - \partial_u f_{\rm f})\big)\,, \qquad \tilde{q}(u,v) = \big(\frac{1}{2}f_{\rm f} + u(\partial_u f_{\rm f} - \partial_v f_{\rm f})\big)\,, \\ &h(u,v) = \big(\frac{3}{2}\partial_u f_{\rm f} + \frac{u-v}{2}\partial_u^2 f_{\rm f} + v\partial_{uv}f_{\rm f}\big)\,, \\ &\tilde{h}(u,v) = \big(\frac{3}{2}\partial_v f_{\rm f} + \frac{v-u}{2}\partial_v^2 f_{\rm f} + u\partial_{uv}f_{\rm f}\big) \end{aligned} \tag{4.4.24}$$

The second term is the result of acting with $\partial_\mu D_z^\mu$ on the piece of (4.4.22) proportional to A (in this piece, we can use the free Dirac equation of motion). To simplify the calculation, one may impose the $\hat{A} = 0$ "light-cone" gauge after differentiation with respect to the z_μ is carried out everywhere. The full form of the descendant as a function of $F_{\mu\nu}$ can be then reconstructed using gauge invariance. As a consistency check, we have also performed the calculation in arbitrary gauge, and verified that all unwanted $\hat{A}$ terms drop out. The final result takes the form

$$\partial \cdot J_{\rm f} = \Big[k_1(\hat\partial_1, \hat\partial_2, \hat\partial_3)\partial_1^\mu + k_2(\hat\partial_1, \hat\partial_2, \hat\partial_3)\partial_2^\mu + k_3(\hat\partial_1, \hat\partial_2, \hat\partial_3)\partial_3^\mu\Big]\,\bar\psi(x_1)\hat\gamma(iF_{\mu\nu}(x_3)z^\nu)\psi(x_1)$$
$$+ k_4(\hat\partial_1, \hat\partial_2, \hat\partial_3)\bar\psi(x_1)(iF^{\mu\nu}(x_3)\gamma_{\mu\nu})\hat\gamma\psi(x_2) + k_5(\hat\partial_1, \hat\partial_2, \hat\partial_3)\bar\psi(x_1)(iF_{\mu\nu}(x_3)z^\nu)\gamma^\mu\psi(x_2)\,,$$

where we defined

$$\begin{aligned}
k_1(u,v,w) &= \frac{2}{w}h(u+w,v) - \frac{1}{w}\Big(\frac{3}{2} - (v+w)\partial_u + v\partial_v + w\partial_w\Big)g(u,v,w)\,,\\
k_2(u,v,w) &= -\frac{2}{w}\tilde h(u,v+w) - \frac{1}{w}\Big(\frac{3}{2} + u\partial_u - (u+w)\partial_v + w\partial_w\Big)g(u,v,w)\,,\\
k_3(u,v,w) &= \frac{1}{w}(h(u+w,v) - \tilde h(u,v+w)) - \frac{1}{w}\Big(\frac{3}{2} + u\partial_u + v\partial_v - (u+v)\partial_w\Big)g(u,v,w)\\
k_4(u,v,w) &= \frac{1}{2}(h(u+w,v) - \tilde h(u,v+w))\,,\\
k_5(u,v,w) &= \frac{1}{w}(q(u+w,v) - \tilde q(u,v+w)) + 2\tilde h(u,v+w) + \frac{u+v+w}{w}g(u,v,w)\,.
\end{aligned} \tag{4.4.25}$$

As a check, note that for $s = 2$ we are left with $\bar\psi F_{\mu\nu} z^\mu \gamma^\nu \psi$, which vanishes upon using the equations of motion.

In 3d Euclidean space, the γ matrices are just Pauli matrices, and we have the following identities ($\epsilon_{123} = 1$):

$$\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\epsilon_{\mu\nu\rho}\gamma^\rho\,, \tag{4.4.26}$$

$$\gamma_{\mu\nu}\gamma_\rho = i\epsilon_{\mu\nu\rho} + \gamma_\mu\delta_{\nu\rho} - \gamma_\nu\delta_{\mu\rho}\,, \tag{4.4.27}$$

$$\gamma_\mu\gamma_\nu\gamma_\rho = i\epsilon_{\mu\nu\rho} - \delta_{\mu\rho}\gamma_\nu + \delta_{\nu\rho}\gamma_\mu + \delta_{\mu\nu}\gamma_\rho\,. \tag{4.4.28}$$

Using these, we can write

$$\partial \cdot J_{\rm f} = [k_1\partial_1^\mu + k_2\partial_2^\mu + k_3\partial_3^\mu]\,\bar\psi(\gamma\cdot z)(iF_{\mu\nu}z^\nu)\psi - k_4\bar\psi F^{\mu\nu}\epsilon_{\mu\nu\rho}z^\rho\psi + (k_5 + 2k_4)\bar\psi(iF_{\mu\nu}z^\nu)\gamma^\mu\psi\,.$$

Upon using the gauge field equations of motion

$$(F_{\mu\nu})^i{}_j = \frac{2\pi}{k}\epsilon_{\mu\nu\rho}\bar{\psi}_j\gamma^\rho\psi^i\,, \tag{4.4.29}$$

we find

$$\begin{aligned}\frac{k}{2\pi}\partial\cdot J_{\rm f} &= [k_1\partial_1^\mu + k_2\partial_2^\mu + k_3(\partial_3^\mu+\partial_4^\mu)]\,\bar{\psi}(x_1)\hat{\gamma}i\epsilon_{\mu\nu\lambda}z^\nu(\bar{\psi}(x_3)\gamma^\lambda\psi(x_4))\psi(x_2)\\ &\quad - 2k_4\bar{\psi}(x_1)(\bar{\psi}(x_3)\hat{\gamma}\psi(x_4))\psi(x_1) + (k_5+2k_4)\bar{\psi}(x_1)i\epsilon_{\mu\nu\rho}z^\nu\gamma^\mu(\bar{\psi}(x_3)\gamma_\rho\psi(x_4))\psi(x_2)\,.\end{aligned} \tag{4.4.30}$$

Note that ∂_3 will have to be "point-split" from now on: $\partial_3\to\partial_3+\partial_4$ (and similarly when $\hat{\partial}_3$ appears in $k_1\,\ldots,k_5$). To write the result (4.4.30) as a sum of double-trace primaries, we can use the Fierz identity[13]

$$\psi\bar{\psi} = -\frac{1}{2}(\bar{\psi}\psi) - \frac{1}{2}(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\,. \tag{4.4.31}$$

After using this identity, we can write the descendant as:

$$\begin{aligned}&\frac{k}{2\pi}\partial\cdot J_{\rm f} = i\epsilon_{\mu\nu\rho}z^\rho\Big[k_4\bar{\psi}_i(x_1)\gamma_\mu\psi^i(x_4)\bar{\psi}_j(x_3)\gamma_\nu\psi^j(x_2)\\ &+\frac{1}{2}(k_1\partial_1^\mu+k_2\partial_2^\mu+k_3(\partial_3^\mu+\partial_4^\mu))(\bar{\psi}_i(x_1)\hat{\gamma}\psi^i(x_4)\bar{\psi}_j(x_3)\gamma_\nu\psi^j(x_2)+\bar{\psi}_i(x_1)\gamma_\nu\psi^i(x_4)\bar{\psi}_j(x_3)\hat{\gamma}\psi^j(x_2))\Big]\\ &+(\frac{1}{2}(k_1\hat{\partial}_1+k_2\hat{\partial}_2+k_3(\hat{\partial}_3+\hat{\partial}_4)+k_5+3k_4)\bar{\psi}_i(x_1)\psi^i(x_4)\bar{\psi}_j(x_3)\hat{\gamma}\psi^j(x_2)\\ &-(\frac{1}{2}(k_1\hat{\partial}_1+k_2\hat{\partial}_2+k_3(\hat{\partial}_3+\hat{\partial}_4)+k_5+k_4)\bar{\psi}_i(x_1)\hat{\gamma}\psi^i(x_4)\bar{\psi}_j(x_3)\psi^j(x_2)\,.\end{aligned} \tag{4.4.32}$$

It is now convenient to use the following identities, which follow from the free Dirac equation:

$$i\epsilon^{\mu\nu\rho}\partial_{2,4\mu}\gamma_\nu = -\partial_{2,4\rho}\,, \tag{4.4.33}$$

$$i\epsilon^{\mu\nu\rho}\partial_{1,3\mu}\gamma_\nu = +\partial_{1,3\rho}\,, \tag{4.4.34}$$

where the subscripts indicate the field we act on, and the sign difference is due to the difference of Dirac equation for ψ and $\bar{\psi}$. We also have:

$$i\epsilon^{\mu\nu\rho}z_\rho\partial_{1,3\mu}\hat{\gamma} = i\epsilon^{\mu\nu\rho}z_\rho\gamma_\mu\hat{\partial}_{1,3} + z_\nu\hat{\partial}_{1,3}\,, \tag{4.4.35}$$

$$i\epsilon^{\mu\nu\rho}z_\rho\partial_{2,4\mu}\hat{\gamma} = i\epsilon^{\mu\nu\rho}z_\rho\gamma_\mu\hat{\partial}_{2,4} - z_\nu\hat{\partial}_{2,4}\,. \tag{4.4.36}$$

[13]Spinor indices are uncontracted on the left-hand side, so the right-hand side is a 2 by 2 matrix.

Using the above identities, we can put (4.4.32) into the form

$$
\begin{aligned}
\partial \cdot J_{\mathrm{f}} = & \frac{2\pi}{k} i\epsilon_{\mu\nu\rho} z^{\rho} \big(k_4 + \frac{1}{2}(k_1\hat{\partial}_1 + k_3\hat{\partial}_4 - k_2\hat{\partial}_2 - k_3\hat{\partial}_3)\big) \bar{\psi}_i \gamma_\mu \psi^i \bar{\psi}_j \gamma_\nu \psi^j \\
& + \frac{2\pi}{k}\big(\frac{1}{2}(k_1\hat{\partial}_1 + k_2\hat{\partial}_2 + k_3\hat{\partial}_3 + k_3\hat{\partial}_4) + k_5 + 3k_4 + k_1\hat{\partial}_1 - k_3\hat{\partial}_4\big) \bar{\psi}_i \psi^i \bar{\psi}_j \hat{\gamma} \psi^j \\
& + \frac{2\pi}{k}\big(-\frac{1}{2}(k_1\hat{\partial}_1 + k_2\hat{\partial}_2 + k_3\hat{\partial}_3 + k_3\hat{\partial}_4) - k_5 - k_4 + k_3\hat{\partial}_3 - k_2\hat{\partial}_2\big) \bar{\psi}_i \hat{\gamma} \psi^i \bar{\psi}_j \psi^j \,.
\end{aligned}
\tag{4.4.37}
$$

To make contact with the decomposition into primaries, it is convenient to define the following object:

$$
\tilde{J}^{(s)}_{\mu} = i\epsilon_{\mu\nu}{}^{\rho} z^{\nu} j^{(s)}_{\rho} \,, \tag{4.4.38}
$$

where on the right-hand side $j^{(s)}_{\rho}$ denotes the spin s current with one free index (and all remaining indices contracted with the null polarization vector). Using the explicit form of the currents (4.2.17), one can show that

$$
\tilde{J}^{(s)}_{\mu} = f_s(\hat{\partial}_1, \hat{\partial}_2) \bar{\psi}(x_1) \gamma_{\mu\nu} z^{\nu} \psi(x_2) + \tilde{f}_s(\hat{\partial}_1, \hat{\partial}_2) z_{\mu} \bar{\psi}(x_1) \psi(x_2) \,, \tag{4.4.39}
$$

where $f_s(u,v)$ is the spin-s part of the generating function in (4.2.17), and $\tilde{f}_s(u,v)$ is given by:

$$
\tilde{f}_s(u,v) = \frac{1}{s} \frac{2uv(\partial_u f_s(u,v) - \partial_v f_s(u,v)) + (s-1)(u-v) f_s(u,v)}{u+v} \,. \tag{4.4.40}
$$

The divergence of $\tilde{J}_s$ notably only has the trivial tensor structure:

$$
\partial^{\mu} \tilde{J}^{(s)}_{\mu} = ((v-u) f_s + (u+v)\tilde{f}_s) \bar{\psi}\psi \,. \tag{4.4.41}
$$

Then we see that the second and third line of (4.4.37) are guaranteed to decompose into products of $\hat{\partial}$-derivatives of the spin-s currents and $\hat{\partial}$-derivatives of the scalar operator $\tilde{j}_0 = \bar{\psi}\psi$ or of the divergence $\partial^{\mu} \tilde{J}^{(s)}_{\mu}$. Noting that $\partial^{\mu} \tilde{J}^{(s)}_{\mu} = i\epsilon^{\mu\nu\rho} z_{\rho} \partial_{\nu} j^{(s)}_{\mu}$, we see that the second and third line of (4.4.37) produce precisely the terms that arise in the decomposition (4.3.17) and (4.3.19). To analyze the terms in the first line of (4.4.37), it is convenient to use explicit light-cone coordinates with $z^{\mu} = \delta^{\mu}_{-}$. Then one of the γ-matrices becomes $\gamma_{-} = \hat{\gamma}$ and the other γ_3. Rewriting $\gamma^3 = i\gamma^{-+}$, we see from (4.4.39) that the factor $\bar{\psi}\gamma_{-+}\psi$ has the structure of $\tilde{J}^{(s)}_{-} \sim \epsilon_{-+\rho} j^{\rho}_{(s)}$, minus the "scalar-like" term in (4.4.39), that will give rise to terms of the same form as the second and third line of (4.4.37). The end result of the analysis is that (4.4.37) precisely takes the form predicted in section 3:

$$\begin{aligned} &\partial \cdot J_{\rm f} = \tilde{\mathcal{K}}^{(a)} + \mathcal{K}^{(b)}\,, \\ &\tilde{\mathcal{K}}^{(a)} = \sum_{s_1} C_{s_1,\tilde{0},s}[j_{s_1}][\tilde{j}_0]\,, \qquad \mathcal{K}^{(b)} = \sum_{s_1,s_2} C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]\,, \end{aligned} \tag{4.4.42}$$

where the double-trace operators $[j_{s_1}][\tilde{j}_0]$ and $[j_{s_1}][j_{s_2}]$ are given respectively in (4.3.19) and (4.3.17), and the overall $C_{s_1,s_2,s}$ coefficients are fixed by our explicit calculation to be

$$C_{s_1,s_2,s} = -\frac{2\pi i\lambda}{N}\frac{4(s+s_1-s_2)!(s-s_2+s_1)!}{(s+s_1+s_2-1)!(s-s_1-s_2-1)!}\frac{s_1!s_2!}{s!}\,, \quad s_1+s_2 = s-2j\,, \quad j>0 \tag{4.4.43}$$

and

$$C_{s_1,\tilde{0},s} = \frac{2\pi\lambda}{N}\frac{4(s+s_1)s_1!}{(s-1)!}\,, \quad s_1 = s-2j\,, \quad j>0\,. \tag{4.4.44}$$

Note that $C_{s_1,s_2,s}$ is the same as for the CS-boson theory (up to the sign), as required by the bose/fermi duality. The result to all orders in λ is obtained using (4.3.7), and was given in (4.3.30) and (4.3.32).

The form (4.4.30) (or (4.4.37)) of the divergence can also be used directly to compute the anomalous dimensions using the master formula (4.3.27) and the free-fermion propagators. This way we can extract the anomalous dimension to order λ^2 and exactly in N, and we find

$$\gamma_s = \frac{\lambda^2}{N}\left\{0, 0, \frac{32}{105}+\frac{8}{105N}, \frac{12}{35}+\frac{4}{105N}, \frac{1504}{3465}+\frac{24}{385N}, \frac{4192}{9009}+\frac{32}{693N}, \ldots\right\} + O(\lambda^3)\,. \tag{4.4.45}$$

Remarkably, this is identical to the (non-critical) CS-scalar result (4.4.15), including the $1/N^2$ term (4.4.17). Note that setting $N=1$ in these expressions, we obtain the anomalous dimensions in the $U(1)_k$ CS-fermion theory to order $1/k^2$.

Critical fermion

Let us now study the "critical" fermionic theory where we add the $(\bar{\psi}\psi)^2$ interaction in addition to the Chern-Simons gauge field. At least at large N, the theory has a UV fixed point whose $1/N$ expansion can be developed using the action (4.2.5). At the UV fixed point, σ_f becomes a scalar primary with $\Delta = 1 + O(1/N)$, and the $\bar{\psi}\psi$ operator is formally removed by the σ_f equation of motion.

It is evident that the equations of motion for $\bar{\psi}$ and ψ are modified by terms involving the σ_f field.

Omitting terms which are quadratic in the gauge field or σ_f, the equations of motion are

$$\begin{aligned}
&\not{\partial}\psi = i\not{A}\psi - \frac{1}{N}\sigma\psi\,, \qquad \partial_\mu\bar{\psi}\gamma^\mu = -i\bar{\psi}\not{A} + \frac{1}{N}\sigma\bar{\psi}\,, \\
&\partial^2\psi = \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\psi + i(\partial\cdot A)\psi + 2iA\cdot\partial\psi - \frac{1}{N}(\partial_\mu\sigma)\gamma^\mu\psi\,, \\
&\partial^2\bar{\psi} = \frac{i}{2}\bar{\psi}\gamma^{\mu\nu}F_{\mu\nu} - i\bar{\psi}\partial\cdot A - 2i(\partial^\mu\bar{\psi})A_\mu + \frac{1}{N}\bar{\psi}(\not{\partial}\sigma)\,.
\end{aligned} \tag{4.4.46}$$

The calculation of the divergence then picks up an extra term compared to the "regular" CS-fermion theory:

$$\begin{aligned}
&\partial\cdot J_{\rm f} = \mathcal{K}_{\rm reg.CS-fer.} + \mathcal{K}_{\rm crit.fer.}\,, \\
&\mathcal{K}_{\rm crit.fer.} = \frac{1}{N}\Big[(q(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2) - \tilde{q}(\hat{\partial}_1,\hat{\partial}_2+\hat{\partial}_3) + h(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2) - \tilde{h}(\hat{\partial}_1,\hat{\partial}_2+\hat{\partial}_3)\bar{\psi}(x_1)\psi(x_2)\sigma(x_3) \\
&\qquad + (h(\hat{\partial}_1+\hat{\partial}_3,\hat{\partial}_2) + \tilde{h}(\hat{\partial}_1,\hat{\partial}_2+\hat{\partial}_3))\partial_3^\mu\bar{\psi}(x_1)\gamma_{\mu\nu}z^\nu\psi(x_2)\sigma(x_3)\Big]\,,
\end{aligned} \tag{4.4.47}$$

where $q(u,v)$, $\tilde{q}(u,v)$, $h(u,v)$ and $\tilde{h}(u,v)$ were defined in (4.4.25), and $\mathcal{K}_{\rm reg.CS-fer.}$ is the descendent computed in the previous section, given in (4.4.30). After expressing the right-hand side in terms of double-trace primaries, one should impose the condition $\bar{\psi}\psi = 0$. This amounts to dropping $\tilde{\mathcal{K}}^{(a)} \sim \sum_{s_1} C_{s_1,\tilde{0},s}[\tilde{j}_0][j_{s_1}]$ from $\mathcal{K}_{\rm reg.CS-fer.}$, and one gets the final result (after also dropping the $\bar{\psi}\psi$ terms which arise when writing $\mathcal{K}_{\rm crit.fer.}$ in terms of primaries):

$$\partial\cdot J_{\rm f} = \mathcal{K}^{(a)} + \mathcal{K}^{(b)}\,, \tag{4.4.48}$$

where $\mathcal{K}^{(b)} \sim \sum_{s_1,s_2} C_{s_1,s_2,s}[j_{s_1}][j_{s_2}]$ remains the same as in the regular CS-fermion theory of the previous section, and $\mathcal{K}^{(a)} \sim \sum_{s_1} C_{s_1,0,s}[j_{s_1}][\sigma_f]$ coincides with the quasi-bosonic result in eq. (4.3.22), with j_0 replaced by σ_f, and with the undetermined constants found to be

$$C_{s_1,0,s} = \frac{2i}{N}\frac{(s^2-s_1^2)s_1!}{s!}\,. \tag{4.4.49}$$

Note that, redefining $\sigma_f = 4\pi\lambda j_0^{\rm crit.fer.}$, this result correctly maps to the divergence in the regular CS-scalar theory, eq. (4.4.12)-(4.4.14).

4.5 Direct Feynman diagram computation

In this section, we evaluate the coefficients a_s and b_s by a direct diagrammatic calculation of the anomalous dimensions. a_s^F can be determined by a perturbative calculation in the critical bosonic theory (at $\lambda_b = 0$), a_s^B can be determined by a perturbative calculation in the critical fermionic theory (at $\lambda_f = 0$). Once a_s is known, then b_s (which must be the same for bosonic and fermionic theories) can be obtained by a two-loop calculation in the non-critical fermionic theory.

In a $U(N_f)_{k_f}$ Chern-Simons theory with fundamental matter, with k_f defined via a dimensional reduction regularization scheme (see [85] and [113,130]), $\lambda_f = \frac{N_f}{k_f}$ and N_f are related to $\tilde{\lambda}$, $\tilde{N}$ of [118] via:

$$\tilde{N} = 2N_b \frac{\sin(\pi\lambda_b)}{\pi\lambda_b} = 2N_f \frac{\sin(\pi\lambda_f)}{\pi\lambda_f}\,, \tag{4.5.1}$$

$$\tilde{\lambda} = \tan(\pi\lambda_f/2) = -\cot(\pi\lambda_b/2)\,. \tag{4.5.2}$$

This implies:

$$\tau_s^f - 1 = \frac{1}{N_f}\left(\frac{\pi\lambda_f}{4}\right)\tan(\pi\lambda_f/2)\left(a_s^F + b_s^F\cos^2(\pi\lambda_f/2)\right)\,, \tag{4.5.3}$$

$$= \frac{1}{N_f}\frac{\pi^2}{8}\left(a_s^F + b_s^F\right)\lambda_f^2 + O(\lambda_f^4) \tag{4.5.4}$$

for the two-loop fermionic theory, and

$$\tau_s^{cb} - 1 = \frac{1}{2N_b} a_s^F \tag{4.5.5}$$

for the critical bosonic theory. Identical results hold for the critical fermionic and two-loop bosonic theories.

In section 4.5.1, we include a calculation of a_s^B in the critical fermionic theory, which also appeared earlier in [87], and in section 4.5.2 we include a two-loop calculation of the anomalous dimension in the non-critical fermionic theory to determine b_s.

Perturbative calculations of the $1/N$ anomalous dimension for all the higher-spin currents in the critical bosonic theory have been obtained earlier in [173] (see also [33,84]), so we do not include them here.

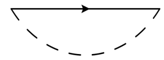


Figure 4.2: The fermion self-energy correction in the critical fermionic theory.

We calculate the anomalous dimension of j_s with null polarization vector z for $s \geq 1$. The free vertex for a spin s current with $s \geq 1$, in momentum space can be written as:

$$\begin{aligned} V_s^0(q,p) &= \gamma^\mu z_\mu f_s(i(q-p)\cdot z, ip\cdot z)\,, && (4.5.6)\\ V_s^0(0,p) &= \not{z}\frac{4^s}{2s!}(-ip\cdot z)^{s-1} && (4.5.7)\\ &= v_s\ (p\cdot z)^{s-1}\not{z} && (4.5.8)\end{aligned}$$

where f_s is determined from the generating function given in equation (4.2.15). The anomalous dimension, $\delta_s = \tau_s - 1$, of j_s is related to the logarithmic divergence of the corrected vertex $V'(q,p)$ via $V_s'(0,p) = -\delta_s V_s^0(0,p)\log\Lambda$.

4.5.1 Critical fermionic theory

We now calculate the $1/N$ anomalous dimension for all the higher-spin currents in the critical fermionic theory. Our conventions are those of [130].

The σ propagator is:

$$\langle\sigma(q)\sigma(-p)\rangle = G(q)\delta^3(p-q)(2\pi)^3 = \frac{G_0}{|q|}\delta^3(p-q)(2\pi)^3\,, \qquad (4.5.9)$$

where $G_0 = 8/N$.

There are essentially three different diagrams which contribute to the $1/N$ logarithmic divergence of the corrected vertex V_s', depicted in Figures 4.2, 4.3 and 4.4.

The fermion self-energy is shown in Figure 4.2. The logarithmic divergence of the self energy is:

$$\int \frac{1}{i\not{p}}G(q-p)\frac{d^3p}{(2\pi)^3} = -\frac{G_0}{6\pi^2}\ i\not{q}\log\Lambda\,, \qquad (4.5.10)$$

which leads to a contribution of $\frac{G_0}{6\pi^2}$ to the anomalous dimension.

Another correction to the vertex is shown in Figure 4.3. The contribution to the corrected vertex

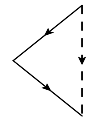


Figure 4.3: A vertex correction in the critical fermionic theory.

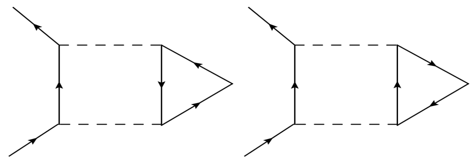


Figure 4.4: These two diagrams provide a third correction to the vertex in the critical fermionic theory when s is even.

V_s' from this diagram is:

$$\begin{aligned}\hat{V}_s'^{(2)}(0,p) &= \int \frac{d^3k}{(2\pi)^3} G(k) \frac{1}{-i(-\not{p}-\not{k})} \gamma^\mu z_\mu f_s(i(-p-k)\cdot z, i(p+k)\cdot z) \frac{1}{i(\not{p}+\not{k})} \quad (4.5.11)\\ &= n_s \frac{G_0}{\pi^2} \log\Lambda \; \gamma^\mu z_\mu f_s(-ip\cdot z, ip\cdot z)\\ &= \left(n_s \frac{G_0}{\pi^2} \log\Lambda\right) \; V_s^0(0,p)\,, \quad (4.5.12)\end{aligned}$$

where

$$n_s = \frac{1}{(4s+2)(2s-1)} \quad (4.5.13)$$

The two diagrams in Figure 4.4 contribute equally to the corrected vertex. Their sum is given by

$$V_s'^{(3)}(0,p) = \int \frac{d^3q}{(2\pi)^3} G(q)(A_s(q)+A_s(-q))G(q) \frac{1}{i\not{p}-i\not{q}}. \quad (4.5.14)$$

with

$$A_s(q) = -N\mathrm{tr} \int \frac{d^3p}{(2\pi)^3} \frac{1}{i\not{p}} V_0(0,p) \frac{1}{i\not{p}} \frac{1}{i\not{p}-i\not{q}}. \quad (4.5.15)$$

We evaluate

$$A(q) = -v_s \left(\frac{s}{2s-1}\right) i \frac{(2s)!}{4^{s+1} s! s!} \frac{(q\cdot z^s}{q}, \quad (4.5.16)$$

for even s and $A(q) = 0$ for odd s.

We find the contribution to the logarithmic divergence from diagram 3 for s even is

$$V_s'^{(3)}(0,p) = 2\frac{G_0^2}{16\pi^2}\left(\frac{s}{(2s-1)(2s+1)}\right)V_s^0(0,p)\log\Lambda, \tag{4.5.17}$$

and $V_s'^{(3)}(0,p) = 0$ for s odd.

Summing all three contributions, the overall logarithmic divergence of the corrected vertex is:

$$\hat{V}_s' = \frac{s-2}{6s-3}\frac{8}{N\pi^2}(-\log\Lambda)\hat{V}_s, \text{ for } s \text{ even, } s>0. \tag{4.5.18}$$

$$\hat{V}_s' = \left(\frac{2(s^2-1)}{3(4s^2-1)}\right)\frac{8}{N\pi^2}(-\log\Lambda)\hat{V}_s, \text{ for } s \text{ odd.} \tag{4.5.19}$$

and the anomalous dimension of the spin s current, with $s>0$ is given by

$$\tau_s^{\text{critical fermionic}} - 1 = \begin{cases} \frac{s-2}{6s-3}\frac{8}{\pi^2}\frac{1}{N}, & s \text{ even}, \\ \frac{2(s^2-1)}{3(4s^2-1)}\frac{8}{\pi^2}\frac{1}{N}, & s \text{ odd}. \end{cases} \tag{4.5.20}$$

A similar calculation shows that the anomalous dimension of the scalar primary σ is given by $-\frac{16}{3\pi^2}\frac{1}{N}$, so the above formula does not apply for $s=0$.

4.5.2 Two-loop Chern-Simons fermionic theory

Our calculation of two-loop anomalous dimensions closely follows [85].

The higher-spin currents in the interacting, non-critical, fermionic theory are the same as those in the free fermionic theory with all derivatives promoted to covariant derivatives. We calculate anomalous dimensions of $j_{(s)}^{+++\cdots}$ with all upper $+$ indices, in light-cone gauge, $A^+ = 0$. In this gauge, the generating function for $j_s^{+++\cdots}$ is the same as in the free theory, and the vertex contains no factors of A_μ.

In light cone gauge the gauge propagator $\langle A_\mu^a(q)A_\nu^b(-p)\rangle = (2\pi)^3\delta(q-p)D_{\mu\nu}(q)\delta^{ab}$ is given by:

$$D_{+3}(q) = -D_{3+}(q) = \frac{4\pi i}{k}\frac{1}{q_-}. \tag{4.5.21}$$

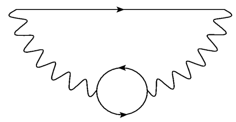


Figure 4.5: The two-loop fermion self-energy correction.

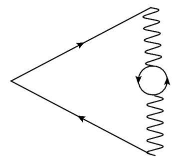


Figure 4.6: A two-loop vertex correction.

The order λ^2 correction to the gauge field propagator at $1/N$ is:

$$\begin{pmatrix} G_{33} & G_{3+} \\ G_{+3} & G_{++} \end{pmatrix} = \frac{2\pi^2\lambda^2}{N^2} q_+^2 \frac{1}{qq_s^4} \begin{pmatrix} -q_-^2 & q_3 q_- \\ q_3 q_- & q_s^2 \end{pmatrix} \quad (4.5.22)$$

Again, there are three diagrams that contribute, depicted in Figures 4.5, 4.6 and 4.7.

The contribution of the two loop $1/N$ self-energy of the fermion to the corrected vertex is given by Figure 4.5. Its contribution to the anomalous dimension can be found a two-point function calculation, we find its contribution to the logarithmic divergence of the corrected vertex to be:

$$V_s'^{(1)} = -\frac{11}{24}\frac{\lambda^2}{N}(-\log\Lambda)V_s^0. \quad (4.5.23)$$

The second diagram contributing to the corrected vertex is shown in Figure 4.6 and is given by

$$\frac{N}{2}\int \frac{d^3q}{(2\pi)^3}\left(G_{\mu\nu}(q)\gamma^\mu \frac{1}{i(\not{p}+\not{q})} v_s\gamma_-(p_-+q_-)^{s-1}\frac{1}{i(\not{p}+\not{q})}\gamma^\nu\right). \quad (4.5.24)$$

The contribution to the anomalous dimension can be evaluated via:

$$V_s'^{(2)} = V_s^0 \frac{N}{2}\frac{1}{2}\,\mathrm{Tr}\left(\gamma^- \int \frac{d^3q}{(2\pi)^3}\left(G_{\mu\nu}(q)\gamma^\mu \frac{1}{i(\not{p}+\not{q})}\gamma_-(p_-+q_-)^{s-1}\frac{1}{i(\not{p}+\not{q})}\gamma^\nu\right)\right). \quad (4.5.25)$$

The logarithmic divergence of the integral is

$$V_s'^{(2)} = \frac{-1}{4}\frac{\lambda^2}{N} p_-^{s-1} v_s \gamma_- \log\Lambda\left(-\frac{1}{2(4s^2-1)} + g(s)\right), \quad (4.5.26)$$

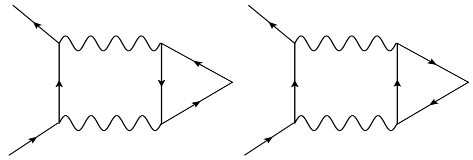


Figure 4.7: These two diagrams contribute equally to the two-loop anomalous dimension when s is even.

where, $g(s)$ is

$$g(s) = \gamma - \psi(s) + 2\psi(2s) = \sum_{n=1}^{s} \frac{1}{n - 1/2}. \tag{4.5.27}$$

Here, $\psi(s)$ is the digamma function. Notice that $g(s) \sim \log s$ for s large.

The last contribution to the corrected vertex is the sum of two diagrams shown in Figure 4.7. Here we evaluate the sum of these diagrams.

The sum of the diagrams is given by:

$$V_s'^{(3)} = \frac{1}{2} \operatorname{tr}\left(\gamma^- \left(-\frac{N}{2}\right) \int \frac{d^3q}{(2\pi)^3} \gamma^\mu \frac{1}{i(\not{p} - \not{q})} \gamma^\nu D_{\mu\alpha}(q) D_{\beta\nu}(q) C^{\alpha\beta}(q)\right), \tag{4.5.28}$$

where

$$C^{\mu\nu}(q) = v_s \int \frac{d^3p}{(2\pi)^3} \operatorname{tr}\left(\frac{1}{i(\not{p} - \not{q})} \gamma^\mu \frac{1}{i\not{p}} \gamma_- p_-^{s-1} \frac{1}{i\not{p}} \gamma^\nu\right) \tag{4.5.29}$$

Evaluating this carefully, we find

$$V_s'^{(3)}(0,p) = \begin{cases} \left(-\frac{2s^2+1}{4s^4-5s^2+1}\right)(\lambda^2/N)(-\log\Lambda) V_s^0(0,p), & s \text{ even}, \\ 0, & s \text{ odd}. \end{cases} \tag{4.5.30}$$

The anomalous dimension of the spin s current gets contributions from only the first two diagrams for s odd and is:

$$\begin{aligned} \tau_s - 1 &= \frac{-11}{24}\frac{\lambda^2}{N} + \frac{1}{4}\left(\frac{-1}{2(4s^2-1)} + g(s)\right)\frac{\lambda^2}{N} \\ &= \left(\frac{-11s^2+2}{6(4s^2-1)} + \frac{1}{4} g(s)\right)\frac{\lambda^2}{N}. \end{aligned}$$

The anomalous dimension for even spin currents is:

$$\begin{aligned}\tau_s - 1 &= \frac{-11}{24}\frac{\lambda^2}{N} + \frac{1}{4}\left(\frac{-1}{2(4s^2-1)} + g(s)\right)\frac{\lambda^2}{N} - \frac{2s^2+1}{4s^4-5s^2+1}\frac{\lambda^2}{N} \\ &= \left(\frac{-11s^4+s^2-8}{6\,(4s^4-5s^2+1)} + \frac{1}{4}g(s)\right)\frac{\lambda^2}{N}\,. \end{aligned} \tag{4.5.31}$$

These anomalous dimensions give rise to the values of a_s and b_s quoted above.

We note that, via a similar calculation, we find that the two-loop anomalous dimension[14] of the scalar $\tilde{j}_0$ is

$$\tau_{\tilde{0}} - 2 = -\frac{4}{3}\frac{\lambda^2}{N}. \tag{4.5.32}$$

which happens to agree with equation (4.5.31) when $s \to 0$.

4.6 Constraining the higher-spin symmetry-breaking three-point functions

In this section, we use our results for the divergence of j_s from sections 4.3.2 and 4.4 to determine the conformally-invariant, non-conserved parity odd three-point functions

$$\langle j_{s_1}(x_1,z_1) j_{s_2}(x_2,z_2) j_{s_3}(x_3,z_3)\rangle.$$

Our analysis in this section uses the results and notation of [174], which we briefly review in appendix 4.8, in which conformally invariant three-point functions are expressed in terms of the structures P_i, Q_i and S_i.

As noted in [174], and in subsequent works [15, 118], there exist exactly three conformally invariant conserved structures for $\langle j_{s_1}(x_1,z_1) j_{s_2}(x_2,z_2) j_{s_3}(x_3,z_3)\rangle$ when conservation with respect to all three currents is imposed. These are the free fermion correlation function, the free boson correlation function, and a parity-odd result, unique to three dimensions.

In [174], based on numerical examples, it was conjectured that the exactly conserved parity odd form exists only when the three spins satisfy the triangle inequality, which takes the form $s_3 \le s_1 + s_2$, if we assume s_3 is the largest of the three spins. Below, we prove this result for arbitrary spins.

[14]We thank Aaron Hui for discussions regarding this calculation.

When the triangle inequality is violated, i.e., $s_3 > s_1+s_2$, a parity-violating form of the three-point function arises in Chern-Simons vector models [118] that is conserved with respect to the first two currents only. Requiring the divergence of j_{s_3} to be a conformal primary, we are able to uniquely determine this form; and using the results of the classical divergence calculation, also its correct normalization.

In subsection 4.6.1 we present recurrence relations that can be easily solved numerically for the parity odd three-point functions of a scalar operator and two other operators of nonzero spin for correlation functions involving the quasi-fermionic scalar ($\tilde{j}_0$) or the quasi-bosonic scalar j_0.

For all spins non-zero, we are able to derive recurrence relations which are valid in a particular limit (the light-like OPE limit of [15]) for arbitrary spins. We are also able to explicitly show that the parity-odd three-point functions are uniquely determined by the divergence of j_{s_3} if the triangle inequality is violated, which implies that, if j_{s_3} is exactly conserved, the parity-odd three-point functions must vanish outside the triangle inequality.

In appendix 4.9, we present some explicit non-conserved parity-odd three-point functions for small spins.

4.6.1 Three-point functions involving a scalar primary

When one of the spins, which we take to be s_2, is zero, it is possible to explicitly determine recurrence relations for the three-point functions.

Quasi-fermionic theory

The most general "parity-odd"[15] three-point function involving a parity-odd, twist-two scalar $\tilde{j}_0$ allowed by conformal invariance is:

$$\langle j_{s_1}(x_1,z_1)\tilde{j}_0 j_{s_3}(x_3,z_3)\rangle = \frac{1}{|x_{12}|^2|x_{23}|^2}\sum_{a=0}^{s_1}\tilde{c}_a Q_1^a (P_2^2)^{s_1-a} Q_3^{s_3-s_1+a}\,, \tag{4.6.1}$$

where the $\tilde{c}_a$ are undetermined coefficients.

[15]Recall that, by "parity-odd", here we mean parity different from the free theory, so that the three-point function must be multiplied by an odd power of λ when it arises in a Chern-Simons vector model. Three-point functions involving $\tilde{j}_0$ in the theory of free fermions involve an epsilon tensor, and hence the S_i's, and are in this case considered to be "parity-even".

The correlation function is not conserved with respect to x_3. Using the results of section 4.3.2 for $\partial \cdot j_s\Big|_{s_1,\tilde{0}}$, we can determine:

$$\langle j_{s_1}(x_1)\tilde{j}_0(x_2)\partial \cdot j_s(x_3)\rangle = \sum_{m=0}^{p} c_m \partial_-^m \langle j_{s_1}(x_1) j_{s_1}(x_3)\rangle \partial_-^{p-m}\langle j_0(x_2) j_0(x_3)\rangle \,, \tag{4.6.2}$$

which implies,

$$\begin{aligned}\partial^\mu_{(3)} D^{(3)}_\mu \langle j_{s_1} j_{\tilde{0}} j_{s_3}\rangle &= \frac{(-1)^{s_3-s_1-1}}{|x_{23}|^4|x_{31}|^2} Q_3^{s_3-s_1-1} P_2^{2s_1} \frac{(s_3-s_1)(s_3+s_1-1)!}{2^{2s_1}(2s_1)!} C_{s_1,0,s_3} n_{s_1} n_{\tilde{0}} (1+\tilde{\lambda}^2)\tilde{N}^2 \\ &= \frac{\tilde{d}_0}{|x_{23}|^4|x_{31}|^2} Q_3^{s_3-s_1-1} P_2^{2s_1} \,.\end{aligned} \tag{4.6.3}$$

Explicitly evaluating the divergence of Equation (4.6.1) and inserting into equation (4.6.3) yields a recurrence relation for the $\tilde{c}_a$:

$$\begin{aligned}&\tilde{c}_{a-1}\left(4a^2 + a(-6s_1+2s_3-7) + 2s_1(s_1+2) - 3s_3 + 3\right) \\ &- \tilde{c}_a 2a(a-s_1+s_3) + \tilde{c}_{a-2}(a-s_1-2)(-2a+2s_1+3) = 0 \,,\end{aligned} \tag{4.6.4}$$

which is valid for $2 \leq a \leq s_1$, along with the following boundary terms:

$$\tilde{c}_{s_1}(s_1(2s_3-1)-s_3) - \tilde{c}_{s_1-1} = 0 \tag{4.6.5}$$

$$(s_3-s_1)(\tilde{c}_0(2(s_1-1)s_1-s_3) + 2\tilde{c}_1(s_1-s_3-1)) = \tilde{d}_0 \,. \tag{4.6.6}$$

This recurrence relation (s_1 equations in s_1 unknowns) has a *unique* solution, which is proportional to $C_{s_1,0,s_3}$. The correlation function therefore necessarily vanishes if j_{s_3} is conserved. It appears to also automatically satisfy conservation with respect to the first current. In Appendix 4.9 we present a few solutions to this recurrence relation explicitly.

The reason we are able to solve for the correlation function uniquely is that the number of conformally invariant structures in equation (4.6.1) is independent of the third spin. So, imposing a constraint on the divergence with respect to s_3 gives us s_1 equations in s_1 unknowns, and hence uniquely determines the correlation function.

Quasi-bosonic theory

We can write the most general conformally invariant parity-odd correlation function involving a twist-one scalar j_0 as:

$$\langle j_{s_1}(x_1,z_1)j_0(x_2)j_{s_3}(x_3,z_3)\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|}\sum_{a=0}^{s_1-1}\tilde{c}_a Q_1^a (P_2^2)^{s_1-a-1}Q_3^{s_3-s_1+a}S_2\,, \tag{4.6.7}$$

where the $\tilde{c}_a$ are undetermined coefficients.

From the constraint that the divergence of j_{s_3} be a conformal primary, we have from section 4.3.2:

$$\begin{aligned}\partial^\mu_{(3)}D^{(3)}_\mu\langle j_{s_1}j_0j_{s_3}\rangle &= 2^p f_n\frac{(2s_1+n)!(p-n)!}{(2s_1)!}n_{s_1}n_0(1+\tilde{\lambda}^2)\tilde{N}^2\left(\epsilon_{\mu\nu-}x_{13}^\mu x_{23}^\nu\right)\\ &\quad(x_{13}^+)^{2s_1-1+n}(x_{23}^+)^{p-n-1}(x_{23}^2)^{n-p-1}(x_{31}^2)^{-2s_1-n-1}\\ &= \frac{|x_{12}|}{|x_{23}|^3|x_{31}|^3}\tilde{d}_0 Q_3^{s_3-s_1-1}P_2^{2s_1-2}S_2\,,\end{aligned} \tag{4.6.8}$$

where $\tilde{d}_0$ is given by

$$\tilde{d}_0 = (-1)^{s_3-s_1+1}(1+\tilde{\lambda}^2)\frac{n_{s_1}n_0\tilde{N}^2C_{s_1,0,s_3}(s_3+s_1-1)!}{2^{2s_1-1}(2s_1)!}s_1. \tag{4.6.9}$$

Using equation (4.6.7), the equation (4.6.8) translates into a recurrence relation for $\tilde{c}_a$.

$$\begin{aligned}&\tilde{c}_a\left(4a^2+a(-6s_1+2s_3+3)+2(s_1-1)s_1+s_3+1\right)\\ &-\tilde{c}_{a-1}(2(a-1)-2s_1+1)(a-s_1)-2(a+1)\tilde{c}_{a+1}(a-s_1+s_3+1)=0\,,\end{aligned} \tag{4.6.10}$$

which is valid for $a=1$ to s_1-2 along with the boundary terms:

$$\tilde{c}_0(s_3-s_1)(2(s_1-1)s_1+s_3+1)-2\tilde{c}_1(s_3-s_1)(-s_1+s_3+1) = \tilde{d}_0 \tag{4.6.11}$$

$$c_{s_1-1}(s_1(2s_3-1)-s_3+2)-3c_{s_1-2} = 0\,. \tag{4.6.12}$$

This has a unique solution (s_1-1 equations in s_1-1 unknowns), which is proportional to $C_{s_1,0,s_3}$. The correlation function necessarily vanishes if j_{s_3} is conserved. A few solutions to this recurrence relation are given in Appendix 4.9.

4.6.2 Three-point functions involving nonzero spins

Let us briefly consider correlation functions involving all nonzero spins. While it is difficult to say much about these in full generality, following [15], we work in the "light-like OPE" limit, which is a constraint on x_{12} and z_1 and z_2 that commutes with the operation of taking the divergence with respect to x_3. In this limit, $P_3 = 0$ and $S_3 = 0$; which we shall use frequently in all the derivations below. One way of taking this limit is to fix the first two polarization tensors to be $z_1^\mu = z_2^\mu = \delta_-^\mu$ and set $x_{12}^+ = 0$.

In the light-like OPE limit, conformal invariance restricts the parity-odd three-point function to be of the form:

$$\langle j_{s_1}(x_1, z_1) j_{s_2}(x_2, z_2) j_{s_3}(x_3, z_3)\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} f_{s_1,s_2,s_3}(P_i, Q_i, S_i) \tag{4.6.13}$$

where

$$\begin{aligned} f_{s_1,s_2,s_3} = & \sum_{n=0}^{\min(s_3-s_1-1,s_2-1)} \tilde{a}_n Q_2^{s_2-n-1} P_1^{2n} Q_3^{s_3-s_1-n-1} P_2^{2s_1} S_1 \\ & + \sum_{m=0}^{\min(s_1+s_2-s_3,s_1-1)} \tilde{b}_m Q_1^m Q_2^{s_1+s_2-s_3-m} P_1^{2(s_3-s_1+m)} P_2^{2(s_1-m-1)} S_2 \\ & + \sum_{n=\max(0,s_1+s_2-s_3)}^{s_2} \tilde{c}_n Q_2^n Q_3^{s_3-s_1-s_2+n} P_1^{2(s_2-n)} P_2^{2(s_1-1)} S_2 \\ & + \sum_{n=0}^{\min(s_3-s_2-1,s_1-1)} \tilde{d}_n Q_1^{s_1-n-1} P_2^{2n} Q_3^{s_3-s_2-1-n} P_1^{2s_2} S_2. \end{aligned} \tag{4.6.14}$$

Here we used the constraints (which simplify in the limit $P_3 = 0$ and $S_3 = 0$)

$$Q_1 Q_2 Q_3 = Q_1 P_1^2 + Q_2 P_2^2 \,, \tag{4.6.15}$$

$$Q_1 S_1 = Q_2 S_2 \,, \tag{4.6.16}$$

to write the function f in a unique way by eliminating all occurrences of $Q_1Q_2Q_3$ and Q_1S_1, so each term is independent, and all powers are positive.

Note that the range of the sums, which was fixed by requiring all exponents to be positive, depends nontrivially on the spins. Let us assume s_3 is the largest spin and $s_3 \geq s_1 + s_2$. Then the total number of undetermined coefficients in (4.6.14) is $2s_2 + s_1 - 1$, which is independent of s_3.

When we take the divergence with respect to x_3 we obtain an expression which is again of the form (4.6.14) but with $s_3 \to s_3' = s_3 - 1$. However, the number of allowed conformal structures in equation (4.6.14) is independent of s_3 outside the triangle inequality. This means that, outside the triangle inequality, imposing a constraint on the divergence with respect to x_3 gives us $2s_2 + s_1 - 1$ equations in $2s_2 + s_1 - 1$ unknowns. Therefore, we expect that exactly conserved parity-odd three-point functions vanish outside the triangle inequality, at least in the light-like OPE limit. (Inside the triangle inequality, the number of independent terms in equation (4.6.14) does depend on s_3 so imposing a constraint on conservation with respect to x_3 does not uniquely determine the three-point function.)

When j_{s_3} is not conserved, the results for the divergence $\partial \cdot j_{s_3}\Big|_{s_1,s_2}$ in section 4.3.2 imply the following:

$$\begin{aligned}
&\langle j_{s_1}(x_1) j_{s_2}(x_2) \partial \cdot j_{s_3}(x_3)\rangle \qquad (4.6.17)\\
= \quad & -n_{s_1} n_{s_2} \tilde{N}^2 C_{s_1,s_2,s_3} 2^p \frac{(s_3+s_1+s_2-1)!}{p(2s_1)!(2s_2)!} \epsilon_{\mu\nu-} x_{13}^{\mu} x_{23}^{\nu} \\
& \times \frac{(x_{23}^+)^{2s_2-1}}{x_{23}^{4s_2+2}} \frac{(x_{13}^+)^{2s_1-1}}{x_{13}^{4s_1+2}} \left(\frac{x_{23}^+}{x_{23}^2} - \frac{x_{13}^+}{x_{13}^2} \right)^{p-1} \left(\frac{x_{23}^+}{x_{23}^2} A - \frac{x_{13}^+}{x_{13}^2} B \right),
\end{aligned}$$

where, $p = s_3 - s_1 - s_2$, $A = s_1(s_1 - s_3 - s_2) + 2s_1 s_2 (-1)^{s_3+s_1+s_2}$ and $B = -2s_1 s_2 + s_2(s_3 + s_1 - s_2)(-1)^{s_3+s_1+s_2}$, which can be written as,

$$\begin{aligned}
&\langle j_{s_1}(x_1) j_{s_2}(x_2) \partial \cdot j_{s_3}(x_3)\rangle = (-1)^{s_3-s_2-s_1} \frac{n_{s_1} n_{s_2} \tilde{N}^2 C_{s_1,s_2,s_3} (s_3+s_1+s_2-1)!}{2^{2s_1+2s_2-1}(s_3-s_1-s_2)(2s_1)!(2s_2)!} \frac{|x_{12}|}{|x_{31}|^3 |x_{23}|^3} \\
&\times \left(A S_2 P_1^{2s_2} P_2^{2s_1-2} Q_3^{s_3-s_2-s_1-1} + B S_1 P_2^{2s_1} P_1^{2s_2-2} Q_3^{s_3-s_2-s_1-1} \right), \qquad (4.6.18)
\end{aligned}$$

in the light-like limit. (If $s_1 = s_2$, this equation has to be multiplied by a factor of 2.) This equation translates into a system of linear equations for the various coefficients in equation (4.6.14). This system of equations appears complicated and difficult to solve in general, though one can obtain solutions for particular spins. In Appendix 4.8, we change variables to obtain an equivalent, but simpler recurrence relation. Using the new variables, we also prove that exactly-conserved parity-odd three-point functions vanish when $s_3 > s_1 + s_2$ even outside the light-like limit. We also present a list of some non-conserved parity-odd three point functions with non-zero spins in Appendix 4.9.

4.7 Appendix: Constraining the divergence of j_s

To constrain the double-trace operators that can appear on the right hand side of the non-conservation equation by requiring the divergence to be a conformal primary, we use the following commutation relations from the conformal algebra:

$$
\begin{aligned}
[M^{\rho\sigma}, P^{\mu}] &= -i\left(\eta^{\mu\sigma}P^{\rho} - \eta^{\mu\rho}P^{\sigma}\right), && (4.7.1)\\
[K^{\nu}, P^{\mu}] &= 2i\left(\eta^{\mu\nu}D + M^{\mu\nu}\right), && (4.7.2)\\
[K^{\nu}, j_s^{\rho\sigma}] &= 0\,, && (4.7.3)\\
[M^{\rho\sigma}, j_s^{\mu}] &= -i\left(\eta^{\mu\sigma}j_s^{\rho} - \eta^{\mu\rho}j_s^{\sigma}\right) - i(s-1)\left(\delta_-^{\sigma}j_s^{\mu\rho} - \delta_-^{\rho}j_s^{\mu\sigma}\right), && (4.7.4)\\
[D, j_s] &= -i\Delta_s j_s\,. && (4.7.5)
\end{aligned}
$$

The last three relations express the fact that j_s is a spin s conformal primary with scaling dimension Δ_s. Recall that $\Delta_s = s+1$, except for the quasi-fermionic scalar $\tilde{j}_0$, for which $\Delta_{\tilde{0}} = 2$. Here, as in section 4.3.2, we are taking all polarization vectors to be given by $z^{\mu} = \delta_-^{\mu}$, so $j_s^{\mu\nu} \equiv j_s^{\mu\nu}{}_{---\ldots}$.

A double-trace operator such as $(\partial_-^2 j_s)\partial_- j_0$ is proportional to

$$
[P_-,\ [P_-, j_s]]\ [P_-, j_0]\,, \tag{4.7.6}
$$

which, using the state-operator correspondence, we can also write schematically as

$$
P_-^2 \left|j_s\right\rangle P_- \left|j_0\right\rangle . \tag{4.7.7}
$$

Let us constrain the double-trace terms in the non-conservation equation involving a quasi-fermionic scalar, given by equation (4.3.19) from the main text:

$$
\partial\cdot j_s\Big|_{s_1,0} \sim \sum_{m=0}^{p} c_m P_-^m \left|j_{s_1}\right\rangle P_-^{p-m} \left|\tilde{j}_0\right\rangle . \tag{4.7.8}
$$

where $p = s - s_1 - 1$.

Acting on this expression with K_+, we obtain

$$
\begin{aligned}
0 &= K_+ \partial \cdot j_s \Big|_{s_1,0} && (4.7.9)\\
&= \sum_{m=0}^{p} c_m \left(\left(K_+ P_-^m \left| j_{s_1} \right\rangle \right) P_-^{p-m} \left| \tilde{j}_0 \right\rangle + P_-^m \left| j_{s_1} \right\rangle \left(K_+ P_-^{p-m} \left| \tilde{j}_0 \right\rangle \right) \right) && (4.7.10)\\
&= \sum_{m=0}^{p} c_m \left(\left([K_+, P_-^m] \left| j_{s_1} \right\rangle \right) P_-^{p-m} \left| \tilde{j}_0 \right\rangle + P_-^m \left| j_{s_1} \right\rangle \left([K_+, P_-^{p-m}] \left| \tilde{j}_0 \right\rangle \right) \right). && (4.7.11)
\end{aligned}
$$

Then we use

$$[K_\delta, P_-^n] = 2inP_-^{n-1} \left(\eta_{-\delta} D + M_{-\delta} \right) + 2n(n-1)\eta_{-\delta} P_-^{n-1} \tag{4.7.12}$$

and the action of the conformal generators on $\left| j_s \right\rangle$ to obtain

$$\sum_{m=1}^{p} \left(m(m+2s_1)c_m \right) P_-^{m-1} \left| j_{s_1} \right\rangle P_-^{p-m} \left| \tilde{j}_0 \right\rangle + \sum_{m=0}^{p-1} (m-p)(m-p-1) c_m P_-^m \left| j_{s_1} \right\rangle P_-^{p-m-1} \left| \tilde{j}_0 \right\rangle = 0, \tag{4.7.13}$$

which implies

$$c_m = \frac{-(m-p-1)(m-p-2)}{m(m+2s_1)} c_{m-1}, \tag{4.7.14}$$

which can be solved to give equation (4.3.21):

$$c_m = (-1)^m \binom{s-s_1}{m} \binom{s+s_1-1}{m+2s_1} C_{s_1,\tilde{0},s_3}. \tag{4.7.15}$$

The resulting expression is also annihilated by K_3 and K_-.

Similar (but more lengthy) calculations determine analogous recurrence relations for contributions to the non-conservation equation involving the quasi-bosonic scalar (4.3.22) and general non-zero spins (4.3.17). These can be solved to give (4.3.23)-(4.3.24), and (4.3.18).

4.8 Appendix: Some results for parity-odd three-point functions

In this appendix, we present slightly simpler recurrence relations for the parity odd three-point functions.

Let us briefly review the notation of [174]. Consider the three point function of three operators

$\mathcal{O}_{s_1}(x_1,z_1)$, $\mathcal{O}_{s_2}(x_2,z_2)$, and $\mathcal{O}_{s_3}(x_3,z_3)$, of spins s_1, s_2 and s_3 and twists τ_1, τ_2 and τ_3. Conformal invariance restricts the three point function of these operators to take on the form

$$\langle \mathcal{O}_{s_1}\mathcal{O}_{s_2}\mathcal{O}_{s_3}\rangle = \frac{1}{|x_{12}|^{\tau_1+\tau_2-\tau_3}|x_{23}|^{-\tau_1+\tau_2+\tau_3}|x_{31}|^{-\tau_2+\tau_1+\tau_3}} f(P_i,Q_i,S_i). \tag{4.8.1}$$

where $f(P_i,Q_i,S_i)$ is a polynomial in the cross ratios P_i, Q_i and S_i defined for $i=1,2,3$, using polarization spinors $z_i^\mu(\sigma_\mu)_{\alpha\beta}\equiv(\lambda_i)_\alpha(\lambda_i)_\beta$ as

$$P_3 = \lambda_1\frac{\not{x}_{12}}{x_{12}^2}\lambda_2\,, \tag{4.8.2}$$

$$Q_3 = \lambda_3\left(\frac{\not{x}_{31}}{x_{31}^2}+\frac{\not{x}_{23}}{x_{23}^2}\right)\lambda_3\,, \tag{4.8.3}$$

$$S_3 = i\frac{1}{|x_{12}||x_{23}||x_{31}|}\left(\lambda_2\not{x}_{12}\not{x}_{23}\lambda_3\right)\left(\lambda_1\frac{\not{x}_{12}}{x_{12}^2}\lambda_2\right), \tag{4.8.4}$$

and cyclic permutations. Here $\not{x}\equiv x^\mu\sigma_\mu$. To match spin, f must be homogeneous of degree s_i in each of the z_i. The cross ratios are not all independent, and satisfy some constraints listed in [174].

In terms of the null polarization vectors z_i, the cross-ratios can be written as:

$$P_3^2 = -2z_1^\mu z_2^\nu\left(\frac{\delta_{\mu\nu}}{x_{12}^2}-\frac{2x_{12}^\mu x_{12}^\nu}{x_{12}^4}\right), \tag{4.8.5}$$

$$Q_3 = 2z_3^\mu\left(\frac{x_{32}^\mu}{x_{32}^2}-\frac{x_{31}^\mu}{x_{31}^2}\right), \tag{4.8.6}$$

$$S_3 = 4\frac{\epsilon_{\mu\nu\rho}}{|x_{31}||x_{12}|^3|x_{23}|}\left(x_{12}^\mu x_{31}^\nu z_1^\rho z_2\cdot x_{12}-\frac{1}{2}(|x_{31}|^2x_{12}^\mu+|x_{12}|^2x_{31}^\mu)z_1^\nu z_2^\rho\right). \tag{4.8.7}$$

Parity odd three-point functions are linear in the S_i, while parity even three-point functions do not contain the S_i.

If some of the operators are conserved currents, we must also require that the appropriate divergence of the three-point function vanishes. We note that taking divergences with respect to x_3 of a correlation function involving a twist-1 operator is facilitated using the operator $\mathcal{D}_3$ defined in Appendix F of [15], which satisfies:

$$\begin{aligned}\partial_{\lambda_3}\not{\partial}_{x_3}\partial_{\lambda_3}\frac{1}{|x_{12}|^{\tau_1+\tau_2-1}|x_{23}|^{-\tau_1+\tau_2+1}|x_{31}|^{-\tau_2+\tau_1+1}}f(P_1,P_2,P_3,Q_1,Q_2,Q_3)\equiv\\ \frac{1}{|x_{12}|^{\tau_1+\tau_2-3}|x_{23}|^{-\tau_1+\tau_2+3}|x_{31}|^{-\tau_2+\tau_1+3}}\mathcal{D}_3 f(P_1,P_2,P_3,Q_1,Q_2,Q_3).\end{aligned} \tag{4.8.8}$$

In the main text, we defined the divergence of j_s using $\partial \cdot j_s(x,z) \equiv \partial_{x^\mu} D_z^\mu j_s(x,z)$. A useful relation is

$$\partial_\lambda \not{\partial}_x \partial_\lambda = 4\partial^\mu D_\mu . \tag{4.8.9}$$

4.8.1 A simpler form for the recurrence relations

Equation (4.6.14) for $\langle j_{s_1} j_{s_2} j_{s_3}\rangle$ can also be written as:

$$\langle j_{s_1} j_{s_2} j_{s_3}(x_3,z_3)\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|}\sum_{a=0}^{s_3-1}\Big(c_a Q_1^{s_1-1-a} P_2^{2a} P_1^{2(s_3-1-a)} Q_2^{s_2-s_3+1+a} S_2\Big) . \tag{4.8.10}$$

after using the identities: $Q_3 = P_1^2/Q_2 + P_2^2/Q_1$ and $Q_1 S_1 = Q_2 S_2$ to eliminate Q_3 and S_2. To fix the range of a we note that, starting from a polynomial including S_2 and Q_3 with all non-negative exponents, after using identities to eliminate Q_3 and S_1, we could end up with an expression where the exponents of the Q_i are negative; however the exponents of the P_i must still be non-negative. (Note that c_a defined here is unrelated to the c_a that appears in section 4.3.2 or Appendix 4.7.)

While any three-point function of the form (4.6.14) can be written in the form (4.8.10), not every expression in the form (4.8.10) corresponds to a valid three-point function. To see this, note that equation (4.8.10) can also be rewritten as

$$\langle j_{s_1} j_{s_2} j_{s_3}(x_3,z_3)\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|}\sum_{m=0}^{s_3-1} \tilde{c}_m Q_1^{s_1-1-m} Q_3^{s_3-1-m} P_2^{2m} Q_2^{s_2} S_2 , \tag{4.8.11}$$

where

$$\tilde{c}_m = \sum_{a=0}^{m} (-1)^{s_3-1-n} \binom{s_3-1-a}{s_3-1-m} c_a . \tag{4.8.12}$$

If $s_2 = 0$, then even outside the light like OPE limit, the correlation function must be of the form (4.8.11). If $s_3 > s_1$, which we assume in what follows, we must have $\tilde{c}_m = 0$ for $m > s_1 - 1$. This is an extra constraint on the c_n.

For all spins nonzero, there are also constraints on $\tilde{c}_m$ that arise from demanding that the expression can be written in terms of only positive powers of the various cross-ratios P_1 , P_2, Q_1, Q_2 and Q_3, S_1 and S_2. To obtain one such constraint, which is sufficient for our purposes, choose points and polarization spinors so that $Q_3 = 0$ which implies $Q_1 P_1^2 = -Q_2 P_2^2$. Then, outside the triangle inequality, equation (4.6.14) vanishes. However, (4.8.10) does not vanish unless

$$\sum_{a=0}^{s_3-1}(-1)^a c_a = 0\,, \tag{4.8.13}$$

which must be imposed for (4.8.10) to represent a valid three-point function, outside the triangle inequality. (Inside the triangle inequality, we do not need to impose (4.8.13).)

Conserved three-point functions

To take the divergence with respect to x_3, we act on the above expression with the operator $\mathcal{D}_3$ derived in equations F.2 of [15].

In the limit $P_3 = 0$, for expressions independent of Q_3, it takes the simple form (equation I.4 of [15]):

$$\begin{aligned}\mathcal{D}_3 = &-(1+2P_1\partial_{P_1}+2Q_2\partial_{Q_2})Q_1\partial_{P_2}^2 + (1+2P_2\partial_{P_2}+2Q_1\partial_{Q_1})Q_2\partial_{P_1}^2\\ &+\left(P_3^2\partial_{Q_2}+2P_2P_3\partial_{P_1}\right)\partial_{P_2}^2 - \left(P_3^2\partial_{Q_1}+2P_1P_3\partial_{P_2}\right)\partial_{P_1}^2\,.\end{aligned} \tag{4.8.14}$$

We also use the identities 2.20 of [174] to derive $S_1^2 = Q_2P_1^2P_2^2Q_1^{-1}$, which yields:

$$\partial_{P_1}S_1 = S_1P_1^{-1},\ \partial_{P_2}S_1 = S_1P_2^{-1},\ \partial_{Q_1}S_1 = -\frac{1}{2}S_1Q_1^{-1},\ \partial_{Q_2}S_1 = \frac{1}{2}S_1Q_2^{-2}. \tag{4.8.15}$$

These relations are valid only when $P_3 = S_3 = 0$, i.e., in the light-like limit.

We find

$$\begin{aligned}&\mathcal{D}_3\sum_{a=0}^{s_3-1}\left(c_aQ_1^{s_1-a}P_2^{2a}P_1^{2(s_3-1-a)}Q_2^{s_2-s_3+a}S_1\right) =\\ &\sum_{a=0}^{s_3-2}4\left(-c_{a+1}(s_3+s_2-a-1)(a+1)(2a+3)+c_a(1+a+s_1)(2s_3-2a-1)(s_3-a-1)\right)\chi\end{aligned} \tag{4.8.16}$$

where

$$\chi = Q_1^{s_1-a}P_2^{2a}P_1^{2s_3-2a-2}Q_2^{s_2-s_3+1+a}S_1. \tag{4.8.17}$$

If j_{s_3} is exactly conserved, then the condition that the divergence of equation (4.8.10) is equal to

0 gives rise to the following recurrence relation:

$$\frac{c_{a+1}}{c_a} = \frac{(1+a+s_1)(2s_3-2a-1)(s_3-a-1)}{(s_3+s_2-a-1)(a+1)(2a+3)}. \tag{4.8.18}$$

This has a unique solution for any values of s_1, s_2 and s_3. It can be expressed in terms of Pochhammer symbols as

$$c_a = -\frac{(-1)^a c_0(s_1+1)(s_3-1)(2s_3-1)(s_1+2)_{a-1}\left(\frac{3}{2}-s_3\right)_{a-1}(2-s_3)_{a-1}}{3(2)_{a-1}\left(\frac{5}{2}\right)_{a-1}(s_2+s_3-1)(-s_2-s_3+2)_{a-1}} \tag{4.8.19}$$

and the sum is a hypergeometric function

$$\langle j_{s_1} j_{s_2} j_{s_3}(x_3,z_3)\rangle = \frac{Q_1^{s_1-1}P_1^{2(s_3-1)}Q_2^{s_2-s_3+1}S_2}{|x_{12}||x_{23}||x_{31}|}c_0\ {}_3F_2\left(s_1+1,\frac{1}{2}-s_3,1-s_3;\frac{3}{2},-s_2-s_3+1;u\right) \tag{4.8.20}$$

where $u = -\frac{P_2^2 Q_2}{Q_1 P_1^2}$.

If $s_3 > s_1 + s_2$, then, as discussed above, we must also impose the extra constraint (4.8.13) for our solution to represent a valid three-point function expressible in the form (4.6.14). This is

$$c_0\ {}_3F_2\left(s_1+1,\frac{1}{2}-s_3,1-s_3;\frac{3}{2},-s_2-s_3+1;1\right) = 0 \tag{4.8.21}$$

which implies that $c_0 = 0$ and the exactly conserved correlation functions vanish outside the triangle inequality, in the light-like OPE limit. (In section 4.8.2, we argue that these correlation function vanish even outside the light-like OPE limit.)

Non-conserved parity odd three-point functions

Outside the triangle inequality, the parity-odd three-point function is not conserved. In the light-like limit, its divergence with respect to x_3 takes the form:

$$\partial_{x_3^\mu} D_{z_3}^\mu \langle j_{s_1}(x_1) j_{s_2}(x_2) j_{s_3}(x_3,z_3)\rangle = \frac{|x_{12}|}{|x_{23}|^3|x_{31}|^3}\sum_a d_a Q_1^{s_1-a-1}P_2^{2a}P_1^{2s_3-2a-4}Q_2^{s_2-s_3+2+a}S_2 \tag{4.8.22}$$

where,

$$d_a = -c_{a+1}(s_3+s_2-a-1)(a+1)(2a+3) + c_a(1+a+s_1)(2s_3-2a-1)(s_3-a-1) \tag{4.8.23}$$

The result of the divergence calculation, (4.6.17) determines the d_a in terms of C_{s_1,s_2,s_3}:

$$d_{s_1-1} = AK\,, \tag{4.8.24}$$

$$d_{s_1-1+n} = \left(\binom{p-1}{n}A + \binom{p-1}{n-1}B\right)K\,, \tag{4.8.25}$$

$$d_{s_1-1+p} = d_{s_3-s_2-1} = BK\,, \tag{4.8.26}$$

with all other $d_a = 0$, and we define

$$K = (-1)^{s_3-s_2-s_1}\frac{n_{s_1}n_{s_2}\tilde{N}^2 C_{s_1,s_2,s_3}(s_3+s_1+s_2-1)!}{2^{2s_1+2s_2-1}(s_3-s_1-s_2)(2s_1)!(2s_2)!}\,.$$

The spin-dependent constants A and B were defined below equation (4.6.17).

We can now determine a recurrence relation the c_a in terms of d_a (and hence C_{s_1,s_2,s_3}) using equation (4.8.23). Equation (4.8.23) can be written as:

$$c_{a+1} = Ec_a + Fd_a\,, \tag{4.8.27}$$

where

$$E = \frac{(1+a+s_1)(2s_3-2a-1)(s_3-a-1)}{(1+a)(2a+3)(s_3+s_2-a-1)},\ F = -\left((1+a)(2a+3)(s_3+s_2-a-1)\right)^{-1}. \tag{4.8.28}$$

The solution to equation (4.8.23) for c_a depends on two parameters: c_0 and C_{s_1,s_2,s_3} (which enters through the d_a), but imposing the extra constraint (4.8.13) determines the c_0 in terms of C_{s_1,s_2,s_3}. Alternatively, we can obtain a relation between c_0 and C_{s_1,s_2,s_3} by demanding conservation with respect to the other currents, before taking the light-like limit.

Quasi-fermionic scalars

For parity-odd correlation functions involving the (parity-odd) twist-two quasi-fermionic scalar operator, we have:

$$\langle j_{s_1} j_{\tilde{0}} j_{s_3}\rangle = \frac{1}{|x_{12}|^2|x_{23}|^2}\sum_a^{s_3-1} c_a Q_1^{s_1-a}P_2^{2a}P_1^{2(s_3-a)}Q_2^{-s_3+a}. \tag{4.8.29}$$

and, using

$$\tilde{\mathcal{D}}_3 = -(2+2P_1\partial_{P_1}+2Q_2\partial_{Q_2})Q_1\partial_{P_2}^2 + (2P_2\partial_{P_2}+2Q_1\partial_{Q_1})Q_2\partial_{P_1}^2. \tag{4.8.30}$$

we can write its divergence as,

$$\partial_{x_3^\mu} D_{z_3}^\mu \langle j_{s_1} j_{\tilde{0}} j_{s_3}(x_3, z_3)\rangle = \frac{1}{|x_{23}|^4 |x_{31}|^2} \sum_{a=0}^{s_3-1} d_a Q_1^{s_1-a} P_2^{2a} P_1^{2(s_3-a-1)} Q_2^{-s_3+a+1}. \tag{4.8.31}$$

with

$$d_a = (s_3 - a)(s_1 + a)(2s_3 - 2a - 1)c_a - (a+1)(2a+1)(s_3 - a)c_{a+1}. \tag{4.8.32}$$

Comparing to our earlier expression (4.6.3), we find

$$d_{s_1+n} = \binom{s_3 - s_1 - 1}{n} (-1)^{s_3 - s_1 - 1} \frac{(s_3 - s_1)(s_3 + s_1 - 1)!}{2^{2s_1}(2s_1)!} n_{s_1} n_{\tilde{0}} C_{s_1, \tilde{0}} \tag{4.8.33}$$

for $n = 0, \ldots, s_3 - s_1 - 1$.

The recurrence relation depends on two unknown parameters: c_0 and $C_{s_1,0,s_3}$. By requiring the correlation function to vanish when $Q_3 = 0$, we obtain a relation between these two parameters.

4.8.2 Conserved parity-odd three-point functions vanish outside the triangle inequality

Above, we showed that the conserved parity-odd three-point function vanishes outside the triangle inequality in the light-like limit, where $P_3 = 0$ and $S_3 = 0$. Let us extend this to a proof that the conserved parity-odd three-point functions vanishes even outside the light-like limit.

Our strategy is to expand the three-point function as a power series in P_3, where we count $S_3 \sim P_3$. Let $m > 0$, and use induction. If we assume all terms of order P_3^{m-1} vanish, we can show that conservation implies the terms of order P_3^m must also vanish.

The most general three-point function can be written as:

$$\langle j_{s_1} j_{s_2} j_{s_3}(x_3, z_3)\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} f(P_i, Q_i, S_i). \tag{4.8.34}$$

By writing down the most general allowed form for $f(P_i, Q_i, S_i)$ that is order P_3^{2m}, and dropping terms of order P_3^{2m+1} and higher we can see that, if $s_3 > s_1 + s_2$, any term we write down must be proportional to Q_3^z, where $z \geq 1$. Hence the correlation function must vanish in the limit that $Q_3 = 0$.

We can also write the three-point function in terms of Q_1, Q_2, P_1, P_2 only, by explicitly solving the constraints in [174] and allowing negative exponents for the Q_i:

$$\langle j_{s_1} j_{s_2} j_{s_3}(x_3,z_3)\rangle = P_3^{2m}\frac{1}{|x_{12}||x_{23}||x_{31}|}\sum_{a=0}^{s_3-1}\Big(c_a Q_1^{s_1-m-1-a}P_2^{2a}P_1^{2(s_3-1-a)}Q_2^{s_2-m-s_3+1+a}S_2\Big). \tag{4.8.35}$$

Then, imposing conservation, we find an essentially identical recurrence relation to Equation (4.8.18) above. When we also impose the condition that it vanishes when $Q_3 = 0$, as in equation (4.8.13), we find that there is no solution.

4.9 Appendix: List of parity-odd three-point functions

We present the non-zero parity-odd three-point functions outside the triangle inequality, for spins up to 6 in the theory, using values of C_{s_1,s_2,s_3} derived from the classical equations of motion.

Correlation Functions involving a Quasi-Fermionic Scalar The correlation functions listed below are to be multiplied by

$$\tilde{N}\tilde{\lambda}\frac{1}{|x_{12}|^2|x_{23}|^2}.$$

(We remind the reader that our normalization for the scalar, given in equation (4.3.6), is such that this is exact to all orders in $\tilde{\lambda}$.)

$$\begin{aligned}
\langle j_1 j_{\tilde{0}} j_3\rangle &= \frac{Q_3^2\left(2P_2^2+Q_1Q_3\right)}{8\pi^4}\\
\langle j_2 j_{\tilde{0}} j_4\rangle &= -\frac{Q_3^2\left(-10P_2^2Q_1Q_3+P_2^4-Q_1^2Q_3^2\right)}{8\pi^4}\\
\langle j_1 j_{\tilde{0}} j_5\rangle &= \frac{Q_3^4\left(4P_2^2+Q_1Q_3\right)}{8\pi^4}\\
\langle j_3 j_{\tilde{0}} j_5\rangle &= -\frac{Q_3^2\left(P_2^4Q_1Q_3-22P_2^2Q_1^2Q_3^2+16P_2^6-Q_1^3Q_3^3\right)}{8\pi^4}\\
\langle j_2 j_{\tilde{0}} j_6\rangle &= \frac{Q_3^4\left(16P_2^2Q_1Q_3+4P_2^4+Q_1^2Q_3^2\right)}{8\pi^4}\\
\langle j_4 j_{\tilde{0}} j_6\rangle &= -\frac{Q_3^2\left(-24P_2^6Q_1Q_3-125P_2^4Q_1^2Q_3^2-38P_2^2Q_1^3Q_3^3+62P_2^8-Q_1^4Q_3^4\right)}{8\pi^4}
\end{aligned}$$

Correlation functions involving a Quasi-Bosonic Scalar The correlation functions listed below are to be multiplied by

$$\tilde{N}\tilde{\lambda}\frac{1}{|x_{12}||x_{23}||x_{31}|}.$$

$$\langle j_1 j_0 j_3\rangle = -\frac{iQ_3^2 S_2}{16\pi^4} \quad (4.9.1)$$

$$\langle j_2 j_0 j_4\rangle = -\frac{iQ_3^2 S_2\left(4P_2^2+Q_1Q_3\right)}{16\pi^4} \quad (4.9.2)$$

$$\langle j_1 j_0 j_5\rangle = -\frac{iQ_3^4 S_2}{16\pi^4} \quad (4.9.3)$$

$$\langle j_3 j_0 j_5\rangle = -\frac{iQ_3^2 S_2\left(2P_2^2+Q_1Q_3\right)\left(6P_2^2+Q_1Q_3\right)}{16\pi^4} \quad (4.9.4)$$

$$\langle j_2 j_0 j_6\rangle = -\frac{iQ_3^4 S_2\left(6P_2^2+Q_1Q_3\right)}{16\pi^4} \quad (4.9.5)$$

$$\langle j_4 j_0 j_6\rangle = -\frac{iQ_3^2 S_2\left(107P_2^4Q_1Q_3+40P_2^2Q_1^2Q_3^2+102P_2^6+3Q_1^3Q_3^3\right)}{48\pi^4} \quad (4.9.6)$$

All Spins nonzero The correlation functions listed below are to be multiplied by

$$\tilde{N}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}\frac{1}{|x_{12}||x_{23}||x_{31}|}.$$

We also omit an overall numerical normalization constant (which is in principle determinable from C_{s_1,s_2,s_3} and the recurrence relations given above.) These are all valid outside the light-like OPE limit as well. To fix the coefficients of terms that vanish in the light-like limit, we also imposed conservation with respect to x_1 and x_2.

$$\langle j_1 j_2 j_5 \rangle \sim Q_3^2\Big(-6P_1^4 S_2 + 6P_1^2 Q_1 Q_3 S_1 + 15P_1^2 Q_2 Q_3 S_2 - P_1^2 Q_3^2 S_3 + 5P_2^2 Q_2 Q_3 S_1 + P_3^2 Q_3^2 S_1 + Q_2 Q_3^3 S_3\Big) \tag{4.9.7}$$

$$\langle j_1 j_1 j_6 \rangle \sim Q_3^5 (3Q_1 S_1 + 3Q_2 S_2 - 2Q_3 S_3) \tag{4.9.8}$$

$$\langle j_1 j_3 j_6 \rangle \sim Q_3^2\Big(72P_1^6 S_2 - 32P_1^4 Q_1 Q_3 S_1 - 208P_1^4 Q_2 Q_3 S_2 + 9P_1^4 Q_3^2 S_3 - 148P_1^2 P_2^2 Q_2 Q_3 S_1 - 58P_1^2 Q_2 Q_3^3 S_3 + 26P_1 P_2 P_3 Q_2 Q_3^2 S_1 - 18P_2^2 Q_2^2 Q_3^2 S_1 - 6P_3^2 Q_2 Q_3^3 S_1 - 3Q_2^2 Q_3^4 S_3\Big) \tag{4.9.9}$$

$$\langle j_2 j_2 j_6 \rangle \sim Q_3^3\Big(14P_1^4 Q_1 S_2 - 88P_1^3 P_2 P_3 S_2 + 95P_1^2 P_2^2 Q_2 S_2 + 27P_1^2 P_2^2 Q_3 S_3 + 7P_1^2 Q_1^2 Q_3 S_1 + 7P_1 P_2 P_3 Q_2 Q_3 S_2 - 39P_1 P_2 P_3 Q_3^2 S_3 + 21P_2^4 Q_2 S_1 + 7P_2^2 Q_2 Q_3^2 S_3 + 3P_3^2 Q_3^3 S_3\Big) \tag{4.9.10}$$

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