Topological Structures in Supersymmetric Gauge Theories

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Abstract

This thesis explores the nonperturbative physics of three-dimensional gauge field theories with varying amounts of supersymmetry through an analysis of their one-dimensional topological substructures.

In the setting of four supercharges, we consider the topological quantum mechanics of Wilson lines in pure Chern-Simons theory. We reframe the perturbative renormalization of observables in this theory in terms of a localization principle associated with an underlying $\mathcal{N} = 2$ supersymmetry. This perspective allows the otherwise perturbative corrections to be interpreted as nonperturbative consequences of a non-renormalization theorem.

In the setting of eight supercharges, we develop an approach to the study of Coulomb branch operators in 3D $\mathcal{N} = 4$ gauge theories and the deformation quantization of their Coulomb branches. To do so, we leverage the existence of a 1D topological subsector whose operator product expansion takes the form of an associative and noncommutative algebra on the Coulomb branch. For "good" and "ugly" theories in the Gaiotto-Witten classification, we exhibit a trace map on this algebra, which amounts to a procedure for computing exact correlation functions of a class of local operators, including certain monopole operators, on S^3. We introduce a "shift operator" formalism for constructing correlators on S^3 by gluing hemispheres HS^3, and we show how to recover our results by dimensionally reducing the line defect Schur index of 4D $\mathcal{N} = 2$ gauge theories. We use our results to study 3D mirror symmetry and to characterize monopole bubbling effects in nonabelian gauge theories. In the process, we arrive at a physical proof of the Bullimore-Dimofte-Gaiotto abelianization description of the Coulomb branch.

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“What we do here is nothing to what we dream of doing.”

— Marquis de Sade, *Justine*[1]

(borrowed from the epigraph of Sidney Coleman’s Ph.D. thesis)

[1] Dubious attribution.

Contents

Chapter 1

Introduction and Summary

1.1 Motivation

By extending the square root operation to the entirety of the real numbers, all polynomial equations are rendered soluble. The applications of the resulting theory of complex analysis have percolated through mathematics over the past several centuries. While the practical utility of complex numbers in (classical) physics has been appreciated since the early days of electrodynamics, a physicist in the time of William Rowan Hamilton may very well have objected to the ontological necessity of introducing the fictitious $\sqrt{-1}$. After all, numbers measured in the lab are *real*. Must a reasonable description of nature make essential reference to unobservable quantities? It was not until the advent of quantum mechanics that complex numbers took on a more foundational role [4], ensuring that the Schrödinger equation (unlike the heat equation) is invariant under time reversal. Regardless of whether complex numbers are emergent or written into the "source code" of nature, they have come to form part of our "user interface" for understanding the quantum world.

Similar ontological questions echo in the present day. The symmetries of the world as we know it are described by the Poincaré algebra, which unifies spacetime translations and Lorentz transformations. There exist hypothetical extensions of Poincaré symmetry whose signatures have so far eluded observation. One such extension is *supersymmetry.* While the study of supersymmetry faces no shortage of phenomenological justifications (ranging from the solution of the electroweak hierarchy problem to grand unification), the following simple reason may be compelling enough:

Supersymmetry is to Poincaré symmetry as the complex numbers are to the reals.

Indeed, the super-Poincaré algebra arises quite literally from taking the "square root" of translations. Pursuing this analogy, theories with greater degrees of supersymmetry exhibit scalar moduli spaces with increasingly constrained geometries (Kähler, hyperkähler, ...), leading to mathematical structures that generalize the complex numbers, namely quaternions and octonions.[1] Just as Hurwitz's theorem (which establishes $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ as the only normed division algebras over the reals) tells us that there is a limit to how far these generalizations can go, there is a limit to the amount of supersymmetry that a relativistic quantum field theory can have.[2]

As with complex numbers, we may be agnostic about whether supersymmetry is fundamental in nature. And like complex numbers, supersymmetry is often *hidden*, simplifying physical arguments and appearing in unexpected contexts. A rather prosaic example of a supersymmetry is the Becchi-Rouet-Stora-Tyutin (BRST) symmetry familiar from quantizing ordinary gauge theories.

Another example comes from elementary quantum mechanics: the problem of an electrically charged particle in the field of a magnetic monopole [6]. This example turns out to contain important notions that will recur throughout this thesis, so we pause to discuss it in some detail. For simplicity, we take the particle to have unit charge and constrain it to move on a sphere S^2 of unit radius around the monopole. Its Hamiltonian is

$$H = -\frac{1}{2M}(\nabla - i\vec{A})^2|_{r=1}, \tag{1.1}$$

[1] The octonions are an exceptional case, appearing in 10D maximal supersymmetry [5].

[2] Our discussion here is restricted to rigid (as opposed to local) supersymmetry. The limit on the number of supercharges follows from the Weinberg-Witten theorem in 4D, combined with dimensional reduction or "oxidation."

where $\vec{A}$ is an appropriate vector potential. For concreteness, we take

$$\vec{A} = \frac{j(1+\cos\theta)}{r\sin\theta}\hat{\varphi}, \tag{1.2}$$

which is divergenceless and singular along the positive z-axis. Here, j denotes the magnetic charge. The standard angular momentum operators

$$\vec{D} = -i\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}, \cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right), \tag{1.3}$$

satisfying $[D_i, D_j] = i\epsilon_{ijk}D_k$, do not commute with H. Rather, the "good" angular momenta (accounting for the contribution from the electromagnetic field) are

$$\vec{L} = -i\vec{r}\times(\nabla - i\vec{A}) + \frac{j\vec{r}}{r} = \vec{D} + j\left(\frac{\sin\theta\cos\varphi}{1-\cos\theta}, \frac{\sin\theta\sin\varphi}{1-\cos\theta}, -1\right), \tag{1.4}$$

which likewise satisfy $[L_i, L_j] = i\epsilon_{ijk}L_k$. We may then diagonalize H as follows:

$$H = \frac{1}{2M}(\vec{L}^2 - j^2) = \frac{1}{2M}(\ell(\ell+1) - j^2), \tag{1.5}$$

where ℓ is an integer or half-integer. To determine which levels ℓ are allowed, note that rotations transform $|\ell, m\rangle$ only into states of the same ℓ. Writing $|\theta,\varphi\rangle = e^{-iL_3\varphi}e^{-iL_2\theta}|\theta = 0\rangle$, the position-space wavefunction of a state $|\ell, m\rangle$ is therefore

$$\langle\theta,\varphi|\ell,m\rangle = e^{im\varphi}\sum_{m'} d^{\ell}_{m'm}(\theta)\langle\theta = 0|\ell, m'\rangle, \tag{1.6}$$

with coefficients given by Wigner d-matrices. But $L_3|\theta = 0\rangle = -j|\theta = 0\rangle$, so by considering the matrix element $\langle\theta = 0|L_3|\ell, m'\rangle$, we derive the selection rule

$$\langle\theta = 0|\ell, m'\rangle = 0 \text{ for } m' \neq -j. \tag{1.7}$$

In particular, $\langle\theta,\varphi|\ell,m\rangle = 0$ unless

$$\ell \geq |j|. \tag{1.8}$$

These are the allowed levels, each occurring once, with ℓ being an integer or half-integer according to the quantized value of j (which we take to be positive for simplicity).

The above system has a spectral gap proportional to $1/M$. Taking the massless limit $M \to 0$ (and adding a constant $-j/2M$ to H, if we like), all states except for those with $\ell = j$ decouple. To elucidate the physical significance of this limit, note that the theory at finite M can be obtained from the classical Lagrangian[3]

$$L = j(1+\cos\theta)\dot{\varphi} + \frac{M}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2). \tag{1.11}$$

In this form, the $M \to 0$ limit is trivial: (1.11) becomes

$$L_0 = j(1+\cos\theta)\dot{\varphi}. \tag{1.12}$$

The massless theory (1.12) has phase space S^2, equipped with a locally exact symplectic (volume) form ω:

$$\omega = j\sin\theta\, d\varphi\wedge d\theta = d\chi, \quad \chi = j(1+\cos\theta)\, d\varphi. \tag{1.13}$$

The action, $S_0 = \int\chi$, computes j times the area enclosed by a trajectory in phase space.[4] Moreover, the half-integral quantization of the coefficient j follows from the Dirac quantization condition. Namely, any closed curve $C \subset S^2$ can be completed to two different disks D_1 and D_2 with $\partial D_1 = \partial D_2 = C$, and consistency of the path integral requires that

$$e^{i\oint_C \chi} = e^{i\int_{D_{1,2}}\omega} \Longleftrightarrow e^{i\int_{S^2}\omega} = 1 \Longleftrightarrow \int_{S^2}\omega = 4\pi j \in 2\pi\mathbb{Z}. \tag{1.14}$$

The operators $(J_x, J_y, J_z) = j(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, when quantized, satisfy the familiar angular momentum algebra. Finally, since the phase space is compact, quantizing it yields a finite-dimensional Hilbert space with $2j+1$ states $|j,m\rangle$, all eigenstates of J_z.

[3]The term linear in $\dot{\varphi}$ may seem irrelevant, as the corresponding classical Hamiltonian is simply

$$H = \frac{M}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2). \tag{1.9}$$

However, it affects the canonical momenta

$$\pi_\varphi = j(1+\cos\theta) + M\sin^2\theta\dot{\varphi}, \quad \pi_\theta = M\dot{\theta}, \tag{1.10}$$

from which the quantum Hamiltonian (1.1) is obtained by writing (1.9) in terms of (1.10) and quantizing.

[4]In the chosen gauge (1.2), the enclosed area lies in the patch excluding the North pole.

Hopefully, the motivation behind our choice of notation for "j" is now clear. Starting from a charged particle in a monopole field, we have derived one of the simplest examples of a *topological quantum field theory* (TQFT), one with trivial Hamiltonian: the famous Wess-Zumino term for quantization of spin (the Lagrangian (1.12) is a Wess-Zumino term because its variation under global $SU(2)$ is a total derivative).

This simple quantum mechanical problem has another surprise in store: it is secretly supersymmetric. One hint in this direction is that the semiclassical approximation to the (J_z-twisted) path integral of the massless theory (1.12) is exact: it computes the character of the spin-j representation of $SU(2)$. This one-loop exactness has a good explanation [7]. We obtained (1.12) as the massless limit of (1.1), thus projecting out all excited states. But we could just as well have considered the supersymmetric counterpart of the theory (1.1), which contains equal numbers of bosonic and fermionic excited states. The contributions of these excited states to the supersymmetric partition function (or flavored Witten index [8]) cancel regardless of the mass, leaving only the contributions of the supersymmetric ground states. This latter perspective reveals the one-loop exactness of the theory (1.12) to be a consequence of the principle of *supersymmetric localization*.

The above example presents a connection between supersymmetry and topology in the context of gauge theory. This entire thesis consists of variations on this theme. Apart from supersymmetric theories, two other special classes of quantum field theories feature prominently in our discussion. First, topological field theories represent a first step beyond quantum mechanics, being particularly easy to quantize and to axiomatize. Second, conformal field theories exhibit a different kind of enhancement of Poincaré symmetry, allowing kinematics to strongly constrain correlation functions. All three classes of theories — supersymmetric, topological, and conformal — provide similar windows into nonperturbative physics, including rigorous understanding of the path integral, exact results in some situations, and "beyond-Lagrangian" techniques for studying theories with multiple Lagrangian descriptions or none at all. Much of our discussion focuses on cases where these three classes overlap, giving rise to what are known as cohomological and superconformal field theories.

Our investigation is divided into two complementary parts. Chapter 2 concerns nonlocal operators in supersymmetric Chern-Simons theories, while Chapter 3 concerns local operators in supersymmetric non-Chern-Simons theories. The phenomena in these chapters are characterized by the existence of four and eight supercharges, respectively, thus manifesting aspects of the Kähler and hyperkähler geometry mentioned at the beginning. The common thread in both of these chapters, as anticipated by the lengthy example above, is the notion of the *magnetic monopole*. Our main setting is three spacetime dimensions, but we also encounter structures in one, two, and four dimensions, whether through dimensional reduction, boundaries, or defects within a bulk theory.

1.2 Monopoles as Wilson Lines

Chapter 2 presents a synthesis of viewpoints on an old subject. It represents a particular attempt to make good on a quantum-field-theoretic analogue of Hadamard's quote that

> "The shortest path between two truths in the real [non-supersymmetric] domain passes through the complex [supersymmetric] domain."

We argue that the well-known perturbative "Weyl shifts" in pure Chern-Simons theory with simple, compact gauge group can be understood in a unified way by embedding the theory in an $\mathcal{N} = 2$ supersymmetric completion. This is accomplished by introducing an auxiliary fermionic symmetry with the aid of generalized Killing spinors. This point of view explains how the quantum corrections persist nonperturbatively in a wide class of observables that are otherwise not one-loop exact, yields a conceptually simpler explanation for the Weyl shifts than that obtained in early literature (involving path integral anomalies [9]), and clarifies what it means for $\mathcal{N} = 0$ and $\mathcal{N} = 2$ Chern-Simons theory to be "equivalent" at the level of line operators.

What role do monopoles play in any of this? Observables in Chern-Simons theory are configurations of Wilson lines. Wilson lines are usually thought of as labeled by representations of the gauge group G. But more fundamentally, a Wilson line describes the infrared (IR) limit of a quantum particle moving on a coadjoint orbit of G, in the presence of a background magnetic field given by the coadjoint symplectic form. This interpretation is valid in *any* gauge theory, and it lies at the core of our analysis. In fact, we have already seen how the story goes for $G = SU(2)$, where the Wilson line quantum mechanics reduces to nothing other than that of a particle on $S^2 = SU(2)/U(1)$ in a monopole field. Thus the physics of Wilson lines is revealed to be a generalization of that of the humble magnetic monopole. This viewpoint on Wilson lines as 1D sigma models occurs naturally in many contexts, not the least of which being quantum gravity [10, 11].

1.3 Monopoles and Supersymmetry

Among supersymmetric theories, those with eight supercharges occupy a "sweet spot" between computability and nontriviality. In Chapter 3, we describe a comprehensive approach to the study of Coulomb branch operators in 3D $\mathcal{N} = 4$ gauge theories. One outcome of this work is the first computation of exact correlation functions involving arbitrary numbers of local defect operators in 3D gauge theories. These operators are the dynamical counterparts of BPS 't Hooft-Wilson loops in 4D $\mathcal{N} = 2$ gauge theories. Central to our techniques is the fact that all 3D $\mathcal{N} = 4$ superconformal field theories (SCFTs) have two protected 1D topological sectors, one associated with the Higgs branch and one with the Coulomb branch [12, 13]. The operator algebras of these sectors yield intricate information about the geometry of the vacuum manifold. While the Higgs branch sector has a Lagrangian description [14], the Coulomb branch sector presents significant new challenges because it involves monopole operators, which are local disorder operators that cannot be expressed as polynomials in the Lagrangian fields.

In [2, 3], we solve the Coulomb branch topological sector of both abelian and nonabelian 3D $\mathcal{N} = 4$ gauge theories. Our techniques combine supersymmetric localization on hemispheres and the algebraic properties of the resulting hemisphere wavefunctions. We use our results to prove abelian mirror symmetry (an IR duality that exchanges Higgs and Coulomb branches, classical and quantum effects, and order and disorder operators) at the level of two- and three-point functions of half-BPS local operators, to derive precise maps between half-BPS operators across nonabelian 3D mirror symmetry, and to compute previously unknown Coulomb branch chiral rings and their deformation quantizations. Our formalism gives a concrete way to determine the coefficients that quantify nonperturbative "monopole bubbling" effects in nonabelian gauge theories:[5] namely, they should be fixed by algebraic consistency of the operator product expansion (OPE) within the Coulomb branch topological sector. Previously, there existed no general algorithm for obtaining these coefficients (direct localization computations of bubbling in 4D $\mathcal{N} = 2$ theories have been performed only for unitary gauge groups with fundamental and adjoint matter [17, 18, 19, 20]). Finally, taking the commutative limit of our construction (the 1D OPE) provides a ground-up derivation of the abelianization description of the Coulomb branch proposed in [21].

[5] In which the Goddard-Nuyts-Olive (GNO) charge [15] of a singular monopole is screened away from the insertion point by small 't Hooft-Polyakov monopoles [16].

Chapter 2

Topological Sigma Models in Chern-Simons Theory

2.1 Introduction

Our goal in this chapter is to argue that certain properties of three-dimensional Chern-Simons theory can be understood in a unified way by regarding the theory as an effective description of an $\mathcal{N} = 2$ supersymmetric completion.

The application of supersymmetry to topological field theories has a long history. For instance, both the topological invariance and semiclassical exactness of observables in Witten-type (cohomological) TQFTs have long been recognized as consequences of a fermionic BRST symmetry [22]. The BRST supersymmetry is a restatement of the underlying general covariance of the theory: the subtraction of ghost degrees of freedom guarantees the absence of excited states. By contrast, our approach relies on a further auxiliary supersymmetry. The relevant fermions obey the spin-statistics theorem. At finite Yang-Mills coupling, they result in an infinite tower of states with equal numbers of bosonic and fermionic degrees of freedom, which make no net contribution to supersymmetric observables. However, they have the additional effect of shifting the number of vacuum states. We will argue that this shift, combined with the localization principle afforded by the auxiliary fermionic symmetry, provides a natural framework in which to understand some features of correlation functions in bosonic Chern-Simons theory that are obscure from the point of view of perturbation theory.

2.1.1 Old Perspectives

Let us first recall what is known. It has long been understood that induced Chern-Simons terms are one-loop exact because higher-order corrections (via an expansion in $\hbar \sim 1/k$) cannot, in general, respect the quantization condition on the level [23, 24]. One manifestation of this fact is that quantum observables in pure Chern-Simons theory with simple gauge group G and level $k > 0$, possibly involving Wilson loops in irreducible representations of G labeled by highest weights λ, are naturally viewed as functions not of the "bare" parameters (suitably defined), but of

$$k \to k + h, \quad \lambda \to \lambda + \rho \tag{2.1}$$

where h is the dual Coxeter number and ρ is the Weyl vector of G. For example, when $G = SU(2)$, the shifts read $k \to k + 2$ and $j \to j + 1/2$, and the latter appears at the level of representation theory in the $SU(2)$ Weyl character

$$\chi_j(\theta) = \sum_{m=-j}^{j} e^{im\theta} = \frac{\sin[(j+1/2)\theta]}{\sin(\theta/2)}, \tag{2.2}$$

which (up to a j-independent prefactor) takes the form of a sum over $m = \pm(j + 1/2)$, as familiar from equivariant localization formulas (see [25, 26] and references therein). These shifts can be thought of as quantum corrections.

By now, exact results for Chern-Simons theory have been obtained by various methods that give different ways of understanding the level shift: aside from surgery and 2D CFT [27], these include abelianization on circle bundles over Riemann surfaces [28, 29], nonabelian localization [30, 31], and supersymmetric localization [32, 33]. Of particular relevance to the last approach (such as when performing supersymmetric tests of non-supersymmetric dualities [34, 35]) is the fact that correlation functions in pure $\mathcal{N} = 2$ and $\mathcal{N} = 0$ Chern-Simons coincide up to a shift of the above form: in the $\mathcal{N} = 2$ Chern-Simons action at level $k + h$, all superpartners of the gauge field are auxiliary, and performing the Gaussian path integral over these fields leads to an effective $\mathcal{N} = 0$ Chern-Simons action at level k.

While λ, unlike k, does not appear in the bulk Lagrangian, the associated shift similarly lends itself to a Lagrangian point of view via an auxiliary system attached to the Wilson line, obtained by quantizing the coadjoint orbit of λ. In fact, as we will explain, the weights λ in $\mathcal{N} = 2$ Chern-Simons theory are subject to a "non-renormalization principle" similar to that of k, as can be seen by localizing the corresponding 1D $\mathcal{N} = 2$ theories on Wilson lines.[1] The essence of the 1D localization argument appears in the prototypical system of a massless charged particle on S^2 in the field of a magnetic monopole, which we examined in the introduction to this thesis. In [7], it is shown using a hidden supersymmetry that the semiclassical approximation to the path integral for the monopole problem is exact. The upshot is a derivation of the Weyl character formula for $SU(2)$ from supersymmetric quantum mechanics. The same strategy of localizing an apparently purely bosonic theory has many modern incarnations: see, for example, [37]. Part of our discussion involves giving a slightly more modern formulation of the treatment of the monopole problem in [7], while embedding it into Chern-Simons theory.

2.1.2 New Perspectives

Our goal is to explain how supersymmetric localization provides a structural understanding of the aforementioned exact results in the sense that the essential mechanism for both shifts, in the supersymmetric context, is identical in 3D and in 1D.

While the renormalized parameters in (2.1) are one-loop exact, general observables in the $\mathcal{N} = 0$ theory are not, reflecting the fact that Chern-Simons theory is conventionally

[1] 3D $\mathcal{N} \geq 2$ theories are precisely those whose holomorphy properties allow them to be constrained by non-renormalization theorems [36].

formulated as a Schwarz-type rather than a Witten-type TQFT (but note that after a suitable topological twist, gauge-fixed Chern-Simons theory does furnish an example of the latter [33]). The real power of supersymmetry lies in its ability to explain how the shifts (2.1) persist nonperturbatively in a wide class of observables. Enhancing both the 3D Chern-Simons action and the 1D coadjoint orbit action for Wilson loops with $\mathcal{N} = 2$ supersymmetry gives one access to a localization argument that ensures that correlation functions depend only on the bare couplings appearing in the respective actions. This is a sort of non-renormalization principle. These two supersymmetrizations are not independent, as there exists a precise map between fields in the bulk and fields on the line. The supersymmetric, coupled 3D-1D path integral can be evaluated exactly, and after adjusting for induced Chern-Simons terms from integrating out the auxiliary fermions (in 3D and in 1D), we immediately deduce the exact result in the corresponding bosonic theory, including the famous shifts. In this way, a one-loop supersymmetric localization computation reproduces an all-loop result in the bosonic theory. This reasoning leads to a conceptually simpler explanation for (2.1) than that originally obtained from anomalies in the coherent state functional integral [9].

Making the above statements precise requires fixing unambiguous physical definitions of the "bare" parameters k and λ: for example, via the coefficient of the two-point function in the associated 2D current algebra and canonical quantization of the coadjoint orbit theory, respectively.[2] Having done so, the shifts in k and λ arise in a unified fashion from jointly supersymmetrizing the 3D bulk theory and the 1D coadjoint orbit theory, giving rise to three equivalent descriptions of the same theory:

[2]An intrinsically bulk definition of k is as follows. For positive integer k, the Hilbert space of Chern-Simons theory with simply connected G on a Riemann surface Σ is isomorphic to $H^0(\mathcal{M}, \mathcal{L}^k)$ where $\mathcal{M}$ is the moduli space of flat G-connections on Σ and $\mathcal{L}$ is the basic line bundle over $\mathcal{M}$ in the sense of having positive curvature and that all other line bundles over $\mathcal{M}$ take the form $\mathcal{L}^n$ for some integer n [24]. For example, for simple, connected, simply connected G and $\Sigma = T^2$, $\mathcal{M}$ is a weighted projective space of complex dimension $\operatorname{rank} G$ and $\mathcal{L} = \mathcal{O}(1)$ (whose sections are functions of degree one in homogeneous coordinates on $\mathcal{M}$). In the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ settings, fermions have the effect of tensoring $\mathcal{L}^k$ with $K^{1/2}$ or K to give $\mathcal{L}^{k-h/2}$ or $\mathcal{L}^{k-h}$, respectively, where $K = \mathcal{L}^{-h}$ is the canonical bundle of $\mathcal{M}$. Note that these fermions effectively implement the metaplectic correction in geometric quantization [38, 39].

1. The bosonic Chern-Simons theory has level k and Wilson loops

$$\mathrm{Tr}_\lambda\, P\exp\left(i\oint A_\mu dx^\mu\right). \tag{2.3}$$

2. The supersymmetric Chern-Simons theory has level $k+h$ and Wilson loops

$$\mathrm{Tr}_\lambda\, P\exp\left[i\oint (A_\mu dx^\mu - i\sigma ds)\right], \tag{2.4}$$

   where σ is the real scalar in the vector multiplet.

3. The coadjoint orbit description of half-BPS Wilson loops coupled to the bulk supersymmetric theory has level $k+h$ and weight $\lambda+\rho$ from the start; these parameters are not renormalized. The trace in (2.4) is replaced by an appropriate supertrace in a 1D theory containing an auxiliary complex fermion ψ. In the standard presentation of a supersymmetric Wilson loop, the fermion ψ has already been integrated out.

One would in principle expect to be able to match *all* observables between these descriptions, not only those that are protected (BPS) and hence calculable using supersymmetric localization, because the path integral over the auxiliary fields can be performed exactly (shifting $(k+h, \lambda+\rho) \to (k, \lambda)$ and setting $\sigma = 0$, respectively). The main limitation of our analysis is that we are able to demonstrate this equivalence only for correlation functions of Wilson loops that are BPS with respect to the bulk supersymmetry (for which the integration contour implicit in (2.4) is subject to certain constraints).

Our approach involves introducing an auxiliary fermionic symmetry with the aid of generalized Killing spinors, allowing the localization procedure to be carried out on arbitrary Seifert manifolds. The underlying geometric structure that makes this possible is a transversely holomorphic foliation, or THF [40, 41]. It is worth contrasting this approach with that of [33], which avoids assuming the existence of Killing spinors by using a contact structure to define the requisite fermionic symmetry. A contact structure exists on any compact, orientable three-manifold. It is, locally, a one-form κ for which $\kappa \wedge d\kappa \neq 0$; a metric can always be chosen for which $\kappa \wedge d\kappa$ is the corresponding volume form, i.e., such that $*1 = \kappa \wedge d\kappa$ and $*\kappa = d\kappa$. The dual vector field v such that $\iota_v \kappa = 1$ and $\iota_v d\kappa = 0$ is known as the Reeb vector field. It was found in [33] that to carry out the localization, the corresponding Reeb vector field must be a Killing vector field, which restricts this approach to Seifert manifolds (as in [31]); this approach was generalized in [42] to Chern-Simons theories with matter. Therefore, while the geometric basis for our approach differs from that for the cohomological localization of [33, 42], the domain of applicability is the same. Our focus, however, is different: the compensating level shift from auxiliary fermions was ignored in [33], noted in [42], and essential in neither.

We begin by reviewing some background material and setting our conventions in Sections 2.2 and 2.3. We then carry out the analysis for Wilson lines very explicitly for $G = SU(2)$ in Section 2.4 (we comment briefly on the generalization to arbitrary G at the end). Using the description of these lines as 1D $\mathcal{N} = 2$ sigma models, we compute the effective action for fermions at both zero and finite temperature, canonically quantize the system, and present the localization argument in 1D. In Section 2.5, we show how to embed this story in bulk 3D $\mathcal{N} = 2$ Chern-Simons theory. Crucially, while we expect $\mathcal{N} = 0$ and $\mathcal{N} = 2$ Chern-Simons to be equivalent by integrating out the extra fields in the vector multiplet, the equivalence only holds if we take into account *both* the shift of the level and the weight (as discussed further in Section 2.7). In Section 2.6, we describe how to generalize the aforementioned analysis of a Wilson line in flat space to various classes of compact three-manifolds.

2.2 $\mathcal{N} = 0$ Chern-Simons Theory

Let M^3 be a compact, oriented three-manifold and let G be a simple, compact, connected, simply connected Lie group. The latter two assumptions on G ensure that any principal G-bundle P over M^3 is trivial, so that the Chern-Simons gauge field A is a connection on all of P. It then suffices to define the Lorentzian $\mathcal{N}=0$ $G_{k>0}$ Chern-Simons action by

$$S_{\rm CS} = \frac{k}{4\pi}\int_{M^3} \text{Tr}\left(A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right). \tag{2.5}$$

We normalize the trace such that the norm squared of the longest root is two.

In flat Minkowski space, we have the $\mathcal{N}=2$ Lagrangians

$$\mathcal{L}_{\rm CS}|_{\mathbb{R}^{1,2}} = \frac{k}{4\pi}\text{Tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - 2i\lambda\bar{\lambda} - 2D\sigma\right], \tag{2.6}$$

$$\mathcal{L}_{\rm YM}|_{\mathbb{R}^{1,2}} = \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D^2 + i\bar{\lambda}\gamma^\mu D_\mu\lambda - i\bar{\lambda}[\sigma,\lambda]\right). \tag{2.7}$$

These are written in the convention where the generators T^a are Hermitian.[3] We sometimes switch from Lorentzian to Euclidean signature when natural, namely when computing the supersymmetric index (Section 2.4.3) and when working in curved space (Section 2.6).

2.2.1 Perturbation Theory

The level of the pure $\mathcal{N}=2$ CS theory whose correlation functions reproduce those of the corresponding $\mathcal{N}=0$ theory is $k_{\mathcal{N}=2} = k_{\mathcal{N}=0} + h$ ($k_{\mathcal{N}=0} > 0$ by assumption). This we refer to as the "fermionic shift": the IR effective action $S_{\rm eff}[A,m]$ for two adjoint Majorana fermions with real mass m, minimally coupled to a G-gauge field, is $S_{\rm CS}$ at level $h\,\text{sign}(m)$ [43, 44]. Specifically, consider the sum of (2.6) and (2.7). The resulting theory has a mass gap of $m = kg^2/2\pi$. At large k ($m \gg g^2$), we may integrate out all massive superpartners of the gauge field. Assuming unbroken supersymmetry, the result is the low-energy effective theory of zero-energy supersymmetric ground states. Of course, the fact that integrating out λ induces $\mathcal{L}_{\rm CS}^{\mathcal{N}=0}$ at level $-h$ (among other interactions), along with the assumption that $\mathcal{N}=2$ SUSY is preserved quantum-mechanically, is only a heuristic justification for the renormalization of the coefficient of $\mathcal{L}_{\rm CS}^{\mathcal{N}=2}$ to $k-h$. This expectation is borne out by computing the one-loop perturbative renormalization of couplings [45].

[3] WLOG, we may take $k>0$ because time reversal (equivalently, spacetime orientation reversal in Euclidean signature) flips the overall sign of (2.6), i.e., the sign of the bosonic Chern-Simons term, the sign of the gaugino mass term, *and* the sign of the pseudoscalar σ.

The fermionic shift discussed above is entirely separate from any "bosonic shift" that might arise from gauge dynamics (as found in, e.g., [46], which effectively integrates out the topologically massive W-boson). Such a shift does not affect the number of vacuum states. Indeed, it is an artifact of regularization scheme: in the Yang-Mills–Chern-Simons regularization (which preserves supersymmetry, and which we use throughout), the IR level is shifted by $+h$ relative to the bare level, while dimensional regularization yields no such shift [23]. It is, nonetheless, a convenient conceptual slogan that k is renormalized to $k+h$ at one loop in $\mathcal{N}=0$ YM-CS, so that k is not renormalized in $\mathcal{N}\geq 2$ YM-CS. The important point is that for $\mathcal{N}\geq 2$ supersymmetry, integrating out the gauginos in the 3D YM-CS Lagrangian yields a shift of $-h$, which is twice the shift of $-h/2$ in the $\mathcal{N}=1$ case [45].

Given a precise physical definition of the level k, such as those presented in the introduction, a more substantive "bosonic" shift of the form mentioned above is that exhibited by correlation functions of $\mathcal{N}=0$ Chern-Simons theory as functions of k. This can already be seen in the semiclassical limit [27]. At large k, we may expand (2.5) to quadratic order around a flat connection A_0. The semiclassical path integral evaluates to its classical value weighted by the one-loop contribution $e^{i\pi\eta(A_0)/2}T(A_0)$ where $T(A_0)$ is the Ray-Singer torsion of A_0. The Atiyah-Patodi-Singer index theorem implies that the relative η-invariant

$$\frac{1}{2}(\eta(A_0)-\eta(0))=\frac{h}{\pi}I(A_0), \tag{2.8}$$

where $I(A_0)\equiv\frac{1}{k}S_{\mathrm{CS}}(A_0)$, is a topological invariant. The large-k partition function is then

$$Z=e^{i\pi\eta(0)/2}\sum_{\alpha}e^{i(k+h)I(A_0^{(\alpha)})}T(A_0^{(\alpha)}), \tag{2.9}$$

where the sum (assumed finite) runs over gauge equivalence classes of flat connections. This is how the shift $k\to k+h$, which persists in the full quantum answer, appears perturbatively. The phase $\eta(0)$ depends on the choice of metric. However, given a trivialization of the tangent bundle of M^3, the gravitational Chern-Simons action $I_{\mathrm{grav}}(g)$ has an unambiguous definition, and upon adding a counterterm $\frac{\dim G}{24} I_{\text{grav}}(g)$ to the action, the resulting large-k partition function is a topological invariant of the framed, oriented three-manifold M^3 [27].

Thus a framing of M^3 fixes the phase of Z. Aside from the framing anomaly of M^3 itself, there exists a framing ambiguity of links within it. This point will be important in our application: the supersymmetric framing of a BPS Wilson loop differs from the canonical framing, when it exists, because the point splitting that determines the self-linking number must be performed with respect to another BPS loop [32].

To make concrete the utility of supersymmetry, take as an example $\mathcal{N} = 0$ $SU(2)_k$ on S^3. A typical observable in this theory, such as the partition function

$$Z(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right), \tag{2.10}$$

receives contributions from all loops (for a review of large-k asymptotics of Chern-Simons invariants, see [47]). On the other hand, a one-loop supersymmetric localization computation in $\mathcal{N} = 2$ $SU(2)_{k+2}$ on S^3 handily yields the all-loop non-supersymmetric result (2.10), up to a framing phase. The bulk of our discussion will focus on more complicated observables that include Wilson loops.

2.2.2 Beyond Perturbation Theory

As known since [27], there exist completely general nonperturbative techniques for computing observables in the $\mathcal{N} = 0$ theory, based on surgery, and thus checks of any results obtained via supersymmetry.

To give a few examples of nonperturbative results computed by these means (stated in the canonical framing), consider $G_{k>0}$ on S^3. Let S_{ij} be the representation of the modular transformation S on T^2 in the Verlinde basis for $\mathcal{H}_{T^2}$. Then

$$Z(S^3) = S_{00} = \frac{1}{(k+h)^{\operatorname{rank} G/2}} \left(\frac{\operatorname{vol} \Lambda_W}{\operatorname{vol} \Lambda_R}\right)^{1/2} \prod_{\alpha>0} 2 \sin\left(\frac{\pi \alpha(\rho)}{k+h}\right), \tag{2.11}$$

while for an unknotted Wilson loop in an irreducible representation R_i,

$$\langle W \rangle = \frac{Z(S^3; R_i)}{Z(S^3)} = \frac{S_{0i}}{S_{00}} = \prod_{\alpha>0} \frac{\sin(\pi\alpha(\lambda+\rho)/(k+h))}{\sin(\pi\alpha(\rho)/(k+h))}. \tag{2.12}$$

Here, α runs over positive roots and λ is the highest weight of R_i. The expressions in terms of S-matrix elements were deduced in [27], while the explicit formulas in (2.11) and (2.12) are consequences of the Weyl denominator and character formulas [48].[4] In particular, for $SU(2)_k$,

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left[\frac{(2i+1)(2j+1)\pi}{k+2}\right] \tag{2.13}$$

where i, j label the spins of the corresponding representations (thus giving (2.10)), and for an unknotted Wilson loop in the spin-j representation,

$$\langle W \rangle = \frac{S_{0j}}{S_{00}} = \frac{q^{j+1/2} - q^{-(j+1/2)}}{q^{1/2} - q^{-1/2}} = \frac{\sin((2j+1)\pi/(k+2))}{\sin(\pi/(k+2))} \tag{2.14}$$

where $q = e^{2\pi i/(k+2)}$.

In some observables, highest weights of integrable representations of the G_k theory appear not due to explicit Wilson loop insertions, but rather because they are summed over. Indeed, the shift in λ already appears in the partition function on $\Sigma \times S^1$, which computes the dimension of the Hilbert space of the Chern-Simons theory on Σ and hence the number of conformal blocks in the corresponding 2D rational CFT. The answer is famously given by the Verlinde formula, which for arbitrary compact G, reads [28]

$$\dim V_{g,k} = (|Z(G)|(k+h)^{\operatorname{rank} G})^{g-1} \sum_{\lambda\in\Lambda_k} \prod_{\alpha} (1 - e^{2\pi i\alpha(\lambda+\rho)/(k+h)})^{1-g} \tag{2.15}$$

where g is the genus of Σ and Λ_k denotes the set of integrable highest weights of $\widehat{G}_k$. While our focus is on Wilson loops, it turns out that the appearance of $\lambda + \rho$ in $Z(\Sigma \times S^1)$ comes "for free" in our approach, without the need to adjust for any 1D fermionic shifts, which

[4] The result for $Z(S^3)$ follows from consistency between two different ways of gluing together two copies of a solid torus $D^2 \times S^1$: one trivially to get $S^2 \times S^1$, and another with an S transformation on the boundary to get S^3. More generally, by inserting Wilson lines in these solid tori, one obtains the expectation value of the Hopf link as a normalized S-matrix element.

is consistent with the fact that the weights in (2.15) are not associated with Wilson loops. This fact has already been appreciated in prior literature.[5]

2.3 Wilson Loops and Coadjoint Orbits

2.3.1 The Orbit Method

A central ingredient in our analysis is the fact that a Wilson loop over a curve γ in M^3 is a path integral for a 1D Chern-Simons theory whose classical phase space is a coadjoint orbit of G, with the corresponding representation R arising by the orbit method [27]. We will be interested in the case of compact G, where this construction is also known as Borel-Weil-Bott quantization. The philosophy is that one can eliminate both the trace and the path ordering from the definition of a Wilson loop in a nonabelian gauge theory at the cost of an additional path integral over all gauge transformations along γ.

To make this description explicit, we draw from the exposition of [31]. We would like to interpret a Wilson loop as the partition function of a quantum-mechanical system on γ with time-dependent Hamiltonian. In the Hamiltonian formalism, this is a matter of writing

$$W_R(\gamma) = \operatorname{Tr}_R P \exp\left(i \oint_\gamma A\right) = \operatorname{Tr}_{\mathcal{H}} T \exp\left(-i \oint_\gamma H\right) \tag{2.16}$$

where the Hilbert space $\mathcal{H}$ is the carrier space of the representation R, H generates translations along γ, and the time evolution operator is the holonomy of the gauge field. In the path integral formalism, this becomes

$$W_R(\gamma) = \int DU\, e^{iS_\lambda(U, A|_\gamma)} \tag{2.17}$$

where U is an auxiliary bosonic field on γ, λ is the highest weight of R, and the restriction of the bulk gauge field $A|_\gamma$ is a background field in the (operator-valued) path integral over U.

[5]Supersymmetric localization has been used to compute the Verlinde formula in genus zero [49] and in arbitrary genus [50, 51], reproducing the result of [28] for $Z_{T^3} = \dim \mathcal{H}_{T^2}$ when $g = 1$.

Since the definition of a Wilson loop is independent of any metric on γ,[6] it is not surprising that the action S_λ will turn out to describe a topological sigma model.

The Borel-Weil-Bott theorem identifies the irreducible representation R with the space of holomorphic sections of a certain line bundle over the coadjoint orbit $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ of λ, which (in the generic case) is isomorphic to the flag manifold G/T where T is a maximal torus of G. In physical terms, it states that R is the Hilbert space obtained by quantizing $\mathcal{O}_\lambda$. We are therefore led to consider the quantum mechanics of a particle on $\mathcal{O}_\lambda$ given by a 1D sigma model of maps $U : S^1 \to \mathcal{O}_\lambda$, where the compact worldline is identified with $\gamma \subset M^3$. To ensure that $\mathcal{O}_\lambda$ (rather than $T^*\mathcal{O}_\lambda$) appears as the classical phase space, the action for U must be first-order in the time derivative along S^1. Moreover, on general grounds, it should be independent of the metric on S^1.

There is an essentially unique choice of action that fulfills these wishes. For convenience, we identify λ via the Killing form as an element of $\mathfrak{g}$ rather than $\mathfrak{g}^*$, so that $\mathcal{O}_\lambda \subset \mathfrak{g}$ is the corresponding adjoint orbit (henceforth, we shall not be careful to distinguish $\mathfrak{g}$ and $\mathfrak{g}^*$). We assume that λ is a regular weight, so that $\mathcal{O}_\lambda \cong G/G_\lambda$ where $G_\lambda \cong T$. The (left-invariant) Maurer-Cartan form θ is a distinguished $\mathfrak{g}$-valued one-form on G that satisfies $d\theta + \theta \wedge \theta = 0$. We obtain from it two natural forms on G, namely the real-valued presymplectic one-form Θ_λ and the coadjoint symplectic two-form ν_λ:

$$\theta = g^{-1}dg \in \Omega^1(G) \otimes \mathfrak{g}, \qquad \Theta_\lambda = i\,\mathrm{Tr}(\lambda\theta) \in \Omega^1(G), \qquad \nu_\lambda = d\Theta_\lambda \in \Omega^2(G). \tag{2.18}$$

Both Θ_λ and ν_λ descend to forms on $\mathcal{O}_\lambda$. The weight λ naturally determines a splitting of the roots of G into positive and negative, positive roots being those having positive inner product with λ. Endowing $\mathcal{O}_\lambda$ with the complex structure induced by this splitting makes

[6] This is not true of its supersymmetric counterparts.

$\mathcal{O}_\lambda$ a Kähler manifold, with Kähler form ν_λ of type $(1,1)$.[7] Now consider the action

$$S_\lambda(U) = \oint_{S^1} U^*(\Theta_\lambda) = \oint_{S^1} (\Theta_\lambda)_m \frac{dU^m}{d\tau}\, d\tau. \tag{2.19}$$

The second expression (written in local coordinates U^m on $\mathcal{O}_\lambda$) is indeed first-order in derivatives, so that the solutions to the classical EOMs are constant maps U, as desired.

To be concrete, we may think of U as parametrizing gauge transformations. Using the isomorphism $G/G_\lambda \xrightarrow{\sim} \mathcal{O}_\lambda$ given by $gG_\lambda \mapsto g\lambda g^{-1}$, we lift U to a map $g : S^1 \to G$, so that

$$S_\lambda(U) = i\oint_{S^1} \mathrm{Tr}(\lambda g^{-1} dg). \tag{2.20}$$

From (2.20), we see very explicitly that the canonical symplectic form ν_λ on $\mathcal{O}_\lambda$, given in (2.18), takes the form $d\pi_g \wedge dg$ where the components of g are canonical coordinates. The fact that $\lambda \in \mathfrak{g}$ is quantized as a weight of G implies that (2.20) is independent of the choice of lift from $\mathcal{O}_\lambda$ to G. Namely, g is only determined by U up to the right action of G_λ; under a large gauge transformation $g \mapsto gh$ where $h : S^1 \to G_\lambda$, the integrand of (2.20) changes by $d\,\mathrm{Tr}(\lambda \log h)$ and the action changes by an integer multiple of 2π.[8] Thus Θ_λ descends (up to exact form) to $\mathcal{O}_\lambda$. The path integral (2.17) is over all maps U in $L\mathcal{O}_\lambda$, or equivalently, over all maps g in LG/LG_λ (accounting for the gauge redundancy).

To couple (2.20) to the bulk gauge field, we simply promote dg to $d_A g = dg - iA|_\gamma \cdot g$:

$$S_\lambda(U, A|_\gamma) = i\oint_{S^1} \mathrm{Tr}(\lambda g^{-1} d_A g). \tag{2.21}$$

Prescribing the correct gauge transformations under $G \times T$ (with T acting on the right and G acting on the left), the 1D Lagrangian transforms by the same total derivative as before.

[7] This is usually phrased as a choice of Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$, so that the coadjoint orbit is isomorphic to $G_\mathbb{C}/B$ where B is the corresponding Borel subgroup and the roots of B are defined to be the positive roots of G; then representations are labeled by their lowest weights. We instead adhere to the "highest weight" conventions of [31].

[8] From the geometric quantization point of view, the quantization of λ is necessary for the existence of a prequantum line bundle $\mathcal{L}(\lambda)$ over $\mathcal{O}_\lambda$, with curvature ν_λ. Each λ in the weight lattice gives a homomorphism $\rho_\lambda : T \to U(1)$, which can be used to construct an associated line bundle $\mathcal{L}(\lambda) = G \times_{\rho_\lambda} \mathbb{C}$ over G/T, so that the Hilbert space is the space of holomorphic sections of $\mathcal{L}(\lambda)$. Then Θ_λ is a connection on $\mathcal{L}(\lambda)$.

The first-order action (2.20), in the absence of a background gauge field, can be thought of as describing the IR limit of a charged particle on $\mathcal{O}_\lambda$ in a magnetic field ν_λ. In complete analogy to 3D Chern-Simons theory, the irrelevant two-derivative kinetic terms have the effect of renormalizing λ to $\lambda+\rho$ at one loop, and upon supersymmetrizing the theory, the fermion effective action provides a compensating shift by $-\rho$.[9] We will substantiate this interpretation for $G = SU(2)$ in exhaustive detail.

2.3.2 Wilson/'t Hooft Loops in Chern-Simons Theory

While the coadjoint representation of a Wilson loop holds in any gauge theory, it is especially transparent in Chern-Simons theory, where it can be derived straightforwardly via a surgery argument [52]. Consider Chern-Simons on $S^1 \times \mathbb{R}^2$, where the Wilson line wraps the S^1 at a point on the $\mathbb{R}^2$. Cutting out a small tube around γ and performing a gauge transformation $\tilde{g}$, the action changes by

$$\Delta S = -\frac{ik}{2\pi}\int_{\partial M^3} \mathrm{Tr}(A\tilde{g}^{-1}d\tilde{g}). \tag{2.22}$$

Set $\tilde{g} = e^{i\alpha\phi}$ where $e^{2\pi i\alpha} = 1$ (this gauge transformation is singular along the loop; t is the coordinate along γ and ϕ the coordinate around it). To define a gauge-invariant operator, average over $\tilde{g} \to g\tilde{g}$ and $A \to gAg^{-1} - idgg^{-1}$ where $g = g(t)$, whereupon this becomes

$$\Delta S = ik\int_\gamma \mathrm{Tr}(\alpha g(\partial_t - iA_t)g^{-1})\, dt, \tag{2.23}$$

where we have performed the ϕ integral and shrunk the boundary to a point. Finally, replace g by g^{-1}. Hence $k\alpha$ must be quantized as a weight λ. This derivation illustrates that Wilson and 't Hooft/vortex [53, 54, 55] loops are equivalent in pure Chern-Simons theory.

To summarize, consider a bulk theory with gauge group G and the 1D Lagrangian

$$\mathcal{L}_{1\mathrm{D}} = i\,\mathrm{Tr}[\lambda g^{-1}(\partial_t - iA)g] \tag{2.24}$$

[9] As in 3D, the effect of these fermions can be compared to that of the metaplectic correction in geometric quantization, which states that wavefunctions should not be viewed as sections of $\mathcal{L}(\lambda)$, but rather as half-densities valued in $\mathcal{L}(\lambda)$, meaning that they belong to $\mathcal{L}(\lambda)\otimes K^{1/2} \cong \mathcal{L}(\lambda-\rho)$ where $K^{1/2}$ is a square root of the canonical bundle of $\mathcal{O}_\lambda$ [38].

where $g \in G$, $A \equiv A|_\gamma$, and $\lambda \in \mathfrak{t}$ (properly, $\lambda \in \mathfrak{t}^*$). Since λ is Hermitian in our conventions, the factor of i ensures that the coadjoint orbit action is real. The Lagrangian (2.24) transforms by a total derivative under t-dependent $G \times T$ gauge transformations

$$g \to h_\ell g h_r, \quad A \to h_\ell A h_\ell^{-1} - i\partial_t h_\ell h_\ell^{-1}, \tag{2.25}$$

namely $i\,\mathrm{Tr}(\lambda \partial_t \log h_r)$, where h_ℓ is the restriction of a G-gauge transformation in the bulk and $h_r \in T$. Hence λ is quantized to be a weight of G. The T-gauge symmetry restricts the degrees of freedom in g to G/T. Quantizing g in this Lagrangian leads to the Wilson line.

Strictly speaking, the global symmetry of the model (2.20) that we gauge to obtain (2.24) is $G/Z(G)$, since the center is already gauged. This should be contrasted with the global symmetry $G \times G/Z(G)$ of a particle on a group manifold with the usual kinetic term $\mathrm{Tr}((g^{-1}\dot{g})^2)$, which consists of isometries of the bi-invariant Killing metric on G.

2.4 Wilson Loops in $\mathcal{N} = 2$ Chern-Simons Theory

We now show that properly defining half-BPS Wilson loops in $\mathcal{N} = 2$ Chern-Simons theory ensures that their weights are not renormalized, in direct parallel to the non-renormalization of the bulk Chern-Simons level. This involves enhancing the sigma model of the previous section with 1D $\mathcal{N} = 2$ supersymmetry in a way compatible with bulk 3D $\mathcal{N} = 2$ supersymmetry.

2.4.1 Shift from Line Dynamics

$\mathcal{N} = 2$ Coadjoint Orbit

We work in Lorentzian 1D $\mathcal{N} = 2$ superspace with coordinates $(t, \theta, \theta^\dagger)$ (see Appendix A.1). Implicitly, we imagine a quantum-mechanical system on a line embedded in $\mathbb{R}^{1,2}$, but we will not need to pass to 3D until the next section. Our primary case study is $G = SU(2)$. We first construct, without reference to the 3D bulk, an $SU(2)$-invariant and supersymmetric coadjoint orbit Lagrangian from the 1D $\mathcal{N}=2$ chiral superfield

$$\Phi = \phi + \theta\psi - i\theta\theta^\dagger\dot{\phi} \tag{2.26}$$

descending from bulk super gauge transformations and the 1D $\mathcal{N}=2$ vector superfields

$$V_i = a_i + \theta\psi_i - \theta^\dagger\psi_i^\dagger + \theta\theta^\dagger A_i \tag{2.27}$$

obtained from restrictions of the bulk fields to the Wilson line, which extends along the 0 direction in flat space. Here, $i = 1, 2, 3$ label the $\mathfrak{su}(2)$ components in the $\vec{\sigma}/2$ basis; ϕ is a complex scalar and ψ is a complex fermion; a_i, A_i are real scalars and ψ_i are complex fermions; and the relevant SUSY transformations are given in (A.6) and (A.8).

We begin by writing (2.24) in a form more amenable to supersymmetrization, namely in terms of a complex scalar ϕ that parametrizes the phase space $SU(2)/U(1) \cong \mathbb{CP}^1$. Take $\lambda = -j\sigma_3$ with $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, which fixes a Cartan; then

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \tag{2.28}$$

is subject to a $U(1)$ gauge redundancy $g \sim ge^{i\theta\sigma_3}$. We identify variables via the Hopf map $SU(2) \to S^2$, followed by stereographic projection:

$$\phi = -\frac{a}{\bar{b}}. \tag{2.29}$$

This map respects the chosen $U(1)$ gauge equivalence: $(a, b) \to (ae^{i\theta}, be^{-i\theta})$. Let us gauge-fix the $U(1)$ action on the right by taking $b = r$ real. Since $|a|^2 + r^2 = 1$, r is only determined by a up to a sign (reflecting the ambiguity in the action of $SU(2)$ on S^2). Note that the gauge fixing breaks down when $|a| = 1$ ($r = 0$). Accounting for the sign ambiguity, we have

$$\phi = -\frac{a}{\pm\sqrt{1-|a|^2}} \implies a = \mp\frac{\phi}{\sqrt{1+|\phi|^2}}, \ r = \pm\frac{1}{\sqrt{1+|\phi|^2}}. \tag{2.30}$$

The relative minus sign is important for ensuring equivariance of the map from a to ϕ with respect to the action of $SU(2)$. Let us fix the overall sign to "$(a, r) = (+, -)$." This is a one-to-one map between the interior of the unit disk $|a| < 1$ and the ϕ-plane that takes the boundary of the disk to the point at infinity. To couple the ϕ degrees of freedom to the gauge field, we work in the basis $\vec{\sigma}/2$, so that

$$A = \frac{1}{2}\begin{pmatrix} A_3 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 \end{pmatrix} \tag{2.31}$$

where the three $\mathfrak{su}(2)$ components $A_{1,2,3}$ are real. Then the non-supersymmetric 1D coadjoint orbit Lagrangian (2.24) can be written as $\mathcal{L}_{1\mathrm{D}} = j\mathcal{L}$ where $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_A$ and

$$\mathcal{L}_0 = \frac{i}{j}\,\mathrm{Tr}(\lambda g^{-1}\partial_t g) = \frac{i(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{1+|\phi|^2}, \tag{2.32}$$

$$\mathcal{L}_A = \frac{1}{j}\,\mathrm{Tr}(\lambda g^{-1} A g) = -\left[\frac{(A_1 + iA_2)\phi + (A_1 - iA_2)\phi^\dagger - A_3(1-|\phi|^2)}{1+|\phi|^2}\right]. \tag{2.33}$$

Note that with Hermitian generators, the Killing form given by Tr is positive-definite.

By promoting ϕ to Φ, we find that the supersymmetric completion of $\mathcal{L}_0$ is

$$\tilde{\mathcal{L}}_0 = \int d^2\theta\, K = \frac{i(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{1+|\phi|^2} - \frac{\psi^\dagger\psi}{(1+|\phi|^2)^2}, \quad K \equiv \log(1+|\Phi|^2). \tag{2.34}$$

We have covered $\mathbb{CP}^1$ with patches having local coordinates Φ and $1/\Phi$, so that K is the Kähler potential for the Fubini-Study metric in the patch containing the origin.

To gauge $\tilde{\mathcal{L}}_0$ in a supersymmetric way and thereby obtain the supersymmetric completion of $\mathcal{L}$ requires promoting the A_i to V_i, which is more involved. Having eliminated the integration variable g in favor of ϕ, let us denote by g what we called h_ℓ in (2.25). Writing finite and infinitesimal local $SU(2)$ transformations as

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \sim \begin{pmatrix} 1 + \frac{i\epsilon_3}{2} & \frac{i\epsilon_1 + \epsilon_2}{2} \\ \frac{i\epsilon_1 - \epsilon_2}{2} & 1 - \frac{i\epsilon_3}{2} \end{pmatrix}, \tag{2.35}$$

finite and infinitesimal gauge transformations take the form

$$A \to gAg^{-1} - i\dot{g}g^{-1} \Longleftrightarrow \delta_{SU(2)}A_i = \epsilon_{ijk}A_j\epsilon_k + \dot{\epsilon}_i, \tag{2.36}$$

$$\Phi \to \frac{a\Phi + b}{-\bar{b}\Phi + \bar{a}} \Longleftrightarrow \delta_{SU(2)}\Phi = \epsilon_i X_i,\ (X_1, X_2, X_3) \equiv \frac{1}{2}(i(1-\Phi^2), 1+\Phi^2, 2i\Phi), \tag{2.37}$$

where the holomorphic $SU(2)$ Killing vectors X_i satisfy $[X_i\partial_\Phi, X_j\partial_\Phi] = \epsilon_{ijk}X_k\partial_\Phi$. Then

$$\delta_{SU(2)}K = \epsilon_i(\mathcal{F}_i + \bar{\mathcal{F}}_i), \quad (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \equiv \frac{1}{2}(-i\Phi, \Phi, i) \tag{2.38}$$

(any purely imaginary $\mathcal{F}_3$ would do, but our choice leads to the "canonical" Noether currents transforming in the adjoint representation). To implement the Noether procedure, we promote the real ϵ_i to complex chiral superfields Λ_i:

$$\delta_{SU(2)}\Phi = \Lambda_i X_i. \tag{2.39}$$

The corresponding change in $\tilde{\mathcal{L}}_0$ can be read off from

$$\delta_{SU(2)}K = \Lambda_i\mathcal{F}_i + \bar{\Lambda}_i\bar{\mathcal{F}}_i - i(\Lambda_i - \bar{\Lambda}_i)J_i \tag{2.40}$$

where the $SU(2)$ Noether currents (Killing potentials) are the real superfields

$$J_i = \frac{iX_i\Phi^\dagger}{1+|\Phi|^2} - i\mathcal{F}_i \implies (J_1, J_2, J_3) = \frac{1}{2}\left(-\frac{\Phi+\Phi^\dagger}{1+|\Phi|^2}, -\frac{i(\Phi-\Phi^\dagger)}{1+|\Phi|^2}, \frac{1-|\Phi|^2}{1+|\Phi|^2}\right), \tag{2.41}$$

which satisfy $J_i^2 = 1/4$ and

$$\delta_{SU(2)}J_i = -\frac{1}{2}\epsilon_{ijk}(\Lambda_j + \bar{\Lambda}_j)J_k + i(\Lambda_j - \bar{\Lambda}_j)J_jJ_i - \frac{i}{4}(\Lambda_i - \bar{\Lambda}_i). \tag{2.42}$$

This generalizes $\delta_{SU(2)}J_i = -\epsilon_{ijk}\epsilon_j J_k$ for real ϵ_i. Now, if we could find a counterterm Γ such that $\delta_{SU(2)}\Gamma = i(\Lambda_i - \bar{\Lambda}_i)J_i$, then we would be done: the supersymmetric completion of $\mathcal{L}$ would be the minimally gauged supersymmetric $\mathbb{CP}^1$ model $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 + \tilde{\mathcal{L}}_A$ where

$$\tilde{\mathcal{L}}_A = \int d^2\theta\,\Gamma, \quad \delta_{SU(2)}\Gamma = i(\Lambda_i - \bar{\Lambda}_i)J_i. \tag{2.43}$$

Note that $\tilde{\mathcal{L}}$ is invariant under local $SU(2)$ because, in light of (2.40), the total variation of $K + \Gamma$ takes the form of a Kähler transformation. There exists a standard procedure for constructing such a Γ [56]. Its exact form is

$$\Gamma = 2\int_0^1 d\alpha\, e^{i\alpha V_i O_i} V_j J_j \tag{2.44}$$

where $O_i = X_i\partial_\Phi - \bar{X}_i\partial_{\Phi^\dagger}$. For our purposes, it suffices to work in Wess-Zumino gauge, where the bulk vector superfield is nilpotent of degree three ($V_{3\mathrm{D}}^3 = 0$) and its restriction to the line is nilpotent of degree two ($V_{1\mathrm{D}}^2 = 0$): namely, $V_i = \theta\theta^\dagger A_i$. In this gauge, we have $\Gamma = 2V_iJ_i$, so that $\tilde{\mathcal{L}}$ reduces to the non-manifestly supersymmetric Lagrangian $\tilde{\mathcal{L}}_0 + \mathcal{L}_A$. In arbitrary gauge, $\tilde{\mathcal{L}}$ contains terms of arbitrarily high order in the dimensionless bottom component of V.

Effective Action

To compute the effective action generated by integrating out ψ, we add an $SU(2)$-invariant kinetic term for ψ (with a dimensionful coefficient) as an ultraviolet (UV) regulator:

$$\mathcal{L}' = \int d^2\theta\, K' = -\frac{i(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) + 4\dot{\phi}\dot{\phi}^\dagger}{(1+|\phi|^2)^2} - \frac{2i(\dot{\phi}^\dagger\phi - \phi^\dagger\dot{\phi})\psi^\dagger\psi}{(1+|\phi|^2)^3}, \quad K' \equiv \frac{D^\dagger\Phi^\dagger D\Phi}{(1+|\Phi|^2)^2}. \tag{2.45}$$

Note that since $D\Phi = \psi - 2i\theta^\dagger\dot{\phi} + i\theta\theta^\dagger\dot{\psi}$ transforms in the same way under $SU(2)$ as its bottom component ψ, K' is automatically invariant under global $SU(2)$. We want to gauge K'. With chiral superfield gauge transformation parameters, we have (note $DX_i = 2\mathcal{F}_iD\Phi$)

$$\delta_{SU(2)}K' = -i(\Lambda_i - \Lambda_i^\dagger)J_i' - i(D\Lambda_iI_i - D^\dagger\Lambda_i^\dagger I_i^\dagger) \tag{2.46}$$

where J_i' are the bosonic Noether currents associated to K' and the I_i are fermionic:

$$J_i' = -2K'J_i, \quad I_i = \frac{iX_i(D\Phi)^\dagger}{(1+|\Phi|^2)^2}. \tag{2.47}$$

There exists a counterterm Γ' satisfying

$$\delta_{SU(2)}\Gamma' = i(\Lambda_i - \bar{\Lambda}_i)J_i' + i(D\Lambda_iI_i - D^\dagger\bar{\Lambda}_i\bar{I}_i), \tag{2.48}$$

which takes the form

$$\int d^2\theta\,\Gamma' = \frac{2\psi^\dagger\psi}{(1+|\phi|^2)^2}\left[\frac{(A_1+iA_2)\phi + (A_1-iA_2)\phi^\dagger - A_3(1-|\phi|^2)}{1+|\phi|^2}\right] + \cdots \tag{2.49}$$

in Wess-Zumino gauge, such that the Lagrangian

$$\tilde{\mathcal{L}}' = \int d^2\theta \, (K' + \Gamma') \equiv \mathcal{L}_\psi - \frac{4\dot{\phi}\dot{\phi}^\dagger}{(1+|\phi|^2)^2} + \cdots \tag{2.50}$$

(written in Wess-Zumino gauge) is invariant under local $SU(2)$, where

$$\mathcal{L}_\psi = -\frac{i(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi)}{(1+|\phi|^2)^2} - \frac{2\mathcal{L}\psi^\dagger\psi}{(1+|\phi|^2)^2} \tag{2.51}$$

is itself invariant under local $SU(2)$ (it is possible to construct Γ' using a general prescription for the full nonlinear gauging of supersymmetric sigma models with higher-derivative terms).[10] Thus the "$\cdots$" in $\tilde{\mathcal{L}}'$ contains only dimension-two terms not involving ψ, namely the couplings to A_i necessary to make the two-derivative term in ϕ invariant under local $SU(2)$. Making the scale μ of the higher-dimension terms explicit, consider

$$\tilde{\mathcal{L}}_{\rm tot} = j\tilde{\mathcal{L}} - \frac{1}{2\mu}\tilde{\mathcal{L}}' \equiv j\mathcal{L} + \psi^\dagger\mathcal{D}\psi + \frac{2\dot{\phi}\dot{\phi}^\dagger}{\mu(1+|\phi|^2)^2} + \cdots \,, \tag{2.52}$$

where we have integrated by parts. Performing the path integral over ψ generates the one-loop effective action

$$\operatorname{tr}\log\mathcal{D} = \pm\frac{i}{2}\int dt\,\mathcal{L}. \tag{2.53}$$

The regularization-dependent sign is fixed to "$-$" by canonical quantization, leading to a shift $j \to j - 1/2$. The "$\cdots$" terms in $\tilde{\mathcal{L}}'$ decouple at low energies ($\mu \to \infty$).

2.4.2 Shift from Canonical Quantization

Canonical quantization of the $\mathcal{N} = 2$ quantum mechanics provides another perspective on the shift in j. Here, we set $A_i = 0$, whence

$$\tilde{\mathcal{L}}|_{A_i=0} = \tilde{\mathcal{L}}_0, \quad \tilde{\mathcal{L}}'|_{A_i=0} = \mathcal{L}', \tag{2.54}$$

[10] In [1], we describe how to gauge a global symmetry (under which the fields do not transform in a linear representation) at the nonlinear level while preserving global SUSY.

so that the full Lagrangian is $\tilde{\mathcal{L}}_{\text{tot}}|_{A_i=0} = j\tilde{\mathcal{L}}_0 - \frac{1}{2\mu}\mathcal{L}' = L_B + L_F$ where L_B and L_F describe 1D sigma models with S^2 target space:

$$L_B = \frac{ij(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{1+|\phi|^2} + \frac{i\alpha}{2}\left(\frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}^\dagger}{\phi^\dagger}\right) + \frac{2\dot{\phi}\dot{\phi}^\dagger}{\mu(1+|\phi|^2)^2}, \tag{2.55}$$

$$L_F = -\left[j - \frac{i(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{\mu(1+|\phi|^2)}\right]\frac{\psi^\dagger\psi}{(1+|\phi|^2)^2} + \frac{i(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi)}{2\mu(1+|\phi|^2)^2}. \tag{2.56}$$

For later convenience, we have added a total derivative, parametrized by $\alpha \in \mathbb{R}$, to L_B. Its meaning is as follows: L_B describes a charged particle on S^2 in the field of a monopole of charge $\propto j$, with the scale $\mu \in \mathbb{R}_{\geq 0}$ proportional to its inverse mass and α parametrizing the longitudinal gauge of the monopole vector potential. We define the gauges S, E, and N by setting $\alpha = (0, j, 2j)$, respectively. We refer to L_B as the "bosonic system" and to $L_B + L_F$ as the corresponding "supersymmetric system." We now summarize the results of quantizing the theories L_B and $L_B + L_F$. As when computing the effective action, we use the $1/\mu$ terms as a technical aid; they have the effect of enlarging the phase space.

Bosonic System

As a warmup, consider L_B alone. At finite μ, the phase space is $(2+2)$-dimensional and the quantum Hamiltonian can be written as

$$H_j = \frac{\mu}{2}(\vec{L}^2 - j^2) = \frac{\mu}{2}(\ell(\ell+1) - j^2). \tag{2.57}$$

Here, $\vec{L}^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_3^2$ and we have defined the operators

$$\begin{aligned} L_+ &= -\phi^2\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi^\dagger} + \frac{2j|\phi|^2 + \alpha(1-|\phi|^2)}{2|\phi|^2}\phi, \\ L_- &= \frac{\partial}{\partial\phi} + (\phi^\dagger)^2\frac{\partial}{\partial\phi^\dagger} + \frac{2j|\phi|^2 + \alpha(1-|\phi|^2)}{2|\phi|^2}\phi^\dagger, \\ L_3 &= \phi\frac{\partial}{\partial\phi} - \phi^\dagger\frac{\partial}{\partial\phi^\dagger} - (j-\alpha), \end{aligned} \tag{2.58}$$

which satisfy $[L_3, L_\pm] = \pm L_\pm$, $[L_+, L_-] = 2L_3$. The spectrum is constrained to $\ell \geq j$ by an L_3 selection rule, with each level ℓ appearing once; the eigenfunctions of the associated generalized angular momentum are monopole spherical harmonics. As $\mu \to \infty$, all states except those with $\ell = j$ decouple. Rather than taking the decoupling limit $\mu \to \infty$ in L_B, which projects out all but the spin-j states, setting $j = 0$ yields the rigid rotor. Its Hamiltonian is given in terms of the Laplace-Beltrami operator Δ_{S^2}, whose spectrum is $-\ell(\ell+1)$ with degeneracy $2\ell+1$ for $\ell \geq 0$.

The bosonic theory with $\mu = \infty$ ($L_B = j\mathcal{L}_0$, in S gauge) is the well-known Wess-Zumino term for quantization of spin that we have already discussed. The action computes the solid angle enclosed by a trajectory on the sphere, and the Dirac quantization condition requires that the coefficient j be a half-integer. Quantizing the compact phase space S^2 yields $2j+1$ states $|j, m\rangle$, all eigenstates of L_3. Indeed, at $\mu = \infty$, the phase space is $(1+1)$-dimensional and we can write

$$L_+ = -\phi^2\partial_\phi + (2j-\alpha)\phi, \quad L_- = \partial_\phi + \frac{\alpha}{\phi}, \quad L_3 = \phi\partial_\phi - (j-\alpha). \tag{2.59}$$

The wavefunctions are $\phi^{-\alpha}, \ldots, \phi^{2j-\alpha}$, the eigenvalues range from $-j$ to j in integer steps, and $\vec{L}^2 = j(j+1)$.

Supersymmetric System

For $L_B + L_F$, let us keep μ finite (work in the full phase space) and set $\alpha = 0$. Write

$$\chi = \frac{\psi}{\sqrt{\mu}(1+|\phi|^2)}, \tag{2.60}$$

which satisfies $\{\chi, \chi^\dagger\} = 1$ upon quantization. The supercharges are represented by differential operators as

$$Q = \psi\left(\frac{\partial}{\partial\phi} - \frac{(j+1/2)\phi^\dagger}{1+|\phi|^2}\right), \quad Q^\dagger = \psi^\dagger\left(-\frac{\partial}{\partial\phi^\dagger} - \frac{(j-1/2)\phi}{1+|\phi|^2}\right), \tag{2.61}$$

which are adjoints with respect to the Fubini-Study measure. The Hamiltonian is

$$H' = \frac{1}{2}\{Q, Q^\dagger\} = H_{j+\chi^\dagger\chi-1/2} + j\mu\chi^\dagger\chi - \frac{\mu}{2}(j-1/2) = \frac{\mu}{2}(\vec{L}_f^2 - (j+1/2)(j-1/2)) \tag{2.62}$$

where $\vec{L}_f = \vec{L}|_{j+\chi^\dagger\chi-1/2}$. On the Hilbert space $(L^2(S^2,\mathbb{C})\otimes|0\rangle)\oplus(L^2(S^2,\mathbb{C})\otimes\chi^\dagger|0\rangle)$,

$$H' = \begin{pmatrix} H_{j-1/2} - \mu(j-1/2)/2 & 0 \\ 0 & H_{j+1/2} + \mu(j+1/2)/2 \end{pmatrix} \tag{2.63}$$

$$= \frac{\mu}{2}\begin{pmatrix} \ell_b(\ell_b+1) - (j-1/2)(j+1/2) & 0 \\ 0 & \ell_f(\ell_f+1) - (j-1/2)(j+1/2) \end{pmatrix} \tag{2.64}$$

where $\ell_b \geq j - 1/2$ and $\ell_f \geq j + 1/2$. There are $2j$ bosonic ground states at $\ell_b = j - 1/2$. This fixes the sign of the previous path integral calculation.

2.4.3 Shift from 1D Supersymmetric Index

To make contact with bulk Wilson loops, we compute both the non-supersymmetric twisted partition function and the flavored Witten index

$$I_{\mathcal{N}=0} = \mathrm{Tr}(e^{-\beta H}e^{izL_3}), \quad I_{\mathcal{N}=2} = \mathrm{Tr}[(-1)^F e^{-\beta H'}e^{iz(L_f)_3}] \tag{2.65}$$

by working semiclassically in the Euclidean path integral. Let

$$L_{B,E} = \frac{j(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{1+|\phi|^2} + \frac{\alpha}{2}\left(\frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}^\dagger}{\phi^\dagger}\right) + \frac{2\dot{\phi}\dot{\phi}^\dagger}{\mu(1+|\phi|^2)^2}, \tag{2.66}$$

$$L_{F,E} = \left[j + \frac{\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi}}{\mu(1+|\phi|^2)}\right]\frac{\psi^\dagger\psi}{(1+|\phi|^2)^2} + \frac{\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi}{2\mu(1+|\phi|^2)^2} \tag{2.67}$$

denote the Euclideanized versions of L_B and L_F, with dots denoting τ-derivatives. Then

$$I_{\mathcal{N}=0} = \int D\phi^\dagger D\phi\, e^{-\int_0^\beta d\tau\, L_{B,E}}, \quad I_{\mathcal{N}=2} = \int D\phi^\dagger D\phi D\psi^\dagger D\psi\, e^{-\int_0^\beta d\tau\,(L_{B,E}+L_{F,E})}, \tag{2.68}$$

with boundary conditions twisted by e^{izL_3} or $e^{iz(L_f)_3}$ as appropriate. While both $I_{\mathcal{N}=0}$ and $I_{\mathcal{N}=2}$ are known from canonical quantization, our goal here is to introduce the localization argument via what amounts to a derivation of the Weyl character formula (2.2) as a sum of two terms coming from the classical saddle points with a spin-independent prefactor coming from the one-loop determinants.

We first compute $I_{\mathcal{N}=0}$ in the bosonic problem. We set $\mu = \infty$ and work in the E gauge (not to be confused with "E for Euclidean") for convenience, where

$$L_{B,E} = \frac{j(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi})}{1+|\phi|^2} + \frac{j}{2}\partial_\tau \log\left(\frac{\phi}{\phi^\dagger}\right). \tag{2.69}$$

We restrict the path integral to field configurations satisfying $\phi(\tau+\beta) = e^{iz}\phi(\tau)$, for which

$$\int_0^\beta d\tau\, \partial_\tau \log\left(\frac{\phi}{\phi^\dagger}\right) = 2iz. \tag{2.70}$$

With this restriction, the action is extremized when $\phi = \phi_{\rm cl} \in \{0,\infty\}$ (the two fixed points of the L_3 action). We see that $L_{B,E}|_0 = ijz/\beta$ and $L_{B,E}|_\infty = -ijz/\beta$. First expand around $\phi_{\rm cl} = 0$ with perturbation Δ: $\phi = \phi_{\rm cl} + \Delta = \Delta$, where Δ satisfies the twisted boundary condition. Its mode expansion takes the form

$$\Delta = \frac{1}{\sqrt{\beta}}\sum_{n=-\infty}^{\infty} \Delta_n e^{i(2\pi n+z)\tau/\beta}, \tag{2.71}$$

from which we obtain simply

$$\int_0^\beta d\tau\, L_{B,E}|_{O(\Delta^2)} = j\int_0^\beta d\tau\,(\Delta\dot{\Delta}^\dagger - \Delta^\dagger\dot{\Delta}) = -\frac{2ij}{\beta}\sum_{n=-\infty}^{\infty}(2\pi n+z)|\Delta_n|^2. \tag{2.72}$$

Thus the one-loop factor from expanding around $\phi_{\rm cl} = 0$ is

$$Z_{\text{1-loop}}|_0 = \exp\left[-\sum_{n=-\infty}^{\infty}\log(2\pi n+z)\right] = \frac{e^{az+b}}{\sin(z/2)} = -\frac{e^{-iz/2}}{2i\sin(z/2)} \tag{2.73}$$

where the integration constants a, b parametrize the counterterms by which different regularization schemes differ; (2.73) is the only choice consistent with canonical quantization. Free-field subtraction (normalizing the functional determinant, sans zero mode, at finite μ and then taking $\mu \to \infty$) yields the same answer. Indeed, accounting for the $1/\mu$ term in (2.66), the kinetic operator for bosonic fluctuations Δ is $-2(j\partial_\tau + \partial_\tau^2/\mu)$ where the eigenvalues of ∂_τ are $i(2\pi n+z)/\beta$, giving the regularized product

$$Z_{\text{1-loop}}|_0 = \frac{1}{\det(j\partial_\tau + \partial_\tau^2/\mu)} = -\frac{\sinh(\beta\mu j/2)}{2i\sin(z/2)\sinh((\beta\mu j + iz)/2)} \xrightarrow{\beta\mu\to\infty} -\frac{e^{-(j/|j|)iz/2}}{2i\sin(z/2)}. \tag{2.74}$$

Now note that taking $\phi \to 1/\phi$ leaves $L_{B,E}$ in E gauge (2.69) invariant (with the $1/\mu$ term in (2.66) being invariant by itself) while taking $z \to -z$ in the boundary condition for the path integral. Hence

$$Z_{\text{1-loop}}|_\infty = (Z_{\text{1-loop}}|_0)|_{z\to -z} = \frac{e^{iz/2}}{2i\sin(z/2)}, \tag{2.75}$$

and it follows that

$$I_{\mathcal{N}=0} = \sum_{0,\infty} e^{-\beta L_{B,E}} Z_{\text{1-loop}} = \frac{e^{i(j+1/2)z} - e^{-i(j+1/2)z}}{2i\sin(z/2)} = \frac{\sin((j+1/2)z)}{\sin(z/2)}. \tag{2.76}$$

This is, of course, a special case of the Duistermaat-Heckman formula for longitudinal rotations of S^2, with the contribution from each fixed point weighted by the appropriate sign. As a consequence, the index is an even function of z (invariant under the Weyl group $\mathbb{Z}_2$), as it must be, because the Hilbert space splits into representations of $SU(2)$.

We now compute $I_{\mathcal{N}=2}$, keeping μ finite. In the supersymmetric problem, the E gauge corresponds to choosing the Kähler potential $\log(1+|\Phi|^2) - \frac{1}{2}\log|\Phi|^2$, which is invariant under $\Phi \to 1/\Phi$. In component fields, the Lagrangian is $L_{B,E} + L_{F,E}$ with $\alpha = j$. Expanding in both bosonic fluctuations Δ and fermionic fluctuations Ξ ($\psi = \psi_{\text{cl}} + \Xi = \Xi$) gives

$$(L_{B,E} + L_{F,E})|_{O(\Delta^2+\Xi^2)} = j(\Delta\dot{\Delta}^\dagger - \Delta^\dagger\dot{\Delta} + \Xi^\dagger\Xi) + \frac{2}{\mu}\dot{\Delta}\dot{\Delta}^\dagger + \frac{1}{2\mu}(\Xi^\dagger\dot{\Xi} - \dot{\Xi}^\dagger\Xi). \tag{2.77}$$

The part of the Lagrangian quadratic in fluctuations, as written above, is supersymmetric by itself. Twisted boundary conditions in the path integral are implemented by $(L_f)_3$, which satisfies $[(L_f)_3, \phi] = \phi$ and $[(L_f)_3, \psi] = \psi$. The moding for the fermionic fluctuations

$$\Xi = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \Xi_n e^{i(2\pi n+z)\tau/\beta} \tag{2.78}$$

is integral because at $z = 0$, the insertion of $(-1)^F$ would require periodic boundary conditions for fermions on the thermal circle. Hence the fermions contribute a factor of

$$\exp\left[\sum_{n=-\infty}^{\infty} \log\left(\frac{2\pi n + z}{\beta\mu} - ij\right)\right] \tag{2.79}$$

to $Z_{\text{1-loop}}|_0$. To take the $\beta\mu \to \infty$ limit, we perform free-field subtraction:

$$\det(j + \partial_\tau/\mu) = \prod_{n=-\infty}^{\infty} \frac{(2\pi n + z)/\beta\mu - ij}{2\pi n/\beta\mu - ij} = \frac{\sin((i\beta\mu j - z)/2)}{\sin(i\beta\mu j/2)} \xrightarrow{\beta\mu\to\infty} e^{(j/|j|)iz/2}. \quad (2.80)$$

Taking j positive, this reduces to a phase of $e^{iz/2}$. By similar reasoning to that in the bosonic case, we conclude that

$$I_{\mathcal{N}=2} = \frac{\sin(jz)}{\sin(z/2)}. \quad (2.81)$$

Again, this is the only answer consistent with canonical quantization. Thus in the supersymmetric theory, the one-loop shift of j due to the bosons $(+1/2)$ exactly cancels that due to the fermions $(-1/2)$.

Localization in 1D

In both the bosonic and supersymmetric theories, direct comparison to canonical quantization shows that the semiclassical (one-loop) approximation for the index is exact. It is natural to ask why this should be so, and supersymmetry provides an answer. While the exactness in the bosonic case can only be heuristically justified by the Dirac quantization condition on j, it can be rigorously justified by appealing to the supersymmetric case.

In its most basic form, the localization principle starts from the fact that a Euclidean partition function deformed by a total variation of some nilpotent symmetry δ ($\delta^2 = 0$) of both the action and the measure is independent of the coefficient of this deformation:

$$Z(t) = \int \mathcal{D}\Phi\, e^{-S[\Phi]+t\delta V} \implies \frac{dZ(t)}{dt} = \int \mathcal{D}\Phi\, \delta\left(e^{-S[\Phi]+t\delta V} V\right) = 0. \quad (2.82)$$

If the bosonic part of δV is positive-semidefinite, then as $t \to \infty$, the path integral localizes to $\delta V = 0$. For a given field configuration with $\delta V = 0$, one can compute a semiclassical path integral for fluctuations on top of this background, and then integrate over all such backgrounds to obtain the exact partition function.

In our case, the quadratic terms arising from perturbation theory are already $(Q+Q^\dagger)$-exact, without the need to add any localizing terms. Indeed, we compute[11] that $\delta(\delta(\phi^\dagger\phi))$ and $\delta(\delta(\psi^\dagger\psi))$ are precisely the quadratic expressions (2.77) (at $O(\mu^0)$ and $O(\mu^{-1})$, respectively) that we integrate over the fluctuations Δ, Ξ to compute the one-loop factors in the index $I_{\mathcal{N}=2}$. As we take the coefficient of either the $\delta(\delta(\phi^\dagger\phi))$ term or the $\delta(\delta(\psi^\dagger\psi))$ term to ∞, the original Lagrangian $L_{B,E}+L_{F,E}$ becomes irrelevant for the one-loop analysis, but since these terms have the same critical points as the original Lagrangian, the result of the localization analysis coincides with that of the original Lagrangian, proving that the path integral for the latter is one-loop exact.[12] Furthermore, the final result is independent of the coefficient of either term. This has a simple explanation: the regularized bosonic and fermionic functional determinants (2.74) and (2.80) have a product which is independent of $\beta\mu$, namely

$$\frac{\det(j+\partial_\tau/\mu)}{\det(j\partial_\tau+\partial_\tau^2/\mu)} = -\frac{1}{2i\sin(z/2)}. \tag{2.83}$$

Hence the one-loop factor has the same limit whether $\beta\mu\to\infty$ or $\beta\mu\to 0$.

Finite Temperature

We have shown in Lorentzian signature and at zero temperature that integrating out the fermions in the supersymmetric theory with isospin J ($2J$ bosonic ground states) yields an effective bosonic theory with isospin $j=J-1/2$ ($2j+1$ bosonic ground states), which is consistent with the equality of $I_{\mathcal{N}=0}(j)$ in (2.76) and $I_{\mathcal{N}=2}(J)$ in (2.81).

The index, however, is computed at finite temperature. The temperature can only enter the effective action through the dimensionless combination $\beta\mu$, and this dependence must disappear in the limit $\mu\to\infty$. Therefore, the statement of the preceding paragraph must be

[11] In Euclidean signature,

$$\delta\phi = \delta_\epsilon\phi = \epsilon\psi, \quad \delta\phi^\dagger = \delta_{\epsilon^\dagger}\phi^\dagger = -\epsilon^\dagger\psi^\dagger, \quad \delta\psi = \delta_{\epsilon^\dagger}\psi = 2\epsilon^\dagger\dot{\phi}, \quad \delta\psi^\dagger = \delta_\epsilon\psi^\dagger = -2\epsilon\dot{\phi}^\dagger$$

where $\delta\mathcal{O}\equiv[\epsilon Q+\epsilon^\dagger Q^\dagger, \mathcal{O}]$ and $\delta_{\epsilon,\epsilon^\dagger}$ are Grassmann-even.

[12] Note that the bosonic part of the $\delta(\delta(\phi^\dagger\phi))$ term is not positive-semidefinite; indeed, it is imaginary. We are implicitly using a stationary phase argument.

independent of temperature. Let us show this directly at finite temperature by mimicking the index computation, thereby giving an alternative and cleaner derivation of (2.53).

We first perform a field redefinition $\psi' = \psi/(1+|\phi|^2)$ (the associated Jacobian determinant cancels in regularization). Integrating by parts then gives

$$L_{F,E} = \psi'^\dagger \mathcal{D}\psi', \quad \mathcal{D} \equiv \frac{\partial_\tau + \mathcal{L}_{0,E}}{\mu} + j, \quad \mathcal{L}_{0,E} \equiv \frac{\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi}}{1+|\phi|^2}. \tag{2.84}$$

In Euclidean signature, the eigenfunctions of $\mathcal{D}$ are simple:

$$f(\tau) = \exp\left[(\lambda - j)\mu\tau - \int^\tau d\tau'\, \mathcal{L}_{0,E}\right]. \tag{2.85}$$

With periodic (supersymmetric) boundary conditions for the fermions, the eigenvalues are

$$\lambda_n = j + \frac{2\pi i n + \mathcal{A}}{\beta\mu}, \quad \mathcal{A} = \int_0^\beta d\tau\, \mathcal{L}_{0,E}, \quad n \in \mathbb{Z}. \tag{2.86}$$

Free-field subtraction then gives

$$\frac{\det(\partial_\tau/\mu + j + \mathcal{L}_{0,E}/\mu)}{\det(\partial_\tau/\mu + j)} = \prod_{n=-\infty}^{\infty} \frac{j + (2\pi i n + \mathcal{A})/\beta\mu}{j + 2\pi i n/\beta\mu} = \frac{e^{-\mathcal{A}/2}(1 - e^{\mathcal{A}+\beta\mu j})}{1 - e^{\beta\mu j}}. \tag{2.87}$$

Upon taking $\mu \to \infty$, this becomes $e^{(j/|j|)\mathcal{A}/2}$, whose exponent has the correct sign because the Euclidean action appears with a minus sign in the path integral.

Note that while this computation seemingly fixes the sign outright, our regularization crucially assumes a positive sign for μ. Moreover, different regularization schemes lead to different global anomalies in the effective action [57, 58]. These ambiguities can be phrased as a mixed anomaly between the "charge conjugation" symmetry taking $z \to -z$ and invariance under global gauge transformations $z \to z + 2\pi n$ for $n \in \mathbb{Z}$ [58] (as we will see shortly, z can be interpreted as a background gauge field). To fix the sign of the shift unambiguously (i.e., such that the effective action computation is consistent with the index), we appeal to canonical quantization. In other words, in the Hamiltonian formalism, we demand that the $SU(2)$ symmetry be preserved quantum-mechanically.

Background Gauge Field

The quantities (2.65) are useful because the twisted index with vanishing background gauge field is in fact equivalent to the *un*twisted index with *arbitrary* constant background gauge field. To see this, set $\mu = \infty$ for simplicity. To restore the background gauge field, we simply take $L_B \to L_B + j\mathcal{L}_A$, or equivalently

$$L_{B,E} \to L_{B,E} - j\mathcal{L}_A, \tag{2.88}$$

with $\mathcal{L}_A$ in (2.33) (note that $\mathcal{L}_{A,E} = -\mathcal{L}_A$, where the gauge field is always written in Lorentzian conventions). With $A_i = 0$, the bosonic index $I_{\mathcal{N}=0}$ corresponds to the partition function for $L_{B,E}$ on S^1 with twisted boundary conditions implemented by the quantum operator L_3. Clearly, $I_{\mathcal{N}=0}$ can also be viewed as a thermal partition function for a deformed Hamiltonian with periodic boundary conditions:

$$I_{\mathcal{N}=0} = \mathrm{Tr}(e^{-\beta H_z}), \quad H_z \equiv H - \frac{izL_3}{\beta} = H + \frac{ijz}{\beta}\frac{1-|\phi|^2}{1+|\phi|^2}. \tag{2.89}$$

This corresponds to a path integral with the modified Lagrangian

$$L_{B,E} + \frac{ijz}{\beta}\frac{1-|\phi|^2}{1+|\phi|^2}. \tag{2.90}$$

Setting $z = i\beta A_3$, we recover precisely $(L_{B,E} - j\mathcal{L}_A)|_{A_1=A_2=0}$, so we deduce from (2.76) that

$$\int D\phi^\dagger D\phi \, e^{-\int_0^\beta d\tau \, (L_{B,E} - j\mathcal{L}_A)|_{A_1=A_2=0}} = \frac{\sinh((j+1/2)\beta A_3)}{\sinh(\beta A_3/2)}, \tag{2.91}$$

with periodic boundary conditions implicit. But for a constant gauge field, we can always change the basis in group space to set $A_1 = A_2 = 0$. Letting $|A| = (\sum_i A_i^2)^{1/2}$ denote the norm in group space, we conclude that

$$\int D\phi^\dagger D\phi \, e^{-\int_0^\beta d\tau \, (L_{B,E} - j\mathcal{L}_A)} = \frac{\sinh((j+1/2)\beta|A|)}{\sinh(\beta|A|/2)}.\text{[13]} \tag{2.92}$$

[13] While we inferred this result from the $SU(2)$ symmetry of the twisted partition function, it can also be seen directly from a semiclassical analysis of the Euclidean Lagrangian.

Setting $L^{\mathrm{WZ}}_{B,E} \equiv j\mathcal{L}_{0,E}$ and noting that $\mathrm{Tr}_j\, e^{\pm\beta A_i J_i} = \mathrm{Tr}_j\, e^{\beta|A|J_3}$, this result can be written more suggestively as

$$\int D\phi^\dagger D\phi \exp\left[-\int_0^\beta d\tau\, (L^{\mathrm{WZ}}_{B,E} - 2jA_iJ_i)\right] = \mathrm{Tr}_j\, e^{-\beta A_i J_i} \tag{2.93}$$

where on the left, the J_i are interpreted as classical Noether currents and on the right, they are interpreted as quantum non-commuting matrices (the Hermitian generators of $SU(2)$ in the spin-j representation). Hence the path integral for the 1D quantum mechanics with constant background gauge field computes a Wilson loop of spin j with constant gauge field along the S^1, i.e., the character of the spin-j representation. This identification holds even for arbitrary background gauge field because one can always choose a time-dependent gauge such that the gauge field is constant along the loop; the only invariant information is the conjugacy class of the holonomy around the loop. Indeed, a Wilson loop can be thought of as a dynamical generalization of a Weyl character.

The above arguments can be carried over wholesale to the supersymmetric index $I_{\mathcal{N}=2}$, since the $(L_f)_i$ rotate into each other under global $SU(2)$. The fermions modify the representation in which the trace is taken, and (as we will see) the fact that a particular linear combination of the bulk gauge field and the auxiliary scalar σ appears in the quantum mechanics is reflected in the appearance of these fields in the supersymmetric path-ordered expression.

2.5 Coupling to the Bulk

We now take a top-down approach to the quantum mechanics on the line by restricting the 3D $\mathcal{N} = 2$ multiplets to 1D $\mathcal{N} = 2$ multiplets closed under SUSY transformations that generate translations along the line, which we take to extend along the 0 direction in $\mathbb{R}^{1,2}$ (as in the previous section, aside from Section 2.4.3, we work in Lorentzian signature). We thus identify the components of the 1D vector multiplet with restrictions of the bulk fields; in principle, the 1D chiral multiplet Φ of the previous section descends from bulk super gauge transformations.

Our conventions for SUSY in $\mathbb{R}^{1,2}$ are given in Appendix A.2. The linear combination of supercharges that generates translations along the line is $\Omega \equiv (Q_1 + iQ_2)/\sqrt{2}$ (any choice $\Omega = c_1 Q_1 + c_2 Q_2$ with $|c_1|^2 = |c_2|^2 = 1/2$ and $c_1 c_2^*$ purely imaginary would suffice), which satisfies $\{\Omega, \Omega^\dagger\} = -2P_0 = 2H$ for vanishing central charge. Therefore, to restrict to the line, we choose the infinitesimal spinor parameter ξ such that

$$\xi Q = \xi_1 Q_2 - \xi_2 Q_1 = \omega\Omega \implies (\xi_1, \xi_2) = \frac{1}{\sqrt{2}}(i\omega, -\omega) \tag{2.94}$$

where ω is some fiducial Grassmann parameter (note that ξQ has suppressed spinor indices, while $\omega\Omega$ does not). In terms of the linear representations of the supercharges on 3D and 1D $\mathcal{N} = 2$ superspace ((A.13) and (A.1), respectively), we compute that for superfields whose only spacetime dependence is on the 0 direction, 1D $\mathcal{N} = 2$ SUSY transformations are implemented by $\xi\mathcal{Q} - \bar{\xi}\bar{\mathcal{Q}} = \omega\hat{Q} + \bar{\omega}\hat{Q}^\dagger$ with $\theta = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2)$ and $\partial_\theta = \frac{1}{\sqrt{2}}(\partial_{\theta^1} + i\partial_{\theta^2})$.

2.5.1 Linearly Realized SUSY on the Line

With all auxiliary fields necessary to realize SUSY transformations linearly, a 3D $\mathcal{N} = 2$ vector multiplet ($V = V^\dagger$) takes the form

$$\begin{aligned} V = C + \theta\chi - \bar{\theta}\bar{\chi} + \frac{1}{2}\theta^2(M + iN) - \frac{1}{2}\bar{\theta}^2(M - iN) - i\theta\bar{\theta}\sigma - \theta\gamma^\mu\bar{\theta}A_\mu \\ + i\theta^2\bar{\theta}\left(\bar{\lambda} - \frac{1}{2}\gamma^\mu\partial_\mu\chi\right) - i\bar{\theta}^2\theta\left(\lambda - \frac{1}{2}\gamma^\mu\partial_\mu\bar{\chi}\right) + \frac{1}{2}\theta^2\bar{\theta}^2\left(D - \frac{1}{2}\partial^2 C\right) \end{aligned} \tag{2.95}$$

where $V = V^a T^a$, etc., and all bosonic components are real. A 3D $\mathcal{N} = 2$ chiral multiplet ($\bar{D}_\alpha\Phi = 0$) takes the form

$$\Phi = A - i\theta\gamma^\mu\bar{\theta}\partial_\mu A - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 A + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\gamma^\mu\partial_\mu\psi + \theta^2 F \tag{2.96}$$

where the scalar components are complex. Bulk (3D) SUSY acts on the vector and chiral multiplets as in Appendix A.2. For f any complex 3D fermion, it is convenient to set

$$f' \equiv \frac{f_1 + i f_2}{\sqrt{2}}, \quad f'' \equiv \frac{f_1 - i f_2}{\sqrt{2}}. \tag{2.97}$$

We find that the 3D $\mathcal{N} = 2$ vector multiplet restricts to the following 1D $\mathcal{N} = 2$ multiplets:

- a 1D vector $\{-C, \chi', \sigma + A_0\}$,
- a 1D chiral $\{(N + iM)/2, \lambda' - i\partial_0\bar{\chi}''\}$ (and its conjugate antichiral),
- and a 1D chiral $\{(iD - \partial_0\sigma)/2, \partial_0\bar{\lambda}''\}$ (and its conjugate antichiral).

We find that the 3D $\mathcal{N} = 2$ chiral multiplet restricts to the following 1D $\mathcal{N} = 2$ multiplets:

- a 1D chiral $\{A, -\sqrt{2}\psi'\}$
- and a 1D antichiral $\{F, -\sqrt{2}\partial_0\psi''\}$.

The above 1D $\mathcal{N} = 2$ multiplets transform according to (A.6) and (A.8) with $\epsilon = \omega$. Note that χ, λ, ψ in 3D each restrict to two independent complex fermions in 1D.

2.5.2 Nonlinearly Realized SUSY on the Line

We will arrive at a bulk interpretation of the quantum-mechanical variables ϕ, ψ in Wess-Zumino gauge, which partially fixes "super gauge" while retaining the freedom to perform ordinary gauge transformations. To this end, it is useful to work in terms of the correponding nonlinearly realized supersymmetry (SUSY') transformations.

In Wess-Zumino gauge, a 3D $\mathcal{N} = 2$ vector multiplet takes the form

$$V|_{\mathrm{WZ}} = -i\theta\bar{\theta}\sigma - \theta\gamma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{2.98}$$

Bulk (3D) SUSY' acts on the vector multiplet as

$$\delta'\sigma = -(\xi\bar{\lambda} - \bar{\xi}\lambda),$$

$$
\begin{aligned}
\delta' A_\mu &= i(\xi\gamma_\mu\bar{\lambda} + \bar{\xi}\gamma_\mu\lambda), \\
\delta'\lambda &= -i\xi D - i\gamma^\mu\xi D_\mu\sigma - \tfrac{1}{2}\epsilon^{\mu\nu\rho}\gamma_\rho\xi F_{\mu\nu}, \\
\delta'\bar{\lambda} &= i\bar{\xi}D + i\gamma^\mu\bar{\xi}D_\mu\sigma - \tfrac{1}{2}\epsilon^{\mu\nu\rho}\gamma_\rho\bar{\xi}F_{\mu\nu}, \\
\delta' D &= -(\xi\gamma^\mu D_\mu\bar{\lambda} - \bar{\xi}\gamma^\mu D_\mu\lambda) + [\xi\bar{\lambda} + \bar{\xi}\lambda, \sigma]
\end{aligned}
\tag{2.99}
$$

where $D_\mu(\cdot) = \partial_\mu(\cdot) - i[A_\mu, (\cdot)]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Bulk (3D) SUSY' acts on a fundamental chiral multiplet as

$$
\begin{aligned}
\delta' A &= -\sqrt{2}\xi\psi, \\
\delta'\psi &= -\sqrt{2}\xi F + i\sqrt{2}\gamma^\mu\bar{\xi}D_\mu A + i\sqrt{2}\bar{\xi}\sigma A, \\
\delta' F &= i\sqrt{2}\bar{\xi}\gamma^\mu D_\mu\psi - i\sqrt{2}\sigma\bar{\xi}\psi - 2i\bar{\xi}\bar{\lambda}A
\end{aligned}
\tag{2.100}
$$

where $D_\mu(\cdot) = \partial_\mu(\cdot) - iA_\mu(\cdot)$. SUSY' transformations close off shell into the algebra

$$
[\delta'_\zeta, \delta'_\xi](\cdot) = -2i(\xi\gamma^\mu\bar{\zeta} + \bar{\xi}\gamma^\mu\zeta)D_\mu(\cdot) - 2i(\xi\bar{\zeta} - \bar{\xi}\zeta)\sigma\cdot(\cdot)
\tag{2.101}
$$

on gauge-covariant fields where, e.g., $\sigma\cdot(\cdot) \equiv [\sigma, (\cdot)]$ for $\sigma, F_{\mu\nu}, \lambda, \bar{\lambda}, D$ and $\sigma\cdot(\cdot) \equiv \sigma(\cdot)$ for A, ψ, F. The above transformation laws and commutators can be obtained by dimensional reduction from 4D.

The 3D SUSY' transformations restrict to the line as follows. We again use the notation (2.97). For the vector multiplet, defining the SUSY'-covariant derivative $D'_0(\cdot) \equiv D_0(\cdot) - i[\sigma, (\cdot)] = \partial_0(\cdot) - i[\sigma + A_0, (\cdot)]$, which satisfies $\delta' D'_0(\cdot) = D'_0\delta'(\cdot)$ and $D'_0\sigma = D_0\sigma$, we obtain the following (rather degenerate) restricted multiplets in 1D:

- a 1D vector $\{0, 0, \sigma + A_0\}$,
- a 1D adjoint chiral $\{0, \lambda'\}$ (and its complex conjugate),
- and a 1D adjoint chiral $\{(iD - D'_0\sigma)/2, D'_0\bar{\lambda}''\}$ (and its complex conjugate).

For a fundamental chiral multiplet, defining the SUSY'-covariant derivative $D'_0(\cdot) \equiv D_0(\cdot) - i\sigma(\cdot) = \partial_0(\cdot) - i(\sigma + A_0)(\cdot)$, which satisfies $\delta' D'_0(\cdot) = D'_0\delta'(\cdot)$, we obtain a single restricted multiplet in 1D, namely

- a 1D fundamental chiral $\{A, -\sqrt{2}\psi'\}$,

whose scalar component is associated with bulk gauge transformations. All of the above 1D $\mathcal{N} = 2$ chiral multiplets transform according to (A.10) with $\epsilon = \omega$ and $D_0 \to D'_0$. On a 1D chiral multiplet, the 1D SUSY' algebra is realized as

$$[\delta'_\eta, \delta'_\epsilon](\cdot) = -2i(\epsilon\eta^\dagger + \epsilon^\dagger\eta)D'_0(\cdot) \tag{2.102}$$

for $(\cdot) = \phi, \psi$, while δ' acts trivially on a 1D vector multiplet in Wess-Zumino gauge.

One would expect to write a coupled 3D-1D action

$$S_{\text{3D-1D}} = \int d^3x\, \mathcal{L}_{\text{CS}} + j \int dt\, \tilde{\mathcal{L}} \tag{2.103}$$

that is both supersymmetric and gauge-invariant (under SUSY' and ordinary gauge transformations), with the transformation of the 1D action compensating for any boundary terms induced along the line in the transformation of the 3D action. However, in Wess-Zumino gauge, $\mathcal{L}_{\text{CS}}$ in (2.6) has the following SUSY' variation:

$$\delta'\mathcal{L}_{\text{CS}} = \frac{k}{4\pi}\partial_\mu \operatorname{Tr}[i\epsilon^{\mu\nu\rho}(\xi\gamma_\nu\bar{\lambda} + \bar{\xi}\gamma_\nu\lambda)A_\rho + 2(\xi\gamma^\mu\bar{\lambda} - \bar{\xi}\gamma^\mu\lambda)\sigma]. \tag{2.104}$$

This induces a boundary term along the line only if the fields are singular as the inverse of the radial distance to the line. Since they are not, it suffices to show that the 1D action is itself invariant under appropriately defined 1D SUSY' transformations.

2.5.3 Nonlinearly Realized SUSY in the Sigma Model

To carry out this last step, we specialize to $SU(2)$. For the vector multiplet, the bulk and line variables are identified as $a_i = -C_i$, $\psi_i = \chi'_i$, $A_i = \sigma_i + (A_0)_i$. The quantum mechanics

$\tilde{\mathcal{L}}|_{\rm WZ} = \tilde{\mathcal{L}}_0 + \mathcal{L}_A$ is invariant under the 1D SUSY' transformations

$$\begin{aligned} \delta'\phi &= \epsilon\psi, \\ \delta'\psi &= -2i\epsilon^\dagger(\dot{\phi} - \tfrac{i}{2}A_1(1-\phi^2) - \tfrac{1}{2}A_2(1+\phi^2) - iA_3\phi), \end{aligned} \tag{2.105}$$

which satisfy the algebra

$$\begin{aligned} [\delta'_\eta, \delta'_\epsilon]\phi &= -2i(\epsilon\eta^\dagger + \epsilon^\dagger\eta)(\dot{\phi} - \tfrac{i}{2}A_- + \tfrac{i}{2}A_+\phi^2 - iA_3\phi), \\ [\delta'_\eta, \delta'_\epsilon]\psi &= -2i(\epsilon\eta^\dagger + \epsilon^\dagger\eta)(\dot{\psi} + i(A_+\phi - A_3)\psi), \\ [\delta'_\eta, \delta'_\epsilon]\psi^\dagger &= -2i(\epsilon\eta^\dagger + \epsilon^\dagger\eta)(\dot{\psi}^\dagger - i(A_-\phi^\dagger - A_3)\psi^\dagger). \end{aligned} \tag{2.106}$$

The adjoint action of $SU(2)$ on its Lie algebra induces an action on S^2, which explains the appearance of the $SU(2)$ Killing vectors in $\delta'\psi$. Explicitly, at the level of scalar components, the map between the adjoint (gauge parameter) chiral superfield $S = s + \theta f - i\theta\theta^\dagger\dot{s} = S^a\sigma^a/2$ and the (scalar) $SU(2)/U(1)$ coset chiral superfield $\Phi = \phi + \theta\psi - i\theta\theta^\dagger\dot{\phi}$ is

$$\frac{1}{|s|}\begin{pmatrix} s^1 \\ -s^2 \\ s^3 \end{pmatrix} \leftrightarrow \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \leftrightarrow \phi = \frac{s^1 - is^2}{|s| - s^3} \tag{2.107}$$

by stereographic projection (note that this only makes sense for s real). In terms of angles,

$$e^{i\varphi} = \frac{s^1 - is^2}{\sqrt{|s|^2 - (s^3)^2}}, \quad \tan(\theta/2) = \sqrt{\frac{|s| - s^3}{|s| + s^3}}. \tag{2.108}$$

To translate between the adjoint action and linear fractional transformations, one must flip the sign of the second Killing vector: that is, one must identify $\vec{\sigma}/2$ with $(\vec{e}_1, -\vec{e}_2, \vec{e}_3)$. The action of $SU(2)$ is then as expected: writing $\epsilon = \epsilon^a\sigma^a/2$ and $s = s^a\sigma^a/2$, we have with ϵ^i infinitesimal that

$$g = 1 + i\epsilon \implies gsg^{-1} = s + i[\epsilon, s] \implies \delta_{SU(2)}s^i = \epsilon^{ijk}s^j\epsilon^k. \tag{2.109}$$

Under the given map (2.107), this is equivalent to $\delta_{SU(2)}\phi = \epsilon_i x_i$. Now we check that SUSY' acts correctly. Naïvely, we have for the components of S that (with $A = A^a\sigma^a/2$)

$$\begin{aligned} \delta' s &= \epsilon f, \\ \delta' f &= -2i\epsilon^\dagger(\dot{s} - i[A, s]), \end{aligned} \tag{2.110}$$

but to make sense of SUSY' transformations for real s, we must take f real and ϵ purely imaginary (though S itself is not real):

$$\begin{aligned} \delta' s &= i\epsilon f, \\ \delta' f &= -2\epsilon(\dot{s} - i[A, s]), \end{aligned} \tag{2.111}$$

where ϵ, f are real Grassmann variables. In terms of chiral superfields, the desired map is

$$\Phi = \frac{S^1 - iS^2}{|S| - S^3} = \phi + \theta\psi - i\theta\theta^\dagger\dot{\phi}. \tag{2.112}$$

Upon substituting for $\delta' s^a$ and $\delta' f^a$, the δ' variations of $\phi = \phi(s^a, f^a)$ and $\psi = \psi(s^a, f^a)$ are

$$\begin{aligned} \delta'\phi &= i\epsilon\psi, \\ \delta'\psi &= -2\epsilon(\dot{\phi} - \tfrac{i}{2}A^1(1 - \phi^2) - \tfrac{1}{2}A^2(1 + \phi^2) - iA^3\phi), \end{aligned} \tag{2.113}$$

as expected (for our choice of ϵ).

2.6 Generalization to Curved Space

We now describe how to generalize this story to certain compact Euclidean spaces [59]. The fact that the Chern-Simons partition function on Seifert manifolds admits a matrix model representation is well-known [47, 48, 60], and has been discussed in the framework of nonabelian localization in [31, 30]. By now, the computation of observables in $\mathcal{N} = 2$ Chern-Simons theory via supersymmetric localization [61] is also a well-established technique, having been generalized from SCFTs [32] (such as pure $\mathcal{N}=2$ Chern-Simons theory [62]) to non-conformal theories with a $U(1)_R$ symmetry [63, 64].

Both the $\mathcal{N}=0$ and $\mathcal{N}=2$ theories are topological, so their observables are metric-independent. In the $\mathcal{N}=0$ case, the introduction of a metric is usually regarded as a "necessary evil" for the purposes of gauge-fixing and regularization. In the $\mathcal{N}=2$ case, the metric plays a more essential role in computing observables because it determines which observables are compatible with supersymmetry and therefore accessible to localization techniques. Seifert loops (i.e., Wilson loops along the Seifert fiber direction) can give different knots depending on the choice of Seifert fibration. For instance, depending on the choice of Seifert fibration on S^3, the half-BPS sector can contain Wilson loop configurations with the topology of Hopf links or torus links [31].

Well-studied backgrounds include squashed spheres with $SU(2)\times U(1)$ [65, 66] or $U(1)\times U(1)$ [40, 67, 68] isometry, lens spaces [69], and more general Seifert manifolds [33, 42, 51, 70]. With appropriate boundary conditions, localizing on a solid torus $D^2\times S^1$ [71, 72] makes contact with supersymmetric analogues of the gluing and Heegaard decompositions usually encountered in the context of Chern-Simons theory [73, 74, 75, 76]. We now formulate the quantum mechanics on Wilson loops in these general backgrounds (concrete examples can be found in [1]).

2.6.1 Supergravity Background

In the 3D $\mathcal{N}=2$ context, the background supergravity formalism of [77] allows for the construction of a scalar supercharge by partially topologically twisting the $U(1)_R$ symmetry of the $\mathcal{N}=2$ algebra, when M^3 admits a transversely holomorphic foliation (THF) [59, 40]. The relevant supergravity theory is "new minimal" supergravity, defined as the off-shell formulation of 3D supergravity that couples to the $\mathcal{R}$-multiplet of a 3D $\mathcal{N}=2$ quantum field theory with a $U(1)_R$ symmetry. Its rigid limit gives rise to the supersymmetry algebra and multiplets in [40]. The bosonic fields in new minimal supergravity are the metric $g_{\mu\nu}$, the

R-symmetry gauge field $A_\mu^{(R)}$, a two-form gauge field $B_{\mu\nu}$, and the central charge symmetry gauge field C_μ. It is convenient to let H and V_μ denote the Hodge duals of the field strengths of $B_{\mu\nu}$ and C_μ, respectively.

For 3D $\mathcal{N}=2$ theories with a $U(1)_R$ symmetry, [40] classifies the backgrounds that preserve some supersymmetry. In particular, to preserve two supercharges of opposite R-charge, the three-manifold M^3 must admit a nowhere vanishing Killing vector K^μ. If K^μ is real, then M^3 is necessarily an orientable Seifert manifold. We focus on the case of a real, nowhere vanishing Killing vector K^μ, but we do not restrict the orbit to be a Seifert fiber. Under these assumptions, it suffices to consider backgrounds with $V_\mu = 0$, so that the conditions for the existence of a rigid supersymmetry are

$$(\nabla_\mu - iA_\mu^{(R)})\xi = -\frac{1}{2}H\gamma_\mu\xi, \qquad (\nabla_\mu + iA_\mu^{(R)})\tilde{\xi} = -\frac{1}{2}H\gamma_\mu\tilde{\xi}. \tag{2.114}$$

These are the generalized Killing spinor equations, under which ξ and $\tilde{\xi}$ have R-charges ± 1, respectively. The corresponding SUSY' transformations with $V_\mu = 0$ [59] are

$$\begin{aligned}
\delta'\sigma &= -(\xi\tilde{\lambda} - \tilde{\xi}\lambda), \\
\delta' A_\mu &= i(\xi\gamma_\mu\tilde{\lambda} + \tilde{\xi}\gamma_\mu\lambda), \\
\delta'\lambda &= -i\xi(D - \sigma H) - i\gamma^\mu\xi D_\mu\sigma - \tfrac{i}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}\gamma_\rho\xi F_{\mu\nu}, \\
\delta'\tilde{\lambda} &= i\tilde{\xi}(D - \sigma H) + i\gamma^\mu\tilde{\xi} D_\mu\sigma - \tfrac{i}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}\gamma_\rho\tilde{\xi} F_{\mu\nu}, \\
\delta' D &= -D_\mu(\xi\gamma^\mu\tilde{\lambda} - \tilde{\xi}\gamma^\mu\lambda) + [\xi\tilde{\lambda} + \tilde{\xi}\lambda, \sigma] + H(\xi\tilde{\lambda} - \tilde{\xi}\lambda)
\end{aligned} \tag{2.115}$$

for the vector multiplet and

$$\begin{aligned}
\delta' A &= -\sqrt{2}\xi\psi, \\
\delta'\psi &= -\sqrt{2}\xi F + i\sqrt{2}\gamma^\mu\tilde{\xi} D_\mu A + i\sqrt{2}\tilde{\xi}\sigma A - i\sqrt{2}\Delta H\tilde{\xi} A, \\
\delta' F &= i\sqrt{2}D_\mu(\tilde{\xi}\gamma^\mu\psi) - i\sqrt{2}\sigma\tilde{\xi}\psi - 2i\tilde{\xi}\tilde{\lambda}A + i\sqrt{2}(\Delta - 2)H\tilde{\xi}\psi
\end{aligned} \tag{2.116}$$

for a fundamental chiral multiplet of dimension Δ. The covariant derivative is now

$$D_\mu = \nabla_\mu - iA_\mu - irA_\mu^{(R)} \tag{2.117}$$

where r is the R-charge of the field on which it acts. The transformations (2.115) and (2.116) furnish a representation of the algebra $\mathfrak{su}(1|1)$. We also have at our disposal

$$\delta'_\xi \delta'_{\tilde{\xi}} \operatorname{Tr}\left(\frac{1}{2}\lambda\tilde{\lambda} - iD\sigma\right) = \xi\tilde{\xi}\mathcal{L}_{\text{YM}}, \tag{2.118}$$

$$\mathcal{L}_{\text{YM}} = \operatorname{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma - \frac{1}{2}(D-\sigma H)^2 - i\tilde{\lambda}\gamma^\mu D_\mu\lambda + i\tilde{\lambda}[\sigma,\lambda] + \frac{i}{2}H\lambda\tilde{\lambda}\right]$$

as a convenient localizing term, where we have omitted the Yang-Mills coupling.

If the generalized Killing spinor equations have at least one solution, then M^3 admits a THF. The existence of solutions to both equations implies that $K^\mu \equiv \xi\gamma^\mu\tilde{\xi}$ is a nowhere vanishing Killing vector. Assuming that K^μ is real, we can find local ("adapted") coordinates $(\tilde{\psi}, z, \bar{z})$ such that $K = \partial_{\tilde{\psi}}$ and

$$ds^2 = (d\tilde{\psi} + a(z,\bar{z})\,dz + \bar{a}(z,\bar{z})\,d\bar{z})^2 + c(z,\bar{z})^2\,dz\,d\bar{z} \tag{2.119}$$

where a is complex and c is real (following [59], we have normalized the metric such that $|K|^2 = 1$, which does not affect results for supersymmetric observables). Coordinate patches are related by transformations of the form $\tilde{\psi}' = \tilde{\psi} + \alpha(z,\bar{z})$, $z' = \beta(z)$, $\bar{z}' = \bar{\beta}(\bar{z})$ with α real and β holomorphic. We choose the vielbein

$$e^1 = \frac{1}{2}c(z,\bar{z})(dz + d\bar{z}), \quad e^2 = \frac{i}{2}c(z,\bar{z})(dz - d\bar{z}), \quad e^3 = d\tilde{\psi} + a(z,\bar{z})\,dz + \bar{a}(z,\bar{z})\,d\bar{z}, \tag{2.120}$$

for which the corresponding spin connection (determined from $de^a + \omega^a{}_b \wedge e^b = 0$) is

$$\omega^{12} = -\omega^{21} = F_a e^3 + (\omega_{2\text{D}})^{12}, \quad \omega^{23} = -\omega^{32} = -F_a e^1, \quad \omega^{31} = -\omega^{13} = -F_a e^2 \tag{2.121}$$

where we have defined

$$F_a(z,\bar{z}) \equiv \frac{i(\partial_{\bar{z}}a - \partial_z\bar{a})}{c^2}, \quad (\omega_{2\text{D}})^{12} = -(\omega_{2\text{D}})^{21} = -\frac{i}{c}(\partial_z c\,dz - \partial_{\bar{z}}c\,d\bar{z}) \tag{2.122}$$

with $\omega_{2\mathrm{D}}$ being the spin connection associated to e^1, e^2 for the 2D metric $c^2\, dz\, d\bar{z}$. Note that F_a is independent of the choice of chart, while $\omega_{2\mathrm{D}}$ is not. We have on spinors[14] that

$$\nabla_\mu = \partial_\mu - \frac{i}{2} F_a \gamma_\mu \cdot + i \left(F_a e^3_\mu + \frac{1}{2} (\omega_{2\mathrm{D}})^{12}_\mu \right) \gamma^3 \cdot , \tag{2.123}$$

where the dots indicate matrix multiplication rather than spinor contraction (see Appendix A.2). Hence if we take

$$H = -iF_a, \quad A^{(R)} = -\left(F_a e^3 + \frac{1}{2} (\omega_{2\mathrm{D}})^{12} \right), \tag{2.124}$$

then the generalized Killing spinor equations (2.114) are solved by

$$\xi = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\xi} = x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.125}$$

in a basis where $\gamma^a = x\sigma^a x^{-1}$ (here, as in the definition of K^μ, we really mean the commuting spinors $\xi|_0$ and $\tilde{\xi}|_0$). In particular, $\xi, \tilde{\xi}$ are constant in the chosen frame, and since $x \in SU(2)$, we have both $\tilde{\xi} = \xi^\dagger$ and $\xi|_0 \xi^\dagger|_0 = 1$. Regardless of basis, we have

$$K^a = (\xi|_0)\gamma^a(\tilde{\xi}|_0) = \begin{pmatrix} 0 & -1 \end{pmatrix} \gamma^a \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta^{a3}, \tag{2.126}$$

so that $K = \partial_{\tilde{\psi}}$.

2.6.2 Localizing "Seifert" Loops

We now describe how bulk $V_\mu = 0$ SUSY' restricts to BPS Wilson loops. To summarize, our assumption that M^3 admits a real, nowhere vanishing Killing vector restricts it to be a Seifert manifold. On any such manifold, it is possible to define a 3D $\mathcal{N} = 2$ supergravity background with $V_\mu = 0$, in which the Killing spinors take a simple form. Namely, we work in local coordinates $(\tilde{\psi}, z, \bar{z})$ such that $K = \partial_{\tilde{\psi}}$ and the metric takes the standard form (2.119): upon choosing the frame (2.120) and the background fields H and $A^{(R)}$ as in (2.124), the

[14] $\nabla_\mu \psi = (\partial_\mu + \omega^{ab}_\mu \sigma_{ab})\psi$ where $\sigma^{ab} = \frac{1}{8}[\gamma^a, \gamma^b]$.

generalized Killing spinor equation $D_\mu \eta = -\frac{1}{2} H \gamma_\mu \eta$ (with D_μ as in (2.117)) has solutions $\eta = \xi, \tilde{\xi}$ of R-charge ± 1 as in (2.125).

However, the integral curves of the Killing vector field may not be compact. Therefore, local coordinates adapted to the Killing vector do not necessarily define a Seifert fibration of M^3. Thus the Wilson loops that we consider, while supported on the Seifert manifold M^3, are not necessarily Seifert loops. The quotation marks in the title of this subsection serve to emphasize that the term "Seifert loop" (in the sense of [31]) is a misnomer.

To begin, consider a Euclidean 3D $\mathcal{N} = 2$ Wilson loop along a curve γ [62, 68]:

$$W = \mathrm{Tr}_R \, P \exp \left[i \oint_\gamma (A_\mu dx^\mu - i\sigma ds) \right] = \mathrm{Tr}_R \, P \exp \left[i \oint_\gamma d\tau \, (A_\mu \dot{x}^\mu - i\sigma |\dot{x}|) \right]. \tag{2.127}$$

The BPS conditions following from (2.115) take the same form on any background geometry:

$$n^\mu \gamma_\mu \xi - \xi = 0, \quad n^\mu \gamma_\mu \tilde{\xi} + \tilde{\xi} = 0, \tag{2.128}$$

with $n^\mu = \dot{x}^\mu / |\dot{x}|$ being the unit tangent vector to γ. They are satisfied when $n^\mu = -K^\mu$. Hence a BPS Wilson loop preserving both supercharges under consideration lies along an integral curve of K^μ.

To determine how bulk SUSY' restricts to these BPS Wilson loops, note that even after demanding that the Killing spinors $\xi, \tilde{\xi}$ be properly normalized, we still have the freedom to introduce a relative phase between them (the overall phase is immaterial). Therefore, let us keep ξ as in (2.125), with $K^\mu = \xi \gamma^\mu \xi^\dagger$, and write

$$\tilde{\xi} = \rho x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \rho \xi^\dagger, \quad |\rho| = 1. \tag{2.129}$$

The linear combinations of 3D fermions that appear in the 1D multiplets depend on the gamma matrix conventions. For simplicity, we work in the basis $\gamma^a = \sigma^a$ ($a = 1, 2, 3$). According to the above discussion, we fix $(n^1, n^2, n^3) = (0, 0, -1)$. Restoring Grassmann parameters, we have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \rho\bar{\omega} \end{pmatrix}. \tag{2.130}$$

To restrict the SUSY' transformations (2.115) and (2.116), we drop dependence on the 1 and 2 directions and consider only the component of the gauge field along the loop. Along the loop, frame and spacetime indices are equivalent since $e_3^3 = 1$. For the vector multiplet, it is convenient to define the 1D SUSY'-covariant derivative

$$D_3'(\cdot) \equiv \partial_3(\cdot) - i[A_3 + i\sigma, (\cdot)] \tag{2.131}$$

on *both* scalars and spinors, which satisfies $\delta' D_3'(\cdot) = D_3'\delta'(\cdot)$ and $D_3'\sigma = D_3\sigma$. Note that $D_3'(\cdot)$ and $D_3(\cdot) + [\sigma, (\cdot)]$ coincide on scalars, but not on spinors; note also that in 1D, we need not diffeomorphism-covariantize the derivative acting on spinors because the spin connection is trivial. In our supergravity background and frame, we have on spinors that

$$\nabla_i = \partial_i - \frac{1}{2}H\gamma_i + iA_i^{(R)}\gamma_3 \implies \nabla_3\psi_{i_\perp} = \partial_3\psi_{i_\perp} - (-1)^{i_\perp}\left(\frac{1}{2}H - iA_3^{(R)}\right)\psi_{i_\perp} \tag{2.132}$$

where $i_\perp = 1, 2$. Moreover, it follows from the $V_\mu = 0$ SUSY' algebra that the gauginos $\lambda, \tilde{\lambda}$ have R-charges ∓ 1, so (2.117) and (2.132) give

$$D_3\lambda_1 = \partial_3\lambda_1 + \frac{1}{2}H\lambda_1 - i[A_3, \lambda_1], \quad D_3\tilde{\lambda}_2 = \partial_3\tilde{\lambda}_2 - \frac{1}{2}H\tilde{\lambda}_2 - i[A_3, \tilde{\lambda}_2]. \tag{2.133}$$

Specializing to our specific $\xi, \tilde{\xi}$, we obtain from (2.115) (using (2.131) and (2.133)) the following restricted multiplets in 1D:

- A 1D vector $\{0, 0, A_3 + i\sigma\}$ where $\delta'(A_3 + i\sigma) = 0$.
- Two independent 1D adjoint chirals (not related by complex conjugation) $\{0, \lambda_2\}$ and $\{0, \tilde{\lambda}_1\}$ where $\delta'\lambda_2 = \delta'\tilde{\lambda}_1 = 0$.

The remaining fields do not comprise good multiplets for any ρ. It turns out that for a 3D chiral to restrict to a 1D chiral, we must choose $\rho = i$. Indeed, consider a fundamental chiral multiplet of dimension Δ. The corresponding 1D SUSY'-covariant derivative is

$$D_3'(\cdot) \equiv \partial_3(\cdot) - i(A_3 + i\sigma)(\cdot), \tag{2.134}$$

which satisfies $\delta' D_3'(\cdot) = D_3'\delta'(\cdot)$. From the $V_\mu = 0$ SUSY' algebra, we see that A, ψ, F have R-charges $-\Delta, 1-\Delta, 2-\Delta$, respectively, so that

$$\begin{aligned} D_3 A &= (\partial_3 - iA_3 + i\Delta A_3^{(R)})A, \\ D_3\psi_1 &= \nabla_3\psi_1 - iA_3\psi_1 - i(1-\Delta)A_3^{(R)}\psi_1 \end{aligned} \tag{2.135}$$

with $\nabla_3\psi_1$ as in (2.132). Substituting our specific $\xi, \tilde{\xi}$ into (2.116), using (2.134), and choosing both $\rho = i$ and $\Delta = 0$, we obtain a single restricted multiplet in 1D, namely a 1D fundamental chiral $\{A, -\sqrt{2}\psi_2\}$ where

$$\begin{aligned} \delta' A &= \omega(-\sqrt{2}\psi_2), \\ \delta'(-\sqrt{2}\psi_2) &= 2\bar{\omega} D_3' A. \end{aligned} \tag{2.136}$$

The remaining transformation rules do not close.

The key point is that the transformation rules for the restricted 1D multiplets are independent of the supergravity background fields and take exactly the same form as in flat space. The fields are *a priori* complex, and D_3' is not Hermitian because it involves a complexified gauge field. After imposing reality conditions in the path integral, we want σ purely imaginary, A_3 purely real, and D purely imaginary; the fermions remain independent.

2.7 Matching $\mathcal{N} = 0$ and $\mathcal{N} = 2$ Line Operators

So far, we have explained the quantum-mechanical non-renormalization of the weight only for certain classes of BPS observables in pure $\mathcal{N} = 2$ Chern-Simons, which can be computed via localization on three-manifolds that admit a real, nowhere vanishing Killing vector. This amounts to an explanation of the renormalization of the weight for a similarly restricted set of observables in the corresponding $\mathcal{N} = 0$ theory, namely those links deformable to BPS configurations. It may be possible to achieve a more general understanding by localizing on a solid torus [1]. In this section, we make some further comments on the correspondence between $\mathcal{N} = 0$ and $\mathcal{N} = 2$ observables.

2.7.1 (Non-)Renormalization

To substantiate the claim that the natural UV completion of Chern-Simons theory should have $\mathcal{N} = 2$ supersymmetry, it is (as mentioned in the introduction) important to fix unambiguous definitions of the level k and the representation λ. Throughout, we have used the canonical definition that the k in $\mathcal{N} = 0$ G_k Chern-Simons theory is the level of the corresponding 2D WZW model, where it appears in correlation functions and has a precise physical meaning. Relative to this definition, the level that appears in the Chern-Simons braiding matrix with parameter q is $k+h$. This shift is independent of regularization scheme, i.e., the question of how the renormalized coupling depends on the bare coupling. Said differently, our $k \equiv k_{\text{phys}}$ is what determines the dimension of the Hilbert space and changes sign under time reversal, while $k_{\text{phys}} + h$ is what appears in correlation functions. The relation of k_{phys} to some UV parameter k_{bare} (e.g., via $k_{\text{phys}} = k_{\text{bare}} + h$ or $k_{\text{phys}} = k_{\text{bare}}$) is a question of regularization scheme and not physically meaningful.

On the other hand, λ determines the conjugacy class of the holonomy around a Wilson loop to be $e^{-2\pi i\lambda/k}$, as measured by another loop that links it. This relation, derived from the classical EOMs, receives quantum corrections. For example, in the case of $SU(2)$ (and using our convention for λ from Section 2.4), the classical and quantum holonomies are $e^{2\pi i j\sigma_3/k}$ and $e^{2\pi i(j+1/2)\sigma_3/(k+2)}$, respectively. To interpret the statement that "λ is not renormalized" in the $\mathcal{N} = 2$ setting, it should be kept in mind that Wilson loops are typically written not in terms of the *bare* λ, but rather in terms of an effective λ that corresponds to having integrated out the fermions along the line.

2.7.2 Equivalence Relations

It is worth noting that there exists a one-to-one correspondence between line operators in the bosonic and supersymmetric theories only if we take into account both shifts.

As can be seen in canonical quantization, the distinct Wilson lines in pure Chern-Simons theory are in one-to-one correspondence with the ground states of the theory on a torus. To explain what "distinct" means, we must identify the precise equivalence classes of Wilson lines that map to these ground states. $SU(2)_k$ Chern-Simons on a torus has $k+1$ ground states labeled by half-integers $j = 0, \ldots, k/2$. These can equivalently be viewed as the $k+1$ primary operators in the $SU(2)_k$ WZW model. From the 3D point of view, however, a Wilson line can carry any spin j. To respect the 2D truncation, all such lines fall into equivalence classes labeled by the basic lines $j = 0, \ldots, k/2$. The equivalence relations turn out to be a combination of Weyl conjugation and translation [9]:

$$j \sim -j, \quad j \sim j + k. \tag{2.137}$$

For general G, line operators are subject to equivalence relations given by the action of the affine Weyl group at level k ($\mathcal{W} \ltimes k\Lambda_R^\vee$, $\Lambda_R^\vee$ being the coroot lattice of G), whose fundamental domain we refer to as an affine Weyl chamber and which contains all inequivalent weights (corresponding to integrable representations of $\widehat{G}_k$).

Now consider the correlation functions of these lines. Two basic observables of $SU(2)_k$ Chern-Simons on S^3 are the expectation value of an unknotted spin-j Wilson loop and the expectation value of two Wilson loops of spins j, j' in a Hopf link:

$$\langle W_j \rangle_{\mathcal{N}=0} = \frac{S_{0j}}{S_{00}}, \quad \langle W_j W_{j'} \rangle_{\mathcal{N}=0} = \frac{S_{jj'}}{S_{00}}. \tag{2.138}$$

Recall that the modular S-matrix of $SU(2)_k$ is given by (2.13) in a basis of integrable representations. The formulas (2.138) apply only to Wilson loops with j within the restricted range $0, \ldots, k/2$. Indeed, (2.13) is not invariant under the equivalence relations (2.137). Nonetheless, let us naïvely extend these formulas to arbitrary j, j'. The first positive value of j for which $\langle W_j \rangle = 0$ is that immediately above the truncation threshold: $j = (k+1)/2$. More generally, from (2.138), it is clear that a line of spin j and a line of spin $j+k+2$ have identical correlation functions, while lines with $j = n(k/2+1) - 1/2$ for any integer n vanish identically. Here, one should distinguish the *trivial* line $j = 0$, which has $\langle W_0 \rangle = 1$ and trivial braiding with all other lines, from *nonexistent* lines, which have $\langle W_j \rangle = 0$ and vanishing correlation functions with all other lines. On the other hand, a line with j and a line with $j + k/2 + 1$ have the same expectation value and braiding, *up to a sign.* In other words, at the level of correlation functions, $SU(2)_k$ Wilson lines are antiperiodic with period $k/2+1$.

An analogous antiperiodicity phenomenon holds for arbitrary simple, compact G. In the WZW model, the fusion rule eigenvalues (computed from the S-matrix elements) are equal to the finite Weyl characters of G, evaluated on some special Cartan angles that respect the truncation of the relevant representations [78]. For example, in $SU(2)_k$, $\lambda_\ell^{(n)} = S_{\ell n}/S_{0n}$ is the Weyl character $\chi_\ell(\theta)$ in (2.2) evaluated at $\theta/2 = (2n+1)\pi/(k+2)$ for $n = 0, \ldots, k/2$, chosen such that the Weyl character of spin $\ell = (k+1)/2$ vanishes.

The (anti)periodicity of S under $j \to j + (k+2)/2$ can be understood in terms of the renormalized parameters $K = k+2$ and $J = j + 1/2$ (see also [38]). In the $\mathcal{N} = 2$ theory, a J Wilson line has holonomy $e^{2\pi i J \sigma_3/K}$, so the equivalence relations are

$$J \sim -J, \quad J \sim J + K \quad \Longleftrightarrow \quad j \sim -1 - j, \quad j \sim j + k + 2. \tag{2.139}$$

The inequivalent values of j are $-1/2, \ldots, (k+1)/2$. The extremal values $j = -1/2$ and $j = (k+1)/2$ correspond to identically zero line operators, and the remaining values are the same as in the $\mathcal{N} = 0$ formulation. In other words, in contrast to (2.138) for $\mathcal{N} = 0$ $SU(2)_k$ on S^3, we have for $\mathcal{N} = 2$ $SU(2)_K$ on S^3 that

$$\langle W_J \rangle_{\mathcal{N}=2} = \frac{S_{\frac{1}{2}J}}{S_{\frac{1}{2}\frac{1}{2}}}, \quad \langle W_J W_{J'} \rangle_{\mathcal{N}=2} = \frac{S_{JJ'}}{S_{\frac{1}{2}\frac{1}{2}}}, \quad S_{JJ'} \equiv \sqrt{\frac{2}{K}} \sin\left[\frac{(2J)(2J')\pi}{K}\right], \tag{2.140}$$

where the *bare* J must satisfy $J \geq 1/2$ for supersymmetry not to be spontaneously broken. In the supersymmetric theory, labeling lines by $J = 1/2, \ldots, (k+1)/2$, the $J = 0$ line does not exist due to the vanishing Grassmann integral over the zero modes of the fermion in the $\mathcal{N} = 2$ coadjoint orbit sigma model. The conclusion is that the $\mathcal{N} = 2$ theory has the same set of independent line operators as the $\mathcal{N} = 0$ theory. In the $\mathcal{N} = 2$ formulation, the S-matrix $S_{jj'}$ is explicitly invariant under the equivalence relations (2.139).

For general G, let $\Lambda = \lambda + \rho$ and $K = k + h$. Then Λ, modulo the action of the affine Weyl group at level K, takes values in an affine Weyl chamber at level K. Those $\lambda = \Lambda - \rho$ for Λ at the boundary of the chamber correspond to nonexistent lines, while those for Λ in the interior are in one-to-one correspondence with weights in the affine Weyl chamber at level k [9].

It would be interesting to understand both shifts from an intrinsically 2D point of view, namely to translate the equivalence between (J, K) and (j, k) into an equivalence between ordinary and super WZW models [79, 80, 81].

2.8 General Gauge Groups

Using $SU(2)$ as a case study, we have supersymmetrized the coadjoint orbit quantum mechanics on a Wilson line in flat space from both intrinsically 1D and 3D points of view, providing several complementary ways to understand the shift in the representation j. We have described how to extend this understanding to certain compact Euclidean manifolds.

For arbitrary simple groups, one has both generic and degenerate coadjoint orbits, corresponding to quotienting G by the maximal torus T or by a Levi subgroup $L \supset T$ (see [82] and references therein). For example, the gauge group $SU(N+1)$ has for a generic orbit the phase space $SU(N+1)/U(1)^N$, a flag manifold with real dimension N^2+N (corresponding to a regular weight); on the other hand, the most degenerate orbit is $SU(N+1)/(SU(N) \times U(1)) \cong S^{2N+1}/S^1 \cong \mathbb{CP}^N$, which has $2N$ real dimensions and a simple Kähler potential (corresponding to a weight that lies in the most symmetric part of the boundary of the positive Weyl chamber). The quantization of the phase space $\mathbb{CP}^N$ is well-known and can be made very explicit in terms of coherent states [25]. The Fubini-Study metric for $\mathbb{CP}^N$ follows from covering the manifold with $N+1$ patches with the Kähler potential in each patch being the obvious generalization of that for $SU(2)$. In principle, one can carry out a similar analysis with $SU(N+1)$ Killing vectors. We simply remark that in general, the shift of a fundamental weight by the Weyl vector is no longer a fundamental weight, so one would need a qualitatively different sigma model than the original to describe the coadjoint orbit of the shifted weight. An option is not to work in local coordinates at all, along the lines of [31] (however, this approach seems harder to supersymmetrize).

Chapter 3

Topological Sectors in Non-Topological Gauge Theories

In this chapter, we turn our attention to the Coulomb branch, or the vector multiplet moduli space, of 3D $\mathcal{N} = 4$ gauge theories. For the sake of exposition, we proceed by algebraic means, eschewing Lagrangians and avoiding analytic localization computations (or simply quoting the results thereof) wherever possible. This quick route to our results complements the more rigorous approach outlined in [2, 3].

3.1 Overview

Three-dimensional gauge theories contain local defect operators called monopole operators, which are defined by requiring certain singular behavior of the gauge field near the insertion point. These operators play important roles in the dynamics of these theories, such as in establishing IR dualities between theories with different UV descriptions (see, e.g., [83, 84]). Because these operators are not polynomial in the Lagrangian fields, they are difficult to study even in perturbation theory, and most studies so far have focused on determining only their quantum numbers [83, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97].

In supersymmetric theories, one can construct BPS monopole operators by assigning additional singular boundary conditions for some of the scalars in the vector multiplet. For such BPS monopoles, some nonperturbative results are known: for instance, in $\mathcal{N} = 4$ theories, their exact conformal dimension was determined in [84, 98, 99, 100] for "good" or "ugly" theories (a 3D $\mathcal{N} = 4$ gauge theory is classified as good, bad, or ugly if the minimum of the set of monopole operator dimensions, computed using the UV R-symmetry, is ≥ 1, ≤ 0, or $= 1/2$, respectively [98]). We develop nonperturbative techniques for calculating correlation functions of certain BPS monopole operators in 3D $\mathcal{N} = 4$ QFTs.

We focus on Lagrangian 3D $\mathcal{N} = 4$ gauge theories constructed from vector multiplets and hypermultiplets. These theories do not allow for the presence of Chern-Simons terms. For a matter representation of sufficiently large dimension, these theories flow in the IR to interacting SCFTs, whose correlation functions are generally intractable. However, these theories also contain one-dimensional protected subsectors whose correlation functions are topological [12, 13]. As we demonstrate shortly, computations in these subsectors become tractable. While 3D $\mathcal{N} = 4$ SCFTs generally have two distinct protected topological sectors, one associated with the Higgs branch and one with the Coulomb branch, it is the Coulomb branch sector that contains monopole operators and that will be the focus of our work (the Higgs branch sector was studied in [14]). From the 3D SCFT point of view, the information contained in either of the two protected sectors is equivalent to that contained in the ($n \leq 3$)-point functions of certain half-BPS local operators in the SCFT [12, 13].

The Coulomb (Higgs) branch protected sector consists of operators in the cohomology of a supercharge $\mathcal{Q}^C$ ($\mathcal{Q}^H$) that is a linear combination of Poincaré and conformal supercharges. As such, one may think that the protected sector mentioned above is emergent at the IR fixed point, and therefore inaccessible in the UV description. This is indeed true for SCFTs defined on $\mathbb{R}^3$. However, if one defines the QFT on S^3 instead of on $\mathbb{R}^3$, then the protected sector becomes accessible in the UV because on S^3, the square of $\mathcal{Q}^C$ does not contain special conformal generators. Indeed, Poincaré and special conformal generators are mixed together when mapping a CFT from $\mathbb{R}^3$ to S^3. The square of $\mathcal{Q}^C$ includes an isometry of S^3 that fixes a great circle, and this is the circle where the 1D topological quantum mechanics (TQM) lives.

Previous work [14] used supersymmetric localization on S^3 to derive a simple Lagrangian description for the 1D Higgs branch theory. The Coulomb branch case is more complicated because it involves monopole operators. We describe how to compute all observables within the 1D Coulomb branch topological sector of a 3D $\mathcal{N} = 4$ gauge theory by constructing "shift operators" whose algebra is a representation of the OPE of the 1D TQM operators (also known as twisted(-translated) Coulomb branch operators, for reasons that will become clear). Having explicit descriptions of both the Higgs and Coulomb branch 1D sectors allows for more explicit tests of 3D mirror symmetry, including the precise mirror map between half-BPS operators of dual theories.

The 1D TQM provides a "quantization" of the ring of holomorphic functions defined on the Coulomb branch $\mathcal{M}_C$. This can be explained as follows. The 3D theories that we study have two distinguished branches of the moduli space of vacua: the Higgs branch and the Coulomb branch. These are each parametrized, redundantly, by VEVs of gauge-invariant chiral operators whose chiral ring relations determine the branches as complex algebraic varieties. While the Higgs branch chiral ring relations follow from the classical Lagrangian, those for the Coulomb branch receive quantum corrections. The Coulomb branch is constrained by extended SUSY to be a generically singular hyperkähler manifold of quaternionic dimension equal to the rank of G, which, with respect to a choice of complex structure, can be viewed as a complex symplectic manifold. The half-BPS operators that acquire VEVs on the Coulomb branch, namely Coulomb branch operators (CBOs), consist of monopole operators, their dressings by vector multiplet scalars, and operators built from the vector multiplet scalars themselves (monopole operator VEVs encode those of additional scalar moduli, the dual photons). All of the holomorphic functions on $\mathcal{M}_C$ are given by VEVs of the subset of CBOs that are chiral with respect to an $\mathcal{N} = 2$ subalgebra. Under the OPE, these operators form a ring, which is isomorphic to the ring $\mathbb{C}[\mathcal{M}_C]$ of holomorphic functions on $\mathcal{M}_C$. It was argued in [13] that because the operators in the 1D TQM are in one-to-one correspondence with chiral ring CBOs, the 1D TQM is a deformation quantization of $\mathbb{C}[\mathcal{M}_C]$. Indeed, the 1D OPE induces an associative but noncommutative product on $\mathbb{C}[\mathcal{M}_C]$ referred to as a star product, which in the limit $r \to \infty$ (r being the radius of S^3) reduces to the ordinary product of the corresponding holomorphic functions, and that at order $1/r$ gives the Poisson bracket of the corresponding holomorphic functions.

Both the quantization of [13] in the "$Q+S$" cohomology and our quantization on a sphere are realizations of the older idea of obtaining a lower-dimensional theory by passing to the equivariant cohomology of a supercharge, which originally appeared in the context of the Ω-deformation in 4D [101] (see also [102]) and was applied to 3D theories in [103, 104, 105].

Our procedure for solving the 1D Coulomb branch theory uses a combination of cutting and gluing techniques [106, 107], supersymmetric localization, and a consistency requirement that we refer to as polynomiality. We first cut S^3 into two hemispheres $HS^3_\pm$ along an equatorial $S^2 = \partial HS^3_\pm$ orthogonal to the circle along which the 1D operators live (see Figure 3.1). Correlators are then represented by an inner product of wavefunctions generated by the path integral on $HS^3_\pm$ with insertions of twisted CBOs. In [2], it was shown that it suffices to consider such wavefunctions $\Psi_\pm(\mathcal{B}_{\text{BPS}})$ with operator insertions only at the tip of $HS^3_\pm$, and evaluated on a certain class of half-BPS boundary conditions $\mathcal{B}_{\text{BPS}}$. Insertions of twisted CBOs anywhere on the great semicircles of $HS^3_\pm$ can then be realized, up to irrelevant $\mathcal{Q}^C$-exact terms, as simple shift operators acting on this restricted class of wavefunctions. It was shown in [2] that these shift operators can be fully reconstructed from general principles and knowledge of $\Psi_\pm(\mathcal{B}_{\text{BPS}})$. In addition, their algebra provides a faithful representation of the star product. Finally, one can determine expectation values (i.e., one can define an evaluation map on $\mathbb{C}[\mathcal{M}_C]$, known as the trace map in deformation quantization) by gluing $\Psi_+(\mathcal{B}_{\text{BPS}})$ and $\Psi_-(\mathcal{B}_{\text{BPS}})$ with an appropriate measure.

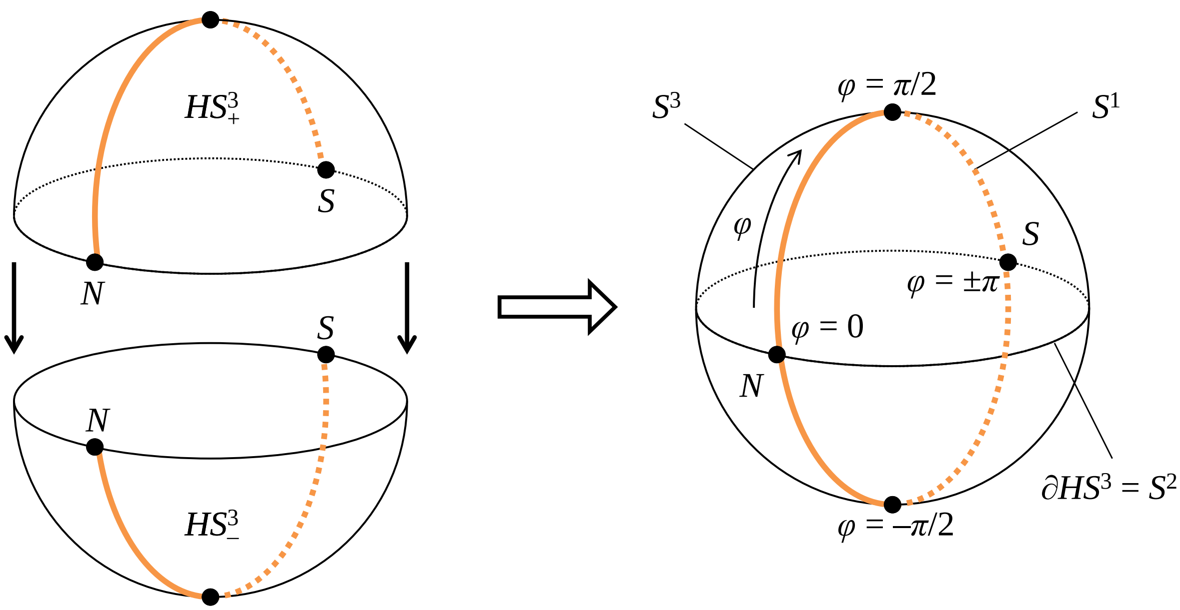

Figure 3.1: Gluing two hemispheres $HS^3_\pm \cong B^3$ to obtain S^3. The 1D TQM lives on the S^1 parametrized by the angle φ (thick orange line). This circle intersects the equatorial $S^2 = \partial HS^3_\pm$ at two points identified with its North (N) and South (S) poles.

The fact that the star product can be determined independently of evaluating correlators is very useful. First, calculating correlators using the above procedure involves solving matrix integrals, while the star product can be inferred from the comparatively simple calculation of the wavefunctions $\Psi_\pm(\mathcal{B}_{\rm BPS})$. Second, the matrix models representing correlators diverge for "bad" theories in the sense of Gaiotto and Witten [98]. Nevertheless, the HS^3 wavefunctions and the star product extracted from them are well-defined even in those cases. Therefore, our formalism works perfectly well even for bad theories, as far as the Coulomb branch and its deformation quantization are concerned.

We also provide a new way of analyzing "monopole bubbling" [16]. Monopole bubbling is a phenomenon whereby the charge of a singular monopole is screened to a lower one by small 't Hooft-Polyakov monopoles. In our setup, this phenomenon manifests itself through the fact that our shift operators for a monopole of given charge contain contributions proportional to those of monopoles of smaller charge, with coefficients that we refer to as bubbling coefficients. While we do not know of a localization-based algorithm for obtaining these coefficients in general, we propose that the requirement that the OPE of any two 1D TQM operators should be a polynomial in the 1D operators uniquely determines the bubbling coefficients, up to operator mixing ambiguities (direct localization computations of bubbling in 4D were undertaken in [17, 18, 108] and refined by [19, 109, 20]).

The main mathematical content of this work is a construction of deformation quantizations of Coulomb branches of 3D $\mathcal{N} = 4$ theories that also satisfy the truncation condition of [13] in the case of good or ugly theories, as a consequence of the existence of the natural trace map (the one-point function). By taking the commutative limit, we recover the ordinary Coulomb branch of the theory in the form of the "abelianization map" proposed by [21]. Therefore, our approach also provides a way to derive the abelianization proposal of [21] from first principles.

Our strategy in this chapter is to deduce our results for correlators of twisted CBOs by dimensionally reducing the Schur index of 4D $\mathcal{N} = 2$ theories enriched by BPS 't Hooft-Wilson loops [110, 108, 111]. The 4D line defect Schur index can be computed by a path integral on $S^3 \times S^1$. To preserve supersymmetry, the defects (which wrap the S^1) should be inserted at points along a great circle in S^3. Upon dimensional reduction on the S^1, the line defects become twisted CBOs in the 3D dimensionally reduced theory.

3.2 Preliminaries

3.2.1 $\mathcal{N} = 4$ Theories on S^3

We study 3D $\mathcal{N} = 4$ gauge theories with gauge group G and matter representation $R \oplus \overline{R}$. They consist of a vector multiplet $\mathcal{V}$ valued in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a hypermultiplet $\mathcal{H}$ valued in a (generally reducible) unitary representation R of G, where $\mathcal{H}$ can be written in terms of half-hypermultiplets taking values in $R \oplus \overline{R}$. The theory could also be deformed by real masses and FI parameters, which we set to zero for simplicity. We denote by $\mathfrak{t}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\mathcal{W}$ the Weyl group, Δ the set of roots, Λ_W the weight lattice, and $\Lambda_W^\vee$ the coweight lattice.

We place such theories supersymmetrically on the round S^3 of radius r. The sphere is a natural setting for deformation quantization of moduli spaces because the Coulomb and Higgs branches in such a background can be viewed as noncommutative, with $1/r$ playing the role of a quantization parameter. As with the 2D Ω-background in flat space [21], the result is an effective compactification of spacetime to a line.

Furthermore, quantized Coulomb and Higgs branch chiral rings are directly related to physical correlation functions. In particular, they encode the OPE data of the BPS operators in the IR superconformal theory, whenever it exists. This relation equips the star product algebra of observables with a natural "trace" operation (the one-point function of the QFT) as well as a natural choice of basis in which operators are orthogonal with respect to the two-point function and have well-defined conformal dimensions at the SCFT point.

The $\mathcal{N} = 4$ supersymmetry algebra on S^3 is $\mathfrak{s} = \mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r$ or its central extension $\widetilde{\mathfrak{s}} = \widetilde{\mathfrak{su}(2|1)}_\ell \oplus \widetilde{\mathfrak{su}(2|1)}_r$, whose central charges correspond to supersymmetric mass and FI deformations of the theory. In the flat-space limit $r \to \infty$, $\mathfrak{s}$ becomes the $\mathcal{N} = 4$ super-Poincaré algebra. The even subalgebra of $\mathfrak{s}$ contains the $\mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r$ isometries of S^3, whose generators we denote by $J_{\alpha\beta}^{(\ell)}$ and $J_{\alpha\beta}^{(r)}$, as well as the R-symmetry subalgebra $\mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$ generated by R_ℓ and R_r. The odd generators are denoted by $\mathcal{Q}_\alpha^{(\ell\pm)}$ and $\mathcal{Q}_\alpha^{(r\pm)}$. The commutation relations of $J_{\alpha\beta}^{(\ell)}$, R_ℓ, $\mathcal{Q}_\alpha^{(\ell\pm)}$ are

$$[J_i^{(\ell)}, J_j^{(\ell)}] = i\epsilon_{ijk} J_k^{(\ell)}, \qquad [J_{\alpha\beta}^{(\ell)}, \mathcal{Q}_\gamma^{(\ell\pm)}] = \frac{1}{2}\left(\varepsilon_{\alpha\gamma}\mathcal{Q}_\beta^{(\ell\pm)} + \varepsilon_{\beta\gamma}\mathcal{Q}_\alpha^{(\ell\pm)}\right), \tag{3.1}$$

$$[R_\ell, \mathcal{Q}_\alpha^{(\ell\pm)}] = \pm\mathcal{Q}_\alpha^{(\ell\pm)}, \qquad \{\mathcal{Q}_\alpha^{(\ell+)}, \mathcal{Q}_\beta^{(\ell-)}\} = -\frac{4i}{r}\left(J_{\alpha\beta}^{(\ell)} + \frac{1}{2}\varepsilon_{\alpha\beta}R_\ell\right), \tag{3.2}$$

where we have set

$$J_{\alpha\beta}^{(\ell)} \equiv \begin{pmatrix} -(J_1^{(\ell)} + iJ_2^{(\ell)}) & J_3^{(\ell)} \\ J_3^{(\ell)} & J_1^{(\ell)} - iJ_2^{(\ell)} \end{pmatrix}. \tag{3.3}$$

The generators of $\mathfrak{su}(2|1)_r$ obey the same relations with $\ell \to r$.

It is convenient to exhibit S^3 as an S^1 fibration over the disk D^2 with the fiber shrinking at the boundary. We embed S^3 in $\mathbb{R}^4$ as $\sum_{i=1}^4 X_i^2 = r^2$ and parametrize the X_i by

$$X_1 + iX_2 = r\cos\theta e^{i\tau}, \quad X_3 + iX_4 = r\sin\theta e^{i\varphi}, \tag{3.4}$$

where $\theta \in [0, \pi/2]$ and $\varphi, \tau \in [-\pi, \pi]$. In these coordinates, $\sin\theta e^{i\varphi}$ parametrizes the unit disk, and $e^{i\tau}$ the S^1 fiber. We use the notation

$$P_\tau = -(J_3^\ell + J_3^r), \quad P_\varphi = -J_3^\ell + J_3^r \tag{3.5}$$

to denote two particular $U(1)$ isometries of S^3, where the fixed-point locus of P_τ is the great φ-circle S^1_φ and P_φ is a rotation along S^1_φ. After conformally mapping to flat space, P_τ would be a rotation that fixes the image of S^1_φ, which is a line.

The 3D $\mathcal{N} = 4$ superconformal algebra $\mathfrak{osp}(4|4)$, with R-symmetry subalgebra $\mathfrak{so}(4) \cong \mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, contains $\mathfrak{s}$ as a subalgebra. This embedding is parametrized by a choice of $\mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$ inside $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, which is specified by the Cartan elements

$$h_a{}^b \in \mathfrak{su}(2)_H, \quad \overline{h}^{\dot{a}}{}_{\dot{b}} \in \mathfrak{su}(2)_C, \tag{3.6}$$

where $a, b, \ldots = 1, 2$ ($\dot{a}, \dot{b}, \ldots = 1, 2$) label the fundamental irrep of $\mathfrak{su}(2)_H$ ($\mathfrak{su}(2)_C$). Here, $h_a{}^b$ and $\overline{h}^{\dot{a}}{}_{\dot{b}}$ are traceless Hermitian matrices satisfying $h_a{}^c h_c{}^b = \delta_a{}^b$ and $\bar{h}^{\dot{a}}{}_{\dot{c}} \bar{h}^{\dot{c}}{}_{\dot{b}} = \delta^{\dot{a}}{}_{\dot{b}}$. They determine a relation between the generators R_ℓ, R_r of $\mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$ and the generators $R_a{}^b$, $\overline{R}^{\dot{a}}{}_{\dot{b}}$ of $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$:

$$\frac{1}{2}(R_\ell + R_r) = \frac{1}{2} h_a{}^b R_b{}^a \equiv R_H, \quad \frac{1}{2}(R_\ell - R_r) = \frac{1}{2} \overline{h}^{\dot{a}}{}_{\dot{b}} \overline{R}^{\dot{b}}{}_{\dot{a}} \equiv R_C. \tag{3.7}$$

Our convention is that

$$h_a{}^b = -\sigma^2, \quad \overline{h}^{\dot{a}}{}_{\dot{b}} = -\sigma^3. \tag{3.8}$$

Different choices of $h, \bar{h}$ are related by conjugation with $SU(2)_H \times SU(2)_C$ and determine which components in the triplets of FI and mass parameters can be present on the sphere.

The vector multiplet transforms in the adjoint representation of G and has components

$$\mathcal{V} = (A_\mu, \lambda_{\alpha a\dot{a}}, \Phi_{\dot{a}\dot{b}}, D_{ab}), \tag{3.9}$$

consisting of the gauge field A_μ, gaugino $\lambda_{\alpha a\dot{a}}$, and scalars $\Phi_{\dot{a}\dot{b}} = \Phi_{\dot{b}\dot{a}}$ and $D_{ab} = D_{ba}$, which transform in the trivial, $(\mathbf{2}, \mathbf{2})$, $(\mathbf{1}, \mathbf{3})$, and $(\mathbf{3}, \mathbf{1})$ irreps of the $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$ R-symmetry, respectively. The hypermultiplet transforms in the R of G and has components

$$\mathcal{H} = (q_a, \widetilde{q}_a, \psi_{\alpha\dot{a}}, \widetilde{\psi}_{\alpha\dot{a}}) \tag{3.10}$$

where $q_a, \widetilde{q}_a$ are scalars transforming as $(\mathbf{2}, \mathbf{1})$ under the R-symmetry and as $R, \overline{R}$ under G, respectively, while $\psi_{\alpha\dot{a}}, \widetilde{\psi}_{\alpha\dot{a}}$ are their fermionic superpartners and transform as $(\mathbf{1}, \mathbf{2})$ under the R-symmetry.

Supersymmetry transformations and supersymmetric actions for $\mathcal{V}$, $\mathcal{H}$ can be found in [14, 2]. The gauged hypermultiplet action $S_{\text{hyper}}[\mathcal{H}, \mathcal{V}]$ preserves the full superconformal symmetry $\mathfrak{osp}(4|4)$. The super Yang-Mills action $S_{\text{YM}}[\mathcal{V}]$ preserves only the subalgebra $\mathfrak{s}$. The theory $S_{\text{hyper}}[\mathcal{H}, \mathcal{V}]$ has flavor symmetry group $G_H \times G_C$, whose Cartan subalgebra we denote by $\mathfrak{t}_H \oplus \mathfrak{t}_C$. The factor G_H acts on the hypermultiplets, while $G_C \cong \text{Hom}(\pi_1(G), U(1))$ (possibly enhanced in the IR) contains the topological $U(1)$ symmetries that act on monopole operators. It is possible to couple the theory to a supersymmetric background twisted vector multiplet in $\mathfrak{t}_C$, which on S^3 leads to a single FI parameter ζ for every $U(1)$ factor of the gauge group (as opposed to an $\mathfrak{su}(2)_H$ triplet on $\mathbb{R}^3$). Similarly, one can introduce real masses for the hypermultiplets by turning on background vector multiplets in $\mathfrak{t}_H$. On S^3, there is a single real mass parameter for every generator in $\mathfrak{t}_H$ (as opposed to an $\mathfrak{su}(2)_C$ triplet on $\mathbb{R}^3$). In the presence of nonzero real mass and FI parameters, $\mathfrak{s}$ is centrally extended by charges Z_ℓ and Z_r for the respective factors of the superalgebra. The central charges are related to the mass/FI parameters by

$$\frac{1}{r}(Z_\ell + Z_r) = i\widehat{m} \in i\mathfrak{t}_H, \qquad \frac{1}{r}(Z_\ell - Z_r) = i\widehat{\zeta} \in i\mathfrak{t}_C. \tag{3.11}$$

3.2.2 $\mathcal{Q}^{H/C}$-Cohomology

We now introduce twisted operators and their corresponding topological sectors. The SUSY algebra $\widetilde{\mathfrak{s}}$ contains two particularly interesting supercharges:[1]

$$\mathcal{Q}^H = \mathcal{Q}_1^{(\ell+)} + \mathcal{Q}_1^{(r-)} + \mathcal{Q}_2^{(\ell-)} + \mathcal{Q}_2^{(r+)}, \tag{3.12}$$

$$\mathcal{Q}^C = \mathcal{Q}_1^{(\ell+)} + \mathcal{Q}_1^{(r+)} + \mathcal{Q}_2^{(\ell-)} + \mathcal{Q}_2^{(r-)}. \tag{3.13}$$

They satisfy the relations

$$(\mathcal{Q}^H)^2 = \frac{4i}{r}(P_\tau + R_C + ir\widehat{\zeta}), \qquad (\mathcal{Q}^C)^2 = \frac{4i}{r}(P_\tau + R_H + ir\widehat{m}). \tag{3.14}$$

The quantities $\widehat{\zeta}$ and $\widehat{m}$ stand for FI and mass deformations, i.e., central charges of $\widetilde{\mathfrak{s}}$. The most important features of $\mathcal{Q}^H$ and $\mathcal{Q}^C$ emerge when we consider their equivariant cohomology classes in the space of local operators. The operators annihilated by $\mathcal{Q}^H$ are the so-called twisted-translated Higgs branch operators (HBOs), whose OPE encodes a quantization of the Higgs branch; such operators were studied in [14]. Correspondingly, the cohomology of $\mathcal{Q}^C$ contains twisted-translated Coulomb branch operators. Such operators must be inserted along the great circle S^1_φ fixed by $(\mathcal{Q}^C)^2$, and their OPE encodes a quantization of the Coulomb branch. Specifically, the properties

$$\{\mathcal{Q}^H, \dots\} = P_\varphi + R_H + ir\widehat{m}, \qquad \{\mathcal{Q}^C, \dots\} = P_\varphi + R_C + ir\widehat{\zeta} \tag{3.15}$$

motivate the definitions of the *twisted translations*

$$\widehat{P}^H_\varphi = P_\varphi + R_H, \qquad \widehat{P}^C_\varphi = P_\varphi + R_C, \tag{3.16}$$

which are $\mathcal{Q}^H$- (or $\mathcal{Q}^C$-) closed operations and can therefore be used to translate cohomology classes along S^1_φ. Furthermore, when $\widehat{m} = 0$ (or $\widehat{\zeta} = 0$), the twisted translation $\widehat{P}^H_\varphi$ (or $\widehat{P}^C_\varphi$) is exact under $\mathcal{Q}^H$ (or $\mathcal{Q}^C$). The twisted-translated cohomology classes then become independent of the position φ along the circle. It follows that each supercharge $\mathcal{Q}^{H/C}$

[1]Here, we set $\beta = 1$ relative to [14, 2].

has an associated 1D topological sector of cohomology classes: the OPE of these twisted HBOs/CBOs is an associative but noncommutative product, since there exists an ordering along the line.

To be concrete, recall that in 3D $\mathcal{N}=4$ SCFTs, half-BPS operators are labeled by their charges (Δ, j, j_H, j_C) under the bosonic subalgebra $\mathfrak{so}(3,2)\oplus\mathfrak{su}(2)_H\oplus\mathfrak{su}(2)_C$ of $\mathfrak{osp}(4|4)$. They are Lorentz scalars ($j=0$) and can be classified as either HBOs ($\Delta=j_H$, $j_C=0$) or CBOs ($\Delta=j_C$, $j_H=0$), which we write with $\mathfrak{su}(2)_{H/C}$ spinor indices as $\mathcal{O}^H_{(a_1\cdots a_{2j_H})}$ and $\mathcal{O}^C_{(\dot{a}_1\cdots\dot{a}_{2j_C})}$. In a Lagrangian theory, HBOs are precisely gauge-invariant polynomials in the hypermultiplet scalars $q_a, \tilde{q}_a$, while CBOs consist of the vector multiplet scalars $\Phi_{\dot{a}\dot{b}}$ and (dressed) monopole operators $\mathcal{M}^b_{\dot{a}_1\cdots\dot{a}_{2j_C}}$.

To define operators in the cohomology at arbitrary φ, one simply applies the appropriate twisted translation in (3.16). For an HBO $\mathcal{O}^H_{a_1\cdots a_{2j_H}}$, the corresponding twisted-translated operator is given by

$$\mathcal{O}^H(\varphi)=u^{a_1}(\varphi)\cdots u^{a_{2j_H}}(\varphi)\mathcal{O}^H_{a_1\cdots a_{2j_H}}(\varphi),\quad u^a=\begin{pmatrix}\cos\frac{\varphi}{2}\\ \sin\frac{\varphi}{2}\end{pmatrix}. \tag{3.17}$$

For a CBO $\mathcal{O}^C_{\dot{a}_1\cdots\dot{a}_{2j_C}}$, the corresponding twisted-translated operator is given by

$$\mathcal{O}^C(\varphi)=v^{\dot{a}_1}(\varphi)\cdots v^{\dot{a}_{2j_C}}(\varphi)\mathcal{O}^C_{\dot{a}_1\cdots\dot{a}_{2j_C}}(\varphi),\quad v^{\dot{a}}=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\varphi/2}\\ e^{-i\varphi/2}\end{pmatrix}. \tag{3.18}$$

The precise expressions for u^a and $v^{\dot{a}}$ in (3.17) and (3.18) follow from our definition of the algebra $\mathfrak{s}$ via the Cartan elements (3.8) for $\mathfrak{su}(2)_{H/C}$. Because the translations in (3.16) are accompanied by R-symmetry rotations, the twisted operators (3.17), (3.18) at $\varphi=0$ and $\varphi\neq 0$ are both in chiral rings, but with respect to distinct Cartan elements of $\mathfrak{su}(2)_H$ (or $\mathfrak{su}(2)_C$). Since cohomology classes at different points φ are not mutually chiral, they may have nontrivial SCFT correlators.[2]

[2]Due to the definition of twisted translations, both u and v from (3.17) and (3.18) are antiperiodic under $\varphi\to\varphi+2\pi$. Therefore, twisted translations give antiperiodic operators on the circle for half-integral R-spins. Antiperiodic observables are only single-valued on the double cover of S^1_φ. We deal with this ambiguity by inserting a "branch point" at $\varphi=\pm\pi$, in the presence of which all observables become single-valued. For

These twisted operators are interesting for at least two reasons. First, their two- and three-point functions fix those of HBOs and CBOs in the full 3D theory, by conformal symmetry and R-symmetry. Second, at any fixed φ, twisted operators in the cohomology of $\mathcal{Q}^{H/C}$ are in one-to-one correspondence with elements of the Higgs/Coulomb branch chiral ring. The position-dependent R-symmetry polarization vector u or v fixes a complex structure on the corresponding branch, so that the operators are chiral with respect to an $\mathcal{N} = 2$ superconformal subalgebra of $\mathfrak{osp}(4|4)$ whose embedding depends on the vector.[3]

So far, we have classified operators in the cohomology within SCFTs. In practice, we need a definition of such operators along RG flows on S^3, where only $\mathfrak{s} \subset \mathfrak{osp}(4|4)$ is preserved. Along the flow, the $\mathfrak{su}(2)_{H,C}$ symmetries are broken to their $\mathfrak{u}(1)_{H,C}$ Cartans. The operators $\mathcal{O}^H_{a_1\cdots a_{2j_H}}$ and $\mathcal{O}^C_{\dot{a}_1\cdots\dot{a}_{2j_C}}$ are still present, but their different $a_i, \dot{a}_i = 1, 2$ components are no longer related by $\mathfrak{su}(2)_{H,C}$, and their correlators therefore need not respect these symmetries away from the fixed point. However, the twisted operators (3.17) and (3.18) are still in the cohomology, and this notion is well-defined along the flow. Furthermore, $\mathcal{O}^H(\varphi)$ and $\mathcal{O}^C(\varphi)$ are not chiral with respect to any $\mathcal{N} = 2$ subalgebra of $\mathfrak{s}$ preserved along the flow; they become chiral with respect to certain such subalgebras of $\mathfrak{osp}(4|4)$, which is only realized at the fixed point. Nevertheless, they are half-BPS under $\mathfrak{s}$.

3.2.3 Monopole Operators

Our primary interest is in monopole operators in the cohomology of $\mathcal{Q}^C$, so it behooves us to give a careful definition of these objects. Half-BPS monopole operators were first defined for 3D $\mathcal{N} = 4$ theories in [84]. The twisted-translated monopole operators that we study are essentially those of [84] undergoing an additional $SU(2)_C$ rotation as we move along S^1_φ [2].

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each observable, we pick its sign at $\varphi = 0$, and if we ever have to move an observable $\mathcal{O}^H$ or $\mathcal{O}^C$ past the branch point, then it picks up an extra sign of $(-1)^{2j_H}$ or $(-1)^{2j_C}$.

[3]Functions on the Coulomb branch are in one-to-one correspondence with Coulomb branch operators. For instance, the operators $\mathcal{O}^C_{\dot{a}_1\cdots\dot{a}_{2j_C}}$ correspond to a spin-j_C multiplet of functions under $\mathfrak{su}(2)_C$. With respect to the $\mathfrak{su}(2)_C$ polarization v, one can identify $v^{\dot{a}_1}\cdots v^{\dot{a}_{2j_C}}\mathcal{O}^C_{\dot{a}_1\cdots\dot{a}_{2j_C}}$ as the holomorphic component of this multiplet of functions, and one can regard the corresponding operator as chiral.

A monopole operator is a point-like source of magnetic flux in 3D spacetime [83]. For semisimple G, the monopole charge b is a cocharacter of G, referred to as the GNO charge [15]. A cocharacter is an element of $\mathrm{Hom}(U(1), G)/G \cong \mathrm{Hom}(U(1), \mathbb{T})/\mathcal{W}$. Passing from the element of $\mathrm{Hom}(U(1), \mathbb{T})/\mathcal{W}$ to the map of algebras $\mathbb{R} \to \mathfrak{t}$, we see that cocharacters can also be identified with Weyl orbits in the coweight lattice $\Lambda_W^\vee \subset \mathfrak{t}$ of G, i.e., in the weight lattice of the Langlands dual group ${}^L G$. Since every Weyl orbit contains exactly one dominant weight (lying in the fundamental Weyl chamber), it is conventional to label monopole charges by dominant weights of ${}^L G$ [21]. Let $b \in \mathfrak{t}$ be such a dominant weight of ${}^L G$. Then a bare monopole operator is defined by a sum over $\mathcal{W}b$, the Weyl orbit of b, of path integrals with singular boundary conditions defined by elements of $\mathcal{W}b$. Specifically, the insertion of a twisted-translated monopole operator at a point $\varphi \in S^1_\varphi$ is defined by the following singular boundary conditions for $F_{\mu\nu}$ and $\Phi_{\dot{a}\dot{b}}$:

$$*F \sim b\frac{y_\mu dy^\mu}{|y|^3}, \qquad \Phi_{\dot{1}\dot{1}} = -(\Phi_{\dot{2}\dot{2}})^\dagger \sim -\frac{b}{2|y|}e^{-i\varphi}, \qquad \Phi_{\dot{1}\dot{2}} \sim 0, \tag{3.19}$$

where it is understood that one must compute not a single path integral, but rather a sum of path integrals over field configurations satisfying (3.19) with b ranging over the full Weyl orbit of a given dominant weight. Here, "$\sim$" means "equal up to regular terms" and y^μ are local Euclidean coordinates centered at the monopole insertion point. The origin of (3.19) is that twisted-translated monopoles are chiral with respect to the $\mathcal{N} = 2$ subalgebra defined by the polarization vector v at any given φ. This requires that the real scalar in the $\mathcal{N} = 2$ vector multiplet diverge as $\frac{b}{2|y|}$ near the monopole [84] and results in nontrivial profiles for the $\mathcal{N} = 4$ vector multiplet scalars near the insertion point. This background can alternatively be viewed as a solution to the $\mathcal{Q}^C$ BPS equations, with a Dirac monopole singularity $*F \sim b\frac{y_\mu dy^\mu}{|y|^3}$.

We denote such twisted-translated monopole operators by $\mathcal{M}^b(\varphi)$, or simply $\mathcal{M}^b$. The $\mathcal{Q}^C$-cohomology, in addition to $\mathcal{M}^b(\varphi)$ and gauge-invariant polynomials in $\Phi(\varphi)$, contains monopole operators dressed by polynomials $P(\Phi)$, which we denote by $[P(\Phi)\mathcal{M}^b]$. Because monopoles are defined by sums over Weyl orbits, $[P(\Phi)\mathcal{M}^b]$ is not merely a product of $P(\Phi)$ and $\mathcal{M}^b$, but rather:

$$[P(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|}\sum_{w\in\mathcal{W}} P(\Phi^w) \times (\text{charge-}(w\cdot b) \text{ monopole singularity}), \tag{3.20}$$

where Φ^w means that as we sum over the Weyl orbit, we act on the $P(\Phi)$ insertion as well. Because $\mathcal{M}^b$ breaks the gauge group at the insertion point to the subgroup $G_b \subset G$ that preserves b, $P(\Phi)$ must be invariant under the G_b action.[4] Also, to avoid overcounting, we divide by the order of the stabilizer of b in $\mathcal{W}$.[5]

At this point, we pause to mention some subtleties inherent to the above definition:

- First, note that the Weyl group acts canonically on the Cartan subalgebra $\mathfrak{t}$, but it does not have a natural action on the full Lie algebra $\mathfrak{g}$ where Φ is valued. Nonetheless, the action of $\mathcal{W} = N(\mathbb{T})/Z(\mathbb{T})$ on a G_b-invariant polynomial $P(\Phi)$ is unambiguous because, for $b \in \mathfrak{t}$, $Z(\mathbb{T}) \subset G_b$. Hence the action of $w \in N(\mathbb{T})/Z(\mathbb{T})$ on $P(\Phi)$ is well-defined, and this is the action that appears in (3.20).

- Second, the action of the Weyl group on the dressing factor differs from its action on b. This is because global (and gauge) symmetries act in opposite ways on order and disorder operators, i.e., on fields and their boundary conditions [2]. In our case, if we act on b by $w \in \mathcal{W}$ (that is, $b \mapsto w\cdot b$), then we should act on Φ by w^{-1}: $\Phi \mapsto \Phi^w = w^{-1}\cdot\Phi$.

To conclude, by b in $[P(\Phi)\mathcal{M}^b]$, we mean some weight of ${}^LG$ within the given Weyl orbit, though not necessarily the dominant one. The polynomial $P(\Phi)$ appearing inside the square brackets is always the one attached to the charge-b singularity (whether or not b is dominant), whereas the Weyl-transformed singularities $w\cdot b$ are multiplied by Weyl-transformed polynomials, as in (3.20).

[4]After localization, Φ takes values in $\mathfrak{t}_\mathbb{C}$, whence the G_b-invariance of $P(\Phi)$ boils down to $\mathcal{W}_b$-invariance. But the summation in (3.20) automatically averages $P(\Phi)$ over $\mathcal{W}_b \subset \mathcal{W}$. Therefore, later on, when we write formulas in terms of $\Phi \in \mathfrak{t}_\mathbb{C}$, we can insert arbitrary polynomials $P(\Phi)$ in $[P(\Phi)\mathcal{M}^b]$.

[5]Sums over Weyl orbits do not require a factor of $|\mathcal{W}_b|^{-1}$, as $\sum_{b'\in\mathcal{W}b} F(b') = \frac{1}{|\mathcal{W}_b|}\sum_{w\in\mathcal{W}} F(w\cdot b)$.

3.2.4 Topological Quantum Mechanics

To summarize, 3D $\mathcal{N} = 4$ SCFTs have two protected subsectors that each take the form of a topological quantum mechanics [12, 13]. The associative operator algebra of the TQM is a deformation quantization of either the Higgs or Coulomb branch chiral ring. Each 1D sector can be described as the equivariant cohomology of an appropriate supercharge. The corresponding cohomology classes are called *twisted* Higgs or Coulomb branch operators. The OPE within each sector takes the form of a noncommutative star product

$$\mathcal{O}_i \star \mathcal{O}_j \equiv \lim_{\varphi \to 0^+} \mathcal{O}_i(0)\mathcal{O}_j(\varphi) = \sum_k \zeta^{\Delta_i + \Delta_j - \Delta_k} c_{ij}{}^k \mathcal{O}_k \tag{3.21}$$

where, for theories placed on S^3, the quantization parameter ζ is the inverse radius of the sphere: $\zeta = 1/r$. In addition to associativity, the star product inherits several conditions from the physical SCFT, namely [13]: truncation or *shortness* (the sum in (3.21) terminates after the term of order $\zeta^{2\min(\Delta_i, \Delta_j)}$) due to the $SU(2)_H$ or $SU(2)_C$ selection rule, *evenness* (swapping $\mathcal{O}_i$ and $\mathcal{O}_j$ in (3.21) takes $\zeta \to -\zeta$) inherited from the symmetry properties of the 3D OPE, and *positivity* from unitarity (reflection positivity) of the 3D SCFT.

The question remains: do there exist effective methods to perform computations within these protected sectors of 3D $\mathcal{N} = 4$ theories on the sphere? The answer is yes, as we now describe. When the SCFT admits a UV Lagrangian description, the Higgs branch sector is directly accessible by supersymmetric localization, but the Coulomb branch sector includes monopole operators, which are disorder operators that cannot be represented in terms of the Lagrangian fields. For this reason, the known methods for computing OPE data within these two sectors look qualitatively different. Combining the prescriptions for computing observables in the Higgs and Coulomb branch sectors gives a way to derive precise maps between half-BPS operators across 3D mirror symmetry, and to compute previously unknown quantizations of Higgs and Coulomb branch chiral rings. For convenience, we review both prescriptions below. The analysis of the Higgs branch sector was the subject of [14], while the analysis of the Coulomb branch sector is the main subject of this chapter.

Higgs Branch Formalism

The operators that comprise the Higgs branch topological sector are gauge-invariant polynomials in antiperiodic scalars $Q(\varphi), \tilde{Q}(\varphi)$ on S^1_φ, which are twisted versions of the hypermultiplet scalars $q_a, \tilde{q}_a$ transforming in the fundamental of $\mathfrak{su}(2)_H$ and in $R, \overline{R}$ of G. The correlation functions of these twisted HBOs $\mathcal{O}_i(\varphi)$ can be computed within a 1D Gaussian theory [14] with path integral

$$Z_\sigma \equiv \int DQ\, D\tilde{Q}\, \exp\left[4\pi r \int d\varphi\, \tilde{Q}(\partial_\varphi + \sigma)Q\right], \tag{3.22}$$

in terms of which the S^3 partition function is

$$Z_{S^3} = \frac{1}{|\mathcal{W}|}\int_{\mathfrak{t}} d\mu(\sigma), \quad d\mu(\sigma) \equiv d\sigma\, \det{}'_{\text{adj}}(2\sinh(\pi\sigma))\, Z_\sigma = d\sigma\, \frac{\det'_{\text{adj}}(2\sinh(\pi\sigma))}{\det_R(2\cosh(\pi\sigma))}. \tag{3.23}$$

Namely, an n-point correlation function $\langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle$ on S^3 can be written as

$$\langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle = \frac{1}{|\mathcal{W}|Z_{S^3}}\int_{\mathfrak{t}} d\mu(\sigma)\, \langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle_\sigma \tag{3.24}$$

in terms of an auxiliary correlator $\langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle_\sigma$ at fixed σ. The latter is computed via Wick contractions with the 1D propagator

$$\langle Q(\varphi_1)\tilde{Q}(\varphi_2)\rangle_\sigma \equiv G_\sigma(\varphi_{12}) \equiv -\frac{\operatorname{sgn}\varphi_{12} + \tanh(\pi\sigma)}{8\pi r}e^{-\sigma\varphi_{12}}, \quad \varphi_{12} \equiv \varphi_1 - \varphi_2, \tag{3.25}$$

derived from (3.22).[6]

Coulomb Branch Formalism

The operators in the Coulomb branch topological sector consist of the twisted scalar $\Phi(\varphi) = \Phi_{\dot{a}\dot{b}}(\varphi)v^{\dot{a}}v^{\dot{b}}$, bare monopoles $\mathcal{M}^b(\varphi)$, and dressed monopoles $[P(\Phi)\mathcal{M}^b(\varphi)]$.

[6] Wick contractions between elementary operators at coincident points are performed using

$$\langle Q(\varphi)\tilde{Q}(\varphi)\rangle_\sigma \equiv G_\sigma(0) = -\frac{\tanh(\pi\sigma)}{8\pi r} \tag{3.26}$$

to resolve normal-ordering ambiguities.

A method for computing all observables within the Coulomb branch TQM was obtained in [2, 3] by constructing a set of shift operators, acting on functions of $\sigma \in \mathfrak{t}$ and $B \in \Lambda_W^\vee$, whose algebra is a representation of the 1D OPE.[7] We find that $\Phi(\varphi)$ is represented by a simple multiplication operator

$$\Phi = \frac{1}{r}\left(\sigma + \frac{i}{2}B\right) \in \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}. \tag{3.27}$$

On the other hand, the shift operator describing a dressed monopole is constructed as

$$[P(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|}\sum_{w\in\mathcal{W}} P(w^{-1}\cdot\Phi)\widetilde{M}^{w\cdot b} \tag{3.28}$$

where $\mathcal{W}_b$ is the stabilizer of b in $\mathcal{W}$, with the Weyl sum reflecting the fact that a physical magnetic charge is labeled by the Weyl orbit of a coweight b. For a given coweight b, we define the abelianized (non-Weyl-averaged) monopole shift operator

$$\widetilde{M}^b = M^b + \sum_{|v|<|b|} Z^{\text{ab}}_{b\to v}(\Phi) M^v, \tag{3.29}$$

where the sum is taken over coweights shorter than b and the rational functions $Z^{\text{ab}}_{b\to v}(\Phi)$, dubbed *abelianized bubbling coefficients*, account for nonperturbative effects in nonabelian gauge theories in which the GNO charge of a singular monopole is screened away from the insertion point by smooth monopoles of vanishing size [16]. It was proposed in [3] that the abelianized bubbling coefficients are fixed by algebraic consistency of the OPE within the Coulomb branch topological sector. Finally, M^b is an abelianized monopole shift operator that represents a bare monopole singularity in the absence of monopole bubbling:

$$M^b = \frac{\prod_{\rho\in R}\left[\frac{(-1)^{(\rho\cdot b)_+}}{r^{|\rho\cdot b|/2}}\left(\frac{1}{2} + ir\rho\cdot\Phi\right)_{(\rho\cdot b)_+}\right]}{\prod_{\alpha\in\Delta}\left[\frac{(-1)^{(\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}}\left(ir\alpha\cdot\Phi\right)_{(\alpha\cdot b)_+}\right]} e^{-b\cdot\left(\frac{i}{2}\partial_\sigma + \partial_B\right)}, \tag{3.30}$$

where $(x)_+ \equiv \max(x, 0)$, $(x)_n \equiv \Gamma(x+n)/\Gamma(x)$, and powers of r encode scaling dimensions.

[7] WLOG, all expressions are given in the "North" picture, as will be explained in Section 3.3.4.

With the above shift operators in hand, the S^3 correlator of twisted CBOs $\mathcal{O}_i(\varphi_i)$, inserted at points φ_i along S^1_φ with $0 < \varphi_1 < \cdots < \varphi_n < \pi$, can be computed as

$$\langle \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \rangle_{S^3} = \frac{1}{|\mathcal{W}| Z_{S^3}} \sum_B \int d\sigma \, \mu(\sigma, B) \Psi_0(\sigma, B) \mathcal{O}_1 \cdots \mathcal{O}_n \Psi_0(\sigma, B) \tag{3.31}$$

where the operators on the right are understood to be the shift operators corresponding to $\mathcal{O}_i$ and $\langle 1 \rangle_{S^3} = 1$ (we do not make a notational distinction between a shift operator and the twisted CBO that it represents). We have introduced the empty hemisphere wavefunction

$$\Psi_0(\sigma, B) \equiv \delta_{B,0} \frac{\prod_{\rho \in R} \frac{1}{\sqrt{2\pi}} \Gamma(\frac{1}{2} - i\rho \cdot \sigma)}{\prod_{\alpha \in \Delta} \frac{1}{\sqrt{2\pi}} \Gamma(1 - i\alpha \cdot \sigma)} \tag{3.32}$$

as well as the gluing measure[8]

$$\mu(\sigma, B) = \prod_{\alpha \in \Delta^+} (-1)^{\alpha \cdot B} \left[\left(\frac{\alpha \cdot \sigma}{r} \right)^2 + \left(\frac{\alpha \cdot B}{2r} \right)^2 \right] \prod_{\rho \in R} (-1)^{\frac{|\rho \cdot B| - \rho \cdot B}{2}} \frac{\Gamma\left(\frac{1}{2} + i\rho \cdot \sigma + \frac{|\rho \cdot B|}{2} \right)}{\Gamma\left(\frac{1}{2} - i\rho \cdot \sigma + \frac{|\rho \cdot B|}{2} \right)}. \tag{3.33}$$

While the matrix model (3.31) converges only for theories with a sufficiently large matter representation (i.e., good and ugly theories [98]), the shift operators can always be used to compute star products in the Coulomb branch TQM.

Note that in writing (3.31), we have assumed (WLOG) that all operators are inserted within the upper hemisphere HS^3_+ ($0 < \varphi < \pi$), in which case the sum over B collapses to the $B = 0$ term. In the "South" picture, the order of the shift operators would be reversed, yielding the same correlator. In general, the correlator can be written as an inner product of hemisphere wavefunctions with arbitrary insertions (see Section 3.3.1).

Note also that the shift operators do not depend on the insertion point. This must be the case because the correlators are topological and depend only on the order of the insertions, which is reflected in nontrivial commutation relations between shift operators.

Finally, in the commutative limit $r \to \infty$, the algebra of shift operators reduces to the Coulomb branch chiral ring and we recover the abelianization description of the Coulomb

[8] The gluing measure is familiar from the results of $\mathcal{N} = (2, 2)$ localization on S^2 [112, 113, 114, 115]. The supercharge used in [113] for localization is precisely our $\mathcal{Q}^C$.

branch proposed in [21]. In this limit, the operators $e^{-b\cdot(\frac{i}{2}\partial_\sigma+\partial_B)}$ turn into generators $e[b]$ of the group ring $\mathbb{C}[\Lambda_W^\vee]$, which act trivially on functions of Φ but satisfy the relations

$$e[b_1]e[b_2] = e[b_1 + b_2]. \tag{3.34}$$

We find that M^b itself has a well-defined $r \to \infty$ limit,[9]

$$\lim_{r\to\infty} M^b \equiv M^b_\infty = \frac{\prod_{\rho\in R}(-i\rho\cdot\Phi)^{(\rho\cdot b)_+}}{\prod_{\alpha\in\Delta}(-i\alpha\cdot\Phi)^{(\alpha\cdot b)_+}} e[b], \tag{3.35}$$

as do the abelianized bubbling coefficients $Z^{\text{ab}}_{b\to v}(\Phi)$.

3.3 Shift Operators

We now turn to our main task: a derivation of the Coulomb branch formalism.

3.3.1 From HS^3 to S^3

Only a very restricted class of twisted CBO correlators on S^3 is amenable to a direct localization computation [2]. A less direct approach is to endow the path integral on S^3 with extra structure by dividing it into path integrals on two open halves. These path integrals individually prepare states in the Hilbert space of the theory on S^2. The advantage of this procedure is that it allows for operator insertions within S^3 to be implemented by acting on these boundary states with operators on their associated Hilbert spaces.

Specifically, the round S^3 is glued from two hemispheres, HS^3_+ and HS^3_-. Gluing corresponds to taking $\langle\Psi_-|\Psi_+\rangle$, where $|\Psi_+\rangle \in \mathcal{H}_{S^2}$ and $\langle\Psi_-| \in \mathcal{H}^\vee_{S^2}$ are states generated at the boundaries of the two hemispheres. For a certain quantization of the theories of interest on S^2, this operation can be represented as an integral over a finite-dimensional space of

[9] The expression (3.35) holds for semisimple G. Otherwise, it would have some residual r-dependence.

half-BPS boundary conditions, which results in a simple gluing formula [2]:

$$\langle \Psi_- | \Psi_+ \rangle = \frac{1}{|\mathcal{W}|} \sum_{B \in \Lambda_W^\vee} \int_{\mathfrak{t}} d\sigma \, \mu(\sigma, B) \, \langle \Psi_- | \sigma, B \rangle \langle \sigma, B | \Psi_+ \rangle. \tag{3.36}$$

Here, $\mu(\sigma, B)$ is the gluing measure given by the one-loop determinant on S^2 (see (3.33)), and $\langle \Psi_- | \sigma, B \rangle$, $\langle \sigma, B | \Psi_+ \rangle$ are the hemisphere partition functions with prescribed boundary conditions determined by $\sigma \in \mathfrak{t}$ and $B \in \Lambda_W^\vee \subset \mathfrak{t}$. The boundary conditions parametrized by σ, B are half-BPS boundary conditions on bulk fields preserving 2D $(2,2)$ SUSY on S^2, namely an $\mathfrak{su}(2|1)$ subalgebra of $\mathfrak{s}$ containing $\mathcal{Q}^C$. These boundary conditions specify the magnetic flux $B \in \Lambda_W^\vee$ through the boundary S^2.

We cut along the great S^2 located at $\varphi = 0$ and $\varphi = \pm\pi$. Correspondingly, we denote by HS^3_+ and HS^3_- the hemispheres with $0 < \varphi < \pi$ and $-\pi < \varphi < 0$, respectively. In terms of the $\mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r$ supercharges $\mathcal{Q}_\alpha^{(\ell\pm)}$ and $\mathcal{Q}_\alpha^{(r\pm)}$, the $\mathfrak{su}(2|1)$ subalgebra preserved by this cut is conjugate to diag $[\mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r]$ and generated by

$$Q_1^\pm \equiv \mathcal{Q}_1^{(\ell\pm)} \pm \mathcal{Q}_1^{(r\pm)}, \quad Q_2^\pm \equiv \mathcal{Q}_2^{(\ell\pm)} \mp \mathcal{Q}_2^{(r\pm)}. \tag{3.37}$$

Our Coulomb branch supercharge $\mathcal{Q}^C = Q_1^+ + Q_2^-$ is indeed part of this algebra.

The empty HS^3 path integral generates an $\mathfrak{su}(2|1)$-invariant state on the boundary S^2. Moreover, the $SO(3)$ isometry is unbroken by insertions of scalar local operators at the tip of HS^3. In fact, it turns out that the full $\mathfrak{su}(2|1)$ symmetry is preserved by insertions of twisted CBOs at the tip of HS^3. On the other hand, insertions of twisted-translated operators along the great semicircle of HS^3 away from the tip generally break the $\mathfrak{su}(2|1)$ symmetry. However, by fusing such operators, we can reduce calculations involving generic insertions to those involving only $\mathfrak{su}(2|1)$-invariant insertions, without any loss of generality. These configurations generate $\mathfrak{su}(2|1)$-invariant states $\Psi_\pm$ on $\partial HS^3_\pm$.

The gluing formula (3.36) holds as long as the states $\Psi_\pm$ are annihilated by $\mathcal{Q}^C$. This is true for the state at the boundary of the empty hemisphere, and remains valid if we insert $\mathcal{Q}^C$-closed observables inside. Such insertions can be represented as operators acting on the empty hemisphere partition function. It remains to determine how $\Phi(\varphi)$ and dressed monopoles act on the hemisphere partition function.

3.3.2 HS^3 Wavefunction *sans* Bubbling

The form of the hemisphere partition function with insertions of local $\mathcal{Q}^C$-closed observables can be motivated (if not directly computed) by supersymmetric localization. Because correlation functions are topological, we can move all insertions to the tip of the hemisphere and replace them by an equivalent composite operator located there. It suffices to consider a bare monopole at the tip, as it is trivial to include insertions of gauge-invariant monomials in the scalar $\Phi(\varphi)$ anywhere along S^1_φ.

We place the monopole of charge b at $(\theta, \varphi) = (\pi/2, \pi/2)$, which is the tip of the hemisphere, by imposing (3.19) there. We also impose appropriate conditions at the boundary of the hemisphere, which we do not write explicitly. The $\mathcal{Q}^C$ BPS equations, which we also do not write explicitly, have an "abelian" solution that exists iff the boundary (flux) coweight B lies in the Weyl orbit of the monopole charge b. This solution has vanishing fields in the hypermultiplet as well as vanishing fermions in the vector multiplet, while the bosons in the vector multiplet take the form:

$$D_{12} = 0, \quad \Phi_{\dot{1}\dot{2}} = irD_{11} = irD_{22} = \frac{\sigma}{r}, \quad \Phi_{\dot{1}\dot{1}} = \Phi_{\dot{2}\dot{2}} = \frac{iB}{2r\sqrt{1-\sin^2\theta\sin^2\varphi}},$$
$$A^\pm = \frac{B}{2}\left(\frac{\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}} \pm 1\right) d\tau, \tag{3.38}$$

where A^- is defined everywhere on the hemisphere except the interval $\pi/2 \leq \varphi \leq \pi$ at $\theta = \pi/2$; similarly, A^+ is defined everywhere except on $0 \leq \varphi \leq \pi/2,\ \theta = \pi/2$. The abelian solution (3.38) has the feature that all fields with nontrivial VEVs on the localization locus are vector multiplet fields valued in $\mathfrak{t}$. In other words, the VEVs look as though the gauge group were actually $\mathbb{T}$, the maximal torus of G.

One can use the $\mathcal{Q}^C$-exact Yang-Mills action for localization [14, 2] and to compute the relevant one-loop determinants in the background of (3.38). From the results of [2], we infer that the hypermultiplet contributes

$$Z^{\text{hyper}}_{\text{1-loop}} = \prod_{\rho\in R} \frac{1}{r^{\frac{|\rho\cdot B|}{2}}} \frac{\Gamma\left(\frac{1+|\rho\cdot B|}{2} - i\rho\cdot\sigma\right)}{\sqrt{2\pi}}. \tag{3.39}$$

An indirect derivation of the vector multiplet one-loop determinant will be presented in Section 3.3.5. The answer is given by

$$Z^{\text{vec}}_{\text{1-loop}} = \prod_{\alpha\in\Delta} r^{\frac{|\alpha\cdot B|}{2}} \frac{\sqrt{2\pi}}{\Gamma\left(1+\frac{|\alpha\cdot B|}{2} - i\alpha\cdot\sigma\right)}. \tag{3.40}$$

Therefore, the contribution from the abelian solution to the hemisphere partition function with a monopole labeled by a coweight $b\in\Lambda^\vee_W\subset\mathfrak{t}$ inserted at the tip is given by

$$Z(b;\sigma,B) = \sum_{b'\in\mathcal{W}b} \delta_{B,b'} \frac{\prod_{\rho\in R}\frac{1}{\sqrt{2\pi}r^{\frac{|\rho\cdot b'|}{2}}}\Gamma\left(\frac{1+|\rho\cdot b'|}{2} - i\rho\cdot\sigma\right)}{\prod_{\alpha\in\Delta}\frac{1}{\sqrt{2\pi}r^{\frac{|\alpha\cdot b'|}{2}}}\Gamma\left(1+\frac{|\alpha\cdot b'|}{2} - i\alpha\cdot\sigma\right)} \equiv \sum_{b'\in\mathcal{W}b} Z_0(b';\sigma,B), \tag{3.41}$$

where the $\delta_{B,b'}$ enforces that the flux sourced by the monopole equals the flux exiting through S^2. We have introduced the notation Z_0 for an "incomplete" partition function that does not include a sum over the Weyl orbit of b. Note that $\Psi_0(\sigma,B)$ in (3.32) is simply $Z_0(0;\sigma,B)$.

In general, Z as given above is not the full answer because for nonabelian G, the BPS equations might have additional solutions. They correspond to screening effects that go by the name of "monopole bubbling" [16].

3.3.3 HS^3 Wavefunction *avec* Bubbling

Close to the monopole insertions, the $\mathcal{Q}^C$ BPS equations behave as Bogomolny equations on $\mathbb{R}^3$ with a monopole singularity at the origin. Such equations have "screening solutions" in addition to the simple abelian "Dirac monopole" solution mentioned above. Such solutions, while characterized by a monopole singularity with $b\in\Lambda^\vee_W$ at the origin, behave at infinity as a Dirac monopole of a different charge $v\in\Lambda^\vee_W$. Such solutions exist only when v is a weight in the representation with highest weight b such that $|v| < |b|$. For given b and v, let ℓ denote the length scale over which the screening takes place. As $\ell \to 0$, the solution approaches a Dirac monopole of charge v everywhere on $\mathbb{R}^3$ except for an infinitesimal neighborhood of the origin. This solution can be thought of as a singular (Dirac) monopole screened by coincident and infinitesimally small smooth ('t Hooft-Polyakov) monopoles; the latter have GNO charges labeled by coroots. It is natural to suppose that such solutions also exist on S^3: while at finite ℓ they are expected to receive $1/r$ corrections compared to the flat-space case, in the $\ell \to 0$ limit, they should be exactly the same bubbling solutions as on $\mathbb{R}^3$.

Since the BPS equations require monopole solutions to be abelian away from the insertion point, only when $\ell \to 0$ can new singular solutions arise. Since such a solution behaves as an abelian Dirac monopole of charge v almost everywhere, it is convenient to factor out $Z(v; \sigma, B)$ computed in the previous subsection, and to say that the full contribution from the "$b \to v$" bubbling locus is given by

$$Z_{\text{mono}}(b, v; \sigma, B) Z(v; \sigma, B), \tag{3.42}$$

where Z_{mono} characterizes the effect of monopole bubbling. To be precise, a monopole insertion is defined by a sum over singular boundary conditions (3.19): therefore, the contribution of the bubbling locus actually takes the form

$$\sum_{\substack{b' \in \mathcal{W} b \\ v' \in \mathcal{W} v}} Z^{\text{ab}}_{\text{mono}}(b', v'; \sigma, B) Z_0(v'; \sigma, B) \tag{3.43}$$

where b and v are coweights representing magnetic charges, and we sum over their Weyl orbits. We refer to the quantity $Z^{\text{ab}}_{\text{mono}}(b', v'; \sigma, B)$ as an "abelianized bubbling coefficient." These coefficients are expected to behave under Weyl reflections in the following way:

$$Z^{\text{ab}}_{\text{mono}}(w \cdot b, w \cdot v; w \cdot \sigma, w \cdot B) = Z^{\text{ab}}_{\text{mono}}(b, v; \sigma, B), \quad w \in \mathcal{W}. \tag{3.44}$$

Now we can write the complete answer for the hemisphere partition function:

$$\langle \sigma, B|\Psi_b\rangle = Z(b;\sigma,B) + \sum_{|v|<|b|}\sum_{\substack{b'\in\mathcal{W}b\\ v'\in\mathcal{W}v}} Z^{\rm ab}_{\rm mono}(b',v';\sigma,B)Z_0(v';\sigma,B), \tag{3.45}$$

where Ψ_b represents the state generated at the boundary of the hemisphere with a physical monopole of charge b inserted at the tip. Here, the first sum runs over dominant coweights v satisfying $|v| < |b|$, while the second sum runs over the corresponding Weyl orbits.

The localization approach to the computation of $Z_{\rm mono}(b, v; \sigma, B)$ is quite technical [17, 18, 108, 19, 20]. We do not attempt a direct computation of $Z_{\rm mono}(b, v; \sigma, B)$ or $Z^{\rm ab}_{\rm mono}(b, v; \sigma, B)$ here. Instead, we describe a roundabout way to find them from the algebraic consistency of our formalism.

3.3.4 Shift Operators for HS^3

We now derive how insertions of local $\mathcal{Q}^C$-closed observables are represented by operators acting on the hemisphere wavefunction, up to the so-far unknown bubbling coefficients. The easiest ones are polynomials in $\Phi(\varphi)$. We can think of them as entering the hemisphere either through the North pole $(\varphi, \theta) = (0, \pi/2)$ or through the South pole $(\varphi, \theta) = (\pi, \pi/2)$. Then we simply substitute the solution (3.38) into the definition of $\Phi(\varphi)$ either for $0 < \varphi < \pi/2$ or for $\pi/2 < \varphi < \pi$. We find that $\Phi(\varphi)$ is represented by the North and South pole operators

$$\Phi_{N/S} = \frac{1}{r}\left(\sigma \pm \frac{i}{2}B\right) \in \mathfrak{t}_{\mathbb{C}}, \tag{3.46}$$

where B should be thought of as measuring $B \in \Lambda^\vee_w$ at the boundary S^2.

From the structure of the partition functions above, it is clear that the shift operators representing insertions of nonabelian monopoles take the following form:

$$\mathcal{M}^b = \sum_{b'\in\mathcal{W}b} M^{b'} + \sum_{|v|<|b|}\sum_{\substack{b'\in\mathcal{W}b\\ v'\in\mathcal{W}v}} Z^{\rm ab}_{\rm mono}(b',v';\sigma,B)M^{v'}. \tag{3.47}$$

Here, M^b is an abelianized (non-Weyl-averaged) shift operator that represents the insertion of a bare monopole singularity with coweight b, and whose definition ignores bubbling phenomena. The inclusion of the $Z^{\text{ab}}_{\text{mono}}$ takes care of screening effects, and summing over Weyl orbits corresponds to passing to cocharacters, i.e., true physical magnetic charges. The expression (3.47) represents nothing other than the abelianization map proposed in [21]: the full nonabelian operator $\mathcal{M}^b$ is written in terms of the abelianized monopoles M^b acting on wavefunctions $\Psi(\sigma, B)$ on $\mathfrak{t} \times \Lambda^\vee_w$.

It remains to determine the expressions for M^b acting on wavefunctions $\Psi(\sigma, B)$. Again, there are separate sets of operators that implement insertions through the North and South poles. These generate isomorphic algebras, and they are uniquely determined by the following set of consistency conditions:

1. They should shift the magnetic flux at which $\Psi(\sigma, B)$ is supported by $b \in \Lambda^\vee_W$.

2. They should commute with Φ at the opposite pole, i.e., $[M^b_N, \Phi_S] = [M^b_S, \Phi_N] = 0$.

3. They should commute with another monopole at the opposite pole, i.e., $[M^b_N, M^{b'}_S] = 0$.

4. When acting on the vacuum wavefunction, the result should agree with (3.41).

This set of conditions determines the North and South shift operators to be

$$M^b_{N/S} = \frac{\prod_{\rho \in R} \left[\frac{(-1)^{(\pm\rho\cdot b)_+}}{r^{|\rho\cdot b|/2}} \left(\frac{1}{2} + ir\rho\cdot\Phi_{N/S}\right)_{(\pm\rho\cdot b)_+}\right]}{\prod_{\alpha\in\Delta} \left[\frac{(-1)^{(\pm\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}} \left(ir\alpha\cdot\Phi_{N/S}\right)_{(\pm\alpha\cdot b)_+}\right]} e^{-b\cdot(\pm\frac{i}{2}\partial_\sigma + \partial_B)}. \tag{3.48}$$

By counting powers of r^{-1}, it follows that the dimension of a charge-b monopole is $\Delta_b = \frac{1}{2}\left(\sum_{\rho\in R} |\rho\cdot b| - \sum_{\alpha\in\Delta} |\alpha\cdot b|\right)$.

The shift operators satisfy an important multiplication property that allows one to generate monopoles of arbitrary charge from a few low-charge monopoles:

$$M^{b_1}_N \star M^{b_2}_N = P_{b_1,b_2}(\Phi) M^{b_1+b_2}_N \text{ for dominant } b_1 \text{ and } b_2, \tag{3.49}$$

and similarly for the South pole operators, where $P_{b_1,b_2}(\Phi)$ is some polynomial in Φ. We use $\star$ to denote products *as operators* (in particular, shift operators act on the Φ-dependent prefactors in $M_{N,S}$), emphasizing that they form an associative noncommutative algebra. In fact, (3.49) holds slightly more generally than for dominant weights: if Δ_+ is some choice of positive roots, then (3.49) holds whenever $(b_1 \cdot \alpha)(b_2 \cdot \alpha) \geq 0$ for all $\alpha \in \Delta_+$. The property (3.49) ensures that in the product of two physical bare monopoles, the highest-charge monopole appears without denominators. If in addition, b_1 and b_2 satisfy $(b_1 \cdot \rho)(b_2 \cdot \rho) \geq 0$ for all matter weights $\rho \in R$, then a stronger equality holds:

$$M_N^{b_1} \star M_N^{b_2} = M_N^{b_1+b_2}. \tag{3.50}$$

Finally, for general b_1 and b_2, we have:

$$M_N^{b_1} \star M_N^{b_2} = \frac{\prod_{\rho \in R}(-i\rho \cdot \Phi_N)^{(\rho\cdot b_1)_+ + (\rho\cdot b_2)_+ - (\rho\cdot(b_1+b_2))_+}}{\prod_{\alpha \in \Delta}(-i\alpha \cdot \Phi_N)^{(\alpha\cdot b_1)_+ + (\alpha\cdot b_2)_+ - (\alpha\cdot(b_1+b_2))_+}} M_N^{b_1+b_2} + O(1/r). \tag{3.51}$$

These are precisely the abelian chiral ring relations of [21]. Note that (3.35) immediately implies (3.51).

3.3.5 From $S^3 \times S^1$ to S^3

Our results for "unbubbled" partition functions can be derived by dimensionally reducing supersymmetric indices of 4D $\mathcal{N} = 2$ theories on $S^3 \times S^1$. The operators constructed from $\Phi(\varphi)$ lift to supersymmetric Wilson loops wrapping the S^1, while monopole operators lift to supersymmetric 't Hooft loops on S^1.[10] For simplicity, let us first set the radius r of S^3 to 1 and denote the circumference of S^1 by β. To restore r, we simply send $\beta \to \beta/r$.

Since the one-loop determinant for hypermultiplets is known from [2], we concern ourselves with determining the vector multiplet contribution. This can be done in a theory with any conveniently chosen matter content. We can always choose the matter content in such a

[10] To preserve supersymmetry, the line defects that decorate the Schur index can only be inserted above points along a great circle of S^3 [110, 111]. While we will not do so here, one can explicitly construct the map between line defects in the 4D Schur index and twisted CBOs on S^3 [2].

way that both the 4D $\mathcal{N}=2$ and the 3D $\mathcal{N}=4$ theories are conformal. The corresponding 4D index is known as the Schur index. The Schur index [116] is defined as

$$\mathcal{I}(p)=\mathrm{Tr}_{\mathcal{H}_{S^3}}(-1)^F p^{E-R} \tag{3.52}$$

where the trace is taken over the Hilbert space of the 4D theory on S^3 and R is the Cartan generator of the $\mathfrak{su}(2)$ R-symmetry, normalized so that the allowed charges are quantized in half-integer units. In the path integral description, $\mathcal{I}(p)$ evaluated when $p=e^{-\beta}$ is given by an $S^3\times S^1$ partition function, with S^1 of circumference β and with an R-symmetry twist by $e^{\beta R}$ as we go once around the S^1. This $S^3\times S^1$ partition function is invariant under all 4D superconformal generators that commute with $E-R$, or in other words, that have $E=R$. One can easily list these generators and check that they form an $\mathfrak{su}(2|1)\oplus\mathfrak{su}(2|1)$ superalgebra. Thus the superconformal index (3.52) is invariant under $\mathfrak{su}(2|1)\oplus\mathfrak{su}(2|1)$. It is also invariant under all continuous deformations of the superconformal theory: in particular, it is independent of g_{YM} and can be computed at weak coupling.

One can additionally insert an 't Hooft loop of GNO charge b (taken to be a dominant coweight) wrapping S^1 at one pole of S^3 and the oppositely charged loop at the opposite pole of S^3. The result for this modified index in a general 4D $\mathcal{N}=2$ gauge theory, up to a sign and ignoring the bubbling effect, is [108]:

$$\mathcal{I}_b(p)=\frac{1}{|\mathcal{W}_b|}\int\left(\prod_{i=1}^{\mathrm{rank}(G)}\frac{d\lambda_i}{2\pi}\right)\left[\prod_{\alpha\in\Delta}\left(1-e^{i\alpha\cdot\lambda}p^{\frac{|\alpha\cdot b|}{2}}\right)\right]\mathrm{PE}[I_v(e^{i\lambda_i},p)]\,\mathrm{PE}[I_h(e^{i\lambda_i},p)], \tag{3.53}$$

where PE is the plethystic exponential defined as $\mathrm{PE}[f(x)]\equiv\exp\left[\sum_{n=1}^{\infty}\frac{f(x^n)}{n}\right]$, I_v is the contribution from the $\mathcal{N}=2$ vector, and I_h is the contribution from the $\mathcal{N}=2$ hyper in the representation R:

$$I_v(e^{i\lambda_i},p)=-2\sum_{\alpha\in\mathrm{adj}}\frac{p^{1+\frac{|\alpha\cdot b|}{2}}}{1-p}e^{i\alpha\cdot\lambda},\quad I_h(e^{i\lambda_i},p)=\sum_{\rho\in R}\frac{p^{\frac{1}{2}+\frac{|\rho\cdot b|}{2}}}{1-p}(e^{i\rho\cdot\lambda}+e^{-i\rho\cdot\lambda}). \tag{3.54}$$

Using the identity $\exp\left[-\sum_{n=1}^{\infty}\frac{a^n}{n(1-q^n)}\right] = (a;q)$ where $(a;q) \equiv \prod_{n=0}^{\infty}(1-aq^n)$ is the q-Pochhammer symbol, we can rewrite $\mathcal{I}_b(p)$ as

$$\mathcal{I}_b(p) = \frac{(p;p)^{2\,\mathrm{rank}(G)}}{|\mathcal{W}_b|}\int_{-\pi}^{\pi}\left(\prod_{i=1}^{\mathrm{rank}(G)}\frac{d\lambda_i}{2\pi}\right)\frac{\prod_{\alpha\in\Delta}\left[\left(1-e^{i\alpha\cdot\lambda}p^{\frac{|\alpha\cdot b|}{2}}\right)(e^{i\alpha\cdot\lambda}p^{1+\frac{|\alpha\cdot b|}{2}};p)^2\right]}{\prod_{\rho\in R}(e^{i\rho\cdot\lambda}p^{\frac{1}{2}+\frac{|\rho\cdot b|}{2}};p)(e^{-i\rho\cdot\lambda}p^{\frac{1}{2}+\frac{|\rho\cdot b|}{2}};p)}. \tag{3.55}$$

We would like to determine the 3D hemisphere partition function. We first reduce the index (3.55) to the S^3 partition function, and then we use the gluing formula (3.36) to recover the hemisphere partition function as the square root of the absolute value of the integrand. One can then fix signs by consistency with gluing.

To dimensionally reduce (3.55), we take the $\beta \to 0$ limit, where in addition to setting $p = e^{-\beta}$, we scale the integration variable accordingly:

$$\lambda = \beta\sigma \in \mathfrak{t}. \tag{3.56}$$

The angular variable λ (parametrizing the maximal torus $\mathbb{T} \subset G$) then "opens up" into an affine variable $\sigma \in \mathfrak{t}$. To take the limit, one needs the following identities [2]:

$$\frac{1}{(p^x;p)} = e^{\frac{\pi^2}{6\beta}}\beta^{x-\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\Gamma(x)(1+O(\beta)), \qquad (p;p) = \sqrt{\frac{2\pi}{\beta}}e^{-\frac{\pi^2}{6\beta}}(1+O(\beta)), \tag{3.57}$$

which give

$$\mathcal{I}_b \approx \frac{e^{-\frac{\pi^2 r}{3\beta}\left(\dim G-\sum_{I=1}^{N_f}\dim R_I\right)}}{|\mathcal{W}_b|}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{\mathrm{rank}(G)}d\sigma_i\right)\prod_{\alpha\in\Delta^+}\left((\alpha\cdot\sigma)^2+\frac{|\alpha\cdot b|^2}{4}\right)$$
$$\times\frac{\prod_{\rho\in R}\left|\frac{\beta^{\frac{|\rho\cdot b|}{2}}}{\sqrt{2\pi}r^{\frac{|\rho\cdot b|}{2}}}\Gamma\left(\frac{1+|\rho\cdot b|}{2}-i\rho\cdot\sigma\right)\right|^2}{\prod_{\alpha\in\Delta}\left|\frac{\beta^{\frac{|\alpha\cdot b|}{2}}}{\sqrt{2\pi}r^{\frac{|\alpha\cdot b|}{2}}}\Gamma\left(1+\frac{|\alpha\cdot b|}{2}-i\alpha\cdot\sigma\right)\right|^2} \tag{3.58}$$

as $\beta \to 0$. In (3.58), we restored the radius r of S^3 by dimensional analysis.

The exponential prefactor in (3.58) is precisely the Cardy behavior of [117]. In the integrand, we recognize the one-loop contribution of the hypermultiplet to the S^3 partition function,

$$Z^{\text{hyper}}_{\text{1-loop, } S^3}(\sigma) = \prod_{\rho \in R} \left| \frac{1}{\sqrt{2\pi} r^{\frac{|\rho \cdot b|}{2}}} \Gamma\left(\frac{1+|\rho \cdot b|}{2} - i\rho \cdot \sigma \right) \right|^2, \tag{3.59}$$

multiplied by $\beta^{\frac{|\rho \cdot b|}{2}}$. The remaining factor in the integrand must be proportional to the one-loop contribution of the vector multiplet to the S^3 partition function. Assuming that the one-loop vector multiplet contribution comes multiplied by $\beta^{-\frac{|\alpha \cdot b|}{2}}$ (by analogy with the hypermultiplet factor), we conclude that it is equal to

$$Z^{\text{vec}}_{\text{1-loop, } S^3}(\sigma) = \frac{\prod_{\alpha \in \Delta^+} \left((\alpha \cdot \sigma)^2 + \frac{|\alpha \cdot b|^2}{4} \right)}{\prod_{\alpha \in \Delta} \left| \frac{1}{\sqrt{2\pi} r^{\frac{|\alpha \cdot b|}{2}}} \Gamma\left(1 + \frac{|\alpha \cdot b|}{2} - i\alpha \cdot \sigma \right) \right|^2}. \tag{3.60}$$

The S^3 partition function is then given by the expression

$$Z_b = \frac{1}{|\mathcal{W}_b|} \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\text{rank}(G)} d\sigma_i \right) Z^{\text{vec}}_{\text{1-loop, } S^3}(\sigma) Z^{\text{hyper}}_{\text{1-loop, } S^3}(\sigma). \tag{3.61}$$

Note that using this method, the overall normalization of Z_b is ambiguous, but we propose that the correct expression is given by (3.61). This expression passes the check that when $b = 0$, it reduces to the S^3 partition function derived in [32], namely

$$Z = Z_0 = \frac{1}{|\mathcal{W}|} \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\text{rank}(G)} d\sigma_i \right) \frac{\prod_{\alpha \in \Delta^+} 4 \sinh^2(\pi \alpha \cdot \sigma)}{\prod_{\rho \in R} 2 \cosh(\pi \rho \cdot \sigma)}. \tag{3.62}$$

Now we use (3.59) and (3.60) to verify the hemisphere one-loop determinants given in (3.39) and (3.40). To do so, we use the gluing formula (3.36) as well as the explicit expression for the gluing measure in (3.33). It immediately follows that the hypermultiplet and vector multiplet contribute (3.39) and (3.40) to the hemisphere partition function, respectively. The hypermultiplet contribution (3.39) was previously determined by an explicit computation of the one-loop determinant on the hemisphere [2]. We have bypassed the analogous computation for the nonabelian vector multiplet by means of the above argument.

3.4 Dressing and Bubbling

We have derived the structure of bare monopoles, up to the bubbling coefficients. We now extend this construction to dressed monopoles, focusing on the case of simple G.

If the dressing polynomial $P(\Phi)$ is G-invariant, then it is a valid $\mathcal{Q}^C$-closed observable on its own. This makes the definition of the corresponding dressed monopole essentially trivial: we simply "collide" two separate observables $P(\Phi)$ and $\mathcal{M}^b$, which within our formalism means multiplying them as operators acting on the hemisphere wavefunction:

$$[P(\Phi)\mathcal{M}^b] \equiv P(\Phi) \star \mathcal{M}^b. \tag{3.63}$$

If $P(\Phi)$ is only invariant under a subgroup G_b, then $P(\Phi)$ does not make sense as a separate observable. In the absence of bubbling, we would simply define the dressed monopole $[P(\Phi)\mathcal{M}^b] = |\mathcal{W}_b|^{-1} \sum_{w\in\mathcal{W}} P(\Phi^w) M^{w\cdot b}$, but the presence of bubbling makes such a simple definition incomplete.

3.4.1 Primitive Monopoles

A general dressed monopole operator takes the form

$$[P(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|} \sum_{w\in\mathcal{W}} P(\Phi^w) M^{w\cdot b} + \cdots \tag{3.64}$$

where the ellipses stand for bubbling contributions. We show that for fixed b, there exists a set of *primitive* dressed monopoles that generate all others via star products.

Definition 1. Dressed monopoles $[P_1(\Phi)\mathcal{M}^b], [P_2(\Phi)\mathcal{M}^b], \ldots, [P_p(\Phi)\mathcal{M}^b]$ are called primitive (of charge b) if they form a basis for the free module of dressed charge-b monopoles over the ring of G-invariant polynomials. This means that by taking linear combinations

$$\sum_{i=1}^{p} Q_i(\Phi) \star [P_i(\Phi)\mathcal{M}^b] \tag{3.65}$$

where the $Q_i(\Phi)$ are G-invariant polynomials, we obtain dressed monopoles with all possible leading terms of the form (3.64), and furthermore, that p is the minimum number that makes this possible. We will always assume $P_1(\Phi) = 1$, so that the first primitive monopole is the bare monopole itself.

Example. *In $SU(2)$ gauge theory, the Weyl group is $\mathbb{Z}_2$, which takes $b \to -b$ and $\Phi \to -\Phi$. A dressed monopole of charge b takes the form $P(\Phi)M^b + P(-\Phi)M^{-b} +$ (bubbling). In this case, there are only two primitive dressed monopoles for each b:*

$$\mathcal{M}^b = M^b + M^{-b} + \textit{(bubbling)}, \qquad [\Phi\mathcal{M}^b] = \Phi(M^b - M^{-b}) + \textit{(bubbling)}. \tag{3.66}$$

Any other dressed monopole can be defined as

$$[P(\Phi)\mathcal{M}^b] \equiv \frac{P(\Phi) + P(-\Phi)}{2} \star \mathcal{M}^b + \frac{P(\Phi) - P(-\Phi)}{2\Phi} \star [\Phi\mathcal{M}^b]. \tag{3.67}$$

To describe primitive monopoles, it suffices to classify the Weyl-invariant leading terms in (3.64).

Proposition 1. *Let G be a simple Lie group, $\mathcal{W}$ its Weyl group, and b a dominant coweight (a magnetic charge). Then there exists a set of primitive monopoles (of magnetic charge b) $[P_1(\Phi)\mathcal{M}^b], [P_2(\Phi)\mathcal{M}^b], \ldots, [P_p(\Phi)\mathcal{M}^b]$, where $p = |\mathcal{W}b|$ is the size of the Weyl orbit of b.*

Proof. Consider ρ^b, a representation of $\mathcal{W}$ spanned as a $\mathbb{C}$-linear space by the Weyl orbit of the coweight b. We write it in terms of shift operators $M^{w\cdot b}$, $w \in \mathcal{W}$, as

$$\rho^b \equiv \mathrm{Span}_{\mathbb{C}}\{M^{w\cdot b} \,|\, w \in \mathcal{W}\}. \tag{3.68}$$

The Cartan subalgebra $\mathfrak{t}$, in which Φ is valued, is an irreducible rank(G)-dimensional representation of $\mathcal{W}$. But recall from the discussion after (3.20) that $w \in \mathcal{W}$ acts in opposite ways on b and Φ. Thus Φ transforms in the dual representation $\mathfrak{t}^*$, and the dressing factor $P(\Phi)$ transforms in the algebra $\mathbb{C}[\mathfrak{t}]$ of polynomial functions on $\mathfrak{t}$. This implies that dressed monopoles are classified by the invariant subspace $\mathfrak{R}_b^{\mathcal{W}}$ inside the following $\mathcal{W}$-module:

$$\mathfrak{R}_b \equiv \mathbb{C}[\mathfrak{t}] \otimes \rho^b. \tag{3.69}$$

It is known [118] that the structure of $\mathbb{C}[\mathfrak{t}]$ as a $\mathcal{W}$-module is

$$\mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}] \tag{3.70}$$

where $\mathbb{C}[\mathcal{W}]$ is the regular representation, realized on a $\mathcal{W}$-invariant subspace of $\mathbb{C}[\mathfrak{t}]$. Hence

$$\mathfrak{R}_b^{\mathcal{W}} \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} \left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}}. \tag{3.71}$$

A $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$-basis of $\mathfrak{R}_b^{\mathcal{W}}$ is a set of primitive dressed monopoles of magnetic charge b.

Since $\mathbb{C}[\mathcal{W}]$ contains each irreducible representation ρ_i of $\mathcal{W}$ exactly $\dim(\rho_i)$ times, we have by Schur's lemma that $\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho_i\right)^{\mathcal{W}} \cong \mathbb{C}^{\dim(\rho_i)}$. Upon decomposing ρ^b into irreducible components, this immediately gives

$$\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}} \cong \mathbb{C}^{\dim(\rho^b)}. \tag{3.72}$$

Hence there are exactly $\dim(\rho^b) = |\mathcal{W}b|$ primitive dressed monopoles of charge b. □

We have now classified the leading terms in dressed monopoles. Such leading terms must be extended by the appropriate bubbling contributions to give physical dressed monopoles. We now turn to the study of these bubbling contributions.

3.4.2 Abelianized Monopole Bubbling

Suppose we have found a set of polynomials $P_1, \ldots, P_{|\mathcal{W}b|}$ such that the dressed monopoles $[P_i(\Phi)\mathcal{M}^b]$ form the primitive set for a given magnetic charge b, in the sense explained in the previous subsection. That is, $|\mathcal{W}_b|^{-1} \sum_{w\in\mathcal{W}} P_i(\Phi^w) M^{w\cdot b}$ for $i = 1, \ldots, |\mathcal{W}b|$ form a basis for $\mathfrak{R}_b^{\mathcal{W}}$ (the space of dressed charge-b monopoles) over $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$ (the algebra of gauge-invariant polynomials in Φ). We show that there exists a special *bubbled* and abelianized monopole shift operator $\widetilde{M}^b = M^b + \cdots$ such that

$$[P_i(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|} \sum_{w\in\mathcal{W}} P_i(\Phi^w)\widetilde{M}^{w\cdot b}. \tag{3.73}$$

The left-hand side has the following structure: for each i,

$$[P_i(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|} \sum_{w\in\mathcal{W}} P_i(\Phi^w)M^{w\cdot b} + \frac{1}{|\mathcal{W}_b|} \sum_{|v|<|b|} \sum_{w\in\mathcal{W}} V_i^{b\to v}(\Phi^w)M^{w\cdot v}. \tag{3.74}$$

The $V_i^{b\to v}$ are rational functions of $\Phi \in \mathfrak{t}_{\mathbb{C}}$ that encode the bubbling data. By combining (3.73) and (3.74), we obtain a system of linear equations for $\widetilde{M}^{w\cdot b}$, $w \in \mathcal{W}$:

$$\sum_{w\in\mathcal{W}} P_i(\Phi^w)\widetilde{M}^{w\cdot b} = \sum_{w\in\mathcal{W}} P_i(\Phi^w)M^{w\cdot b} + \sum_{|v|<|b|} \sum_{w\in\mathcal{W}} V_i^{b\to v}(\Phi^w)M^{w\cdot v}. \tag{3.75}$$

A solution exists because the matrix of coefficients $P_i(\Phi^w)$ is nondegenerate.[11]

The solution to (3.75) takes precisely the form (3.29). The $Z^{\text{ab}}_{b\to v}(\Phi)$ are rational functions of $\Phi \in \mathfrak{t}_{\mathbb{C}}$ that account for monopole bubbling. They do not have any invariance property under the action of $\mathcal{W}$. We may extend them to non-dominant b by postulating the following transformation property:

$$Z^{\text{ab}}_{w\cdot b\to w\cdot v}(\Phi) = Z^{\text{ab}}_{b\to v}(\Phi^w), \tag{3.76}$$

consistent with (3.44). We refer to these functions as *abelianized bubbling coefficients*.

The expression for $\widetilde{M}^{w\cdot b}$ can be obtained from the expression for $\widetilde{M}^b$ by a Weyl reflection:

$$\widetilde{M}^{w\cdot b} = M^{w\cdot b} + \sum_{|v|<|b|} Z^{\text{ab}}_{b\to v}(\Phi^w)M^{w\cdot v} = M^{w\cdot b} + \sum_{|v|<|b|} Z^{\text{ab}}_{w\cdot b\to w\cdot v}(\Phi)M^{w\cdot v}. \tag{3.77}$$

Having established the existence of abelianized and bubbled monopoles $\widetilde{M}^b$, one can very easily construct arbitrary dressed monopoles as in (3.28).

[11] By construction, $|\mathcal{W}_b|^{-1}\sum_{w\in\mathcal{W}} P_i(\Phi^w)M^{w\cdot b}$ for $i = 1, \ldots, \dim(\rho^b)$ form a basis over $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$. This implies that the rows of the matrix in question are linearly independent over Weyl-invariant polynomials $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$, which can be shown to imply linear dependence over rational functions $\mathbb{C}(\mathfrak{t})$.

3.4.3 The Abelianization Map

Before putting the notion of abelianized bubbling to work, let us comment on how it fits into the context of previous studies.

One approach to understanding the geometry of the Coulomb branch of a 3D $\mathcal{N} = 4$ theory was proposed in [21]. Let $\mathcal{M}_C^{\text{abel}} \subset \mathcal{M}_C$ denote the generic points on the Coulomb branch where the gauge group G is broken to its maximal torus $\mathbb{T}$. Using the fact that the chiral ring is independent of gauge couplings, it was argued in [21] that the abelianized chiral ring $\mathbb{C}[\mathcal{M}_C^{\text{abel}}]$ can be determined by integrating out the massive W-bosons at one loop, ignoring nonperturbative effects. This ring is generated by (VEVs of) dressed chiral monopole operators of $\mathbb{T}$, the complex scalars Φ_a $(a = 1, \ldots, r)$, and the inverses of the W-boson complex masses $\alpha(\Phi)$ for all roots $\alpha \in \Delta$. At points on $\mathcal{M}_C$ where a nonabelian subgroup of G is restored, some $\alpha(\Phi) \to 0$ and hence $\mathbb{C}[\mathcal{M}_C^{\text{abel}}]$ becomes ill-defined. Nonperturbative effects cannot be ignored at such points.

These nonperturbative effects are encoded in the so-called abelianization map, which expresses a chiral monopole operator $\mathcal{M}$ in the nonabelian theory as a linear combination of monopole operators M in the low-energy abelian gauge theory, with coefficients being meromorphic functions of the complex abelian vector multiplet scalars. In our notation, this map takes precisely the form (3.47) or, before Weyl-averaging, (3.29). The abelianization map realizes $\mathbb{C}[\mathcal{M}_C]$ as the subring of $\mathbb{C}[\mathcal{M}_C^{\text{abel}}]$ generated by the operators on the RHS of (3.47). To obtain $\mathbb{C}[\mathcal{M}_C]$, all we need are the abelianized bubbling coefficients Z^{ab}. These bubbling coefficients ensure that $\mathbb{C}[\mathcal{M}_C]$ closes without needing to include $\alpha(\Phi)^{-1}$, so that it is well-defined everywhere on $\mathcal{M}_C$.

The shift operators that we construct allow us to directly compute the OPE of chiral monopole operators within a cohomological truncation of a given 3D $\mathcal{N} = 4$ gauge theory. As explained in the introduction, this OPE encodes information about the geometry of the Coulomb branch beyond the chiral ring data. In particular, our shift operators give a concrete realization of the abelianization map of [21], allowing us to determine the bubbling coefficients from the bottom up. In our approach, the bubbling coefficients so obtained can further be used as input to calculate SCFT correlators via the gluing formula of Section 3.3.1.

In fact, previous formulations of the abelianization map do not distinguish between the abelianized bubbling coefficients Z^{ab} and certain coarser, Weyl-averaged counterparts thereof, denoted by Z_{mono}:

$$Z_{\text{mono}}^{b\to v}(\Phi) \equiv \sum_{b'\in\mathcal{W}b} Z_{b'\to v}^{\text{ab}}(\Phi) \implies \mathcal{M}^b = \sum_{b'\in\mathcal{W}b} M^{b'} + \sum_{|v|<|b|} Z_{\text{mono}}^{b\to v}(\Phi) M^v. \tag{3.78}$$

Even the Z_{mono} remain inaccessible to direct localization computations except in a few classes of examples, namely $G = U(N)$ with fundamental and adjoint hypermultiplets [18, 108]. The fact that the previously considered Weyl-averaged bubbling coefficients Z_{mono} can be written in terms of the more basic Z^{ab} is one of the key observations of our work, and the computability of Z^{ab} is one of our main results.

For a bare monopole, decomposing Z_{mono} into Z^{ab} is merely a rewriting of the Weyl sum. However, the refinement of bubbling by abelianized bubbling turns out to be crucial for constructing dressed monopoles. Given a bare monopole, its abelianized bubbling coefficients allow us to construct all of its dressings in a way that guarantees closure of the star algebra. As we discuss next, our claim is that the closure of this algebra, or "polynomiality," determines Z^{ab} uniquely up to operator mixing, in a sense to be made precise. By taking star products of (dressed) monopoles whose bubbling coefficients are known, one can inductively extract Z_{mono} for all pairs of monopole charges (b, v) with $v < b$.

3.5 Bubbling from Polynomiality

The algebra of quantum Coulomb branch operators, $\mathcal{A}_C$, consists of gauge-invariant polynomials $P(\Phi)$ in the $\mathcal{Q}^C$-closed variable $\Phi(\varphi)$ and dressed monopole operators $[F(\Phi)\mathcal{M}^b]$. The subleading (bubbling) terms in $[F(\Phi)\mathcal{M}^b]$ can involve rational functions of Φ, but the leading term must be built solely from the polynomial $F(\Phi)$. This is the assumption of *polynomiality*, which is motivated by the expectation that the VEVs of operators in $\mathcal{A}_C$ should be algebraic functions on the Coulomb branch.

If we neglect to include bubbling terms in the definition of $[F(\Phi)\mathcal{M}^b]$, then polynomiality in general fails: operator products of such observables produce denominators that do not cancel. We conjecture that polynomiality fully determines the algebra $\mathcal{A}_C$, up to the natural ambiguity of operator mixing.

3.5.1 Mixing Ambiguity

In quantum field theories, an arbitrarily chosen basis of observables need not be diagonal with respect to the two-point function, nor does it need to diagonalize the dilatation operator in the case of a CFT. Observables can mix with others of the same dimension, and on curved spaces, they can also mix with those of lower dimension, the difference being compensated for by background fields. The mixing patterns often depend on short-distance ambiguities that must be resolved in the end by diagonalizing the two-point function.

The presence of operator mixing implies that in our problem, it is natural to construct $\mathcal{A}_C$ modulo r-dependent basis changes of the form

$$\mathcal{O} \mapsto \mathcal{O} + \sum_{\substack{n\geq 0 \\ |\mathcal{O}_n|<|\mathcal{O}|}} \frac{1}{r^n}\mathcal{O}_n \tag{3.79}$$

where $\mathcal{O}_n$ has dimension $\Delta_{\mathcal{O}} - n$ (other quantum numbers, if present, should be preserved by such transformations). We have further imposed the condition $|\mathcal{O}_n| < |\mathcal{O}|$ that the GNO charge b of $\mathcal{O}$ "can bubble" into the GNO charge v of $\mathcal{O}_n$, meaning that $|v| < |b|$ and that v belongs to the ${}^LG$-representation of highest weight b. This is because we wish to think of the leading term of a dressed monopole $[P(\Phi)\mathcal{M}^b]$ as canonically defined, and the subleading (bubbling) terms as possibly ambiguous. One might recognize redefinitions of the form (3.79), without the restriction $|\mathcal{O}_n| < |\mathcal{O}|$, as typical "gauge" transformations in deformation quantization [13].

We refer to (3.79) as the mixing ambiguity in the definition of dressed monopoles. Such shifts significantly alter the bubbling coefficients $V_i^{b\to v}(\Phi)$ appearing in $[P(\Phi)\mathcal{M}^b]$: they can be shifted by polynomials or by multiples of other bubbling terms, which translates into complicated rational ambiguities of the *abelianized* bubbling coefficients $Z^{\mathrm{ab}}_{b\to v}(\Phi)$. We argue that $\mathcal{A}_C$ is uniquely determined by polynomiality up to mixing ambiguities of the form (3.79).

3.5.2 Minuscule Monopoles

The simplest case is that in which the algebra $\mathcal{A}_C$ is generated by monopole operators in minuscule representations of ${}^LG$. Such monopoles cannot bubble because for minuscule coweights b, there are no v such that $|v| < |b|$ and $b - v$ is a coroot. For such monopoles, we have simply

$$[P(\Phi)\mathcal{M}^b] = \frac{1}{|\mathcal{W}_b|}\sum_{w\in\mathcal{W}} P(\Phi^w)M^{w\cdot b}. \tag{3.80}$$

Higher-charge monopole operators might contain bubbling terms, but they are easily determined by taking products of lower-charge monopoles. Such cases were previously addressed in the literature using different methods, and essentially comprise the main examples in [21] because abelianization has a simpler structure in these cases.

Theories with minuscule generators include those with gauge group $PSU(N)$, whose Langlands dual is $SU(N)$: the fundamental weights of $SU(N)$ are minuscule and thus cannot bubble. Another example is $U(N)$ gauge theory, since $U(N)$ is self-dual and its fundamental weights are also minuscule.

We now discuss the more interesting theories with no minuscule generators, limiting ourselves to the case of lowest rank (the strategy is similar for general gauge groups).

3.5.3 Rank-One Theories

The only rank-one gauge theory with no minuscule generators is $SU(2)$ gauge theory. The dual group is $SO(3)$, so the lowest monopole operator corresponds to the vector representation of $SO(3)$. In a normalization where the weights of $SU(2)$ are half-integers and products of weights with monopole charges (cocharacters, or dominant coweights) are integers, the minimal monopole has $b = 2$. It can bubble to the identity, because $0 < |b|$ and $b - 0$ is a root. The abelianized monopole operator takes the form

$$\widetilde{M}^2 = M^2 + Z(\Phi) \tag{3.81}$$

with a single abelianized bubbling term, a function $Z(\Phi)$. Knowledge of $Z(\Phi)$ allows one to construct arbitrary dressed monopole operators of charge 2, and ultimately, by taking star products of the latter, monopoles of arbitrary charge.

The dressed monopole is constructed as

$$\begin{aligned}[P(\Phi)\mathcal{M}^2] &= P(\Phi)\widetilde{M}^2 + P(-\Phi)\widetilde{M}^{-2} \\ &= P(\Phi)M^2 + P(-\Phi)M^{-2} + P(\Phi)Z(\Phi) + P(-\Phi)Z(-\Phi).\end{aligned} \tag{3.82}$$

Clearly, the primitive dressed monopoles in this case are

$$\begin{aligned}\mathcal{M}^2 &= M^2 + M^{-2} + Z(\Phi) + Z(-\Phi), \\ [\Phi\mathcal{M}^2] &= \Phi(M^2 - M^{-2}) + \Phi(Z(\Phi) - Z(-\Phi)).\end{aligned} \tag{3.83}$$

Using (3.30), we compute the following star products of these primitive monopoles with the Weyl-invariant polynomial Φ^2:

$$\begin{aligned}\mathcal{M}^2 \star \Phi^2 &= \left[\left(\Phi - \frac{2i}{r}\right)^2 \mathcal{M}^2\right] + \frac{1}{r}A_0(\Phi^2), \\ [\Phi\mathcal{M}^2] \star \Phi^2 &= \left[\left(\Phi - \frac{2i}{r}\right)^2 \Phi\mathcal{M}^2\right] + \frac{1}{r}A_1(\Phi^2),\end{aligned} \tag{3.84}$$

where we have defined

$$\frac{1}{r}A_0(\Phi^2) \equiv \frac{4}{r^2}(Z(\Phi)+Z(-\Phi)) + \frac{4i}{r}\Phi(Z(\Phi)-Z(-\Phi)) \in \mathbb{C}[\Phi^2],$$
$$\frac{1}{r}A_1(\Phi^2) \equiv \frac{4}{r^2}\Phi(Z(\Phi)-Z(-\Phi)) + \frac{4i}{r}\Phi^2(Z(\Phi)+Z(-\Phi)) \in \mathbb{C}[\Phi^2], \tag{3.85}$$

which are necessarily Weyl-invariant polynomials in $\Phi \in \mathfrak{su}(2)$ by the polynomiality condition. Recall that the operator mixing ambiguity allows one to shift the bubbling coefficients $Z(\Phi)+Z(-\Phi)$ and $\Phi(Z(\Phi)-Z(-\Phi))$ by arbitrary Weyl-invariant polynomials whose degrees are fixed by dimensional analysis. Using the freedom to shift $\Phi(Z(\Phi)-Z(-\Phi))$, we can eliminate $A_0(\Phi^2)$. We can then solve (3.85) for $Z(\Phi)$:

$$Z(\Phi) = -\frac{iA_1(\Phi^2)}{8\Phi(\Phi-\frac{i}{r})}. \tag{3.86}$$

We still have the freedom to shift $Z(\Phi)+Z(-\Phi)$ by a Weyl-invariant polynomial in a way that preserves $\frac{1}{r}(Z(\Phi)+Z(-\Phi)) + i\Phi(Z(\Phi)-Z(-\Phi))$ (because we have fixed the latter expression by demanding $A_0 = 0$). This gives one the freedom to shift $Z(\Phi)$ by

$$\Delta Z(\Phi) = \frac{\Phi+\frac{i}{r}}{2\Phi}V(\Phi^2), \tag{3.87}$$

with $V(\Phi^2)$ an arbitrary Weyl-invariant polynomial. Adding this ambiguity to (3.86) gives

$$Z(\Phi) = \frac{-iA_1(\Phi^2) + 4(\Phi^2+\frac{1}{r^2})V(\Phi^2)}{8\Phi(\Phi-\frac{i}{r})}. \tag{3.88}$$

For any $A_1(\Phi^2)$, there exists a unique $V(\Phi^2)$ such that the numerator does not depend on $\Phi \in \mathfrak{su}(2)$. Therefore, by completely fixing the mixing ambiguity, we find that

$$Z(\Phi) = \frac{c}{\Phi(\Phi-\frac{i}{r})}. \tag{3.89}$$

The value of the dimensionful constant c depends on the matter content of the theory. To fix it, we demand that the expression

$$\mathcal{M}^2 \star [\Phi\mathcal{M}^2] - [(\Phi-2i/r)\mathcal{M}^2] \star \mathcal{M}^2 \tag{3.90}$$

satisfy the polynomiality constraint. For simplicity, we take the theory to have N_f fundamental and N_a adjoint hypermultiplets, so that a charge-b monopole has dimension

$$\Delta_b = \frac{|b|}{2}N_f + |b|(N_a - 1). \tag{3.91}$$

A straightforward computation with shift operators shows that (3.90) evaluates to

$$\frac{8ic^2r^3}{(1+r^2\Phi^2)^2} + \left[\frac{1}{2\Phi}\left(\frac{i}{2r}+\frac{\Phi}{2}\right)^{2(N_f-1)}\left(\frac{3i}{2r}+\Phi\right)^{2N_a}\left(\frac{i}{2r}+\Phi\right)^{2N_a} + (\Phi \leftrightarrow -\Phi)\right]. \tag{3.92}$$

If $N_f \geq 1$, then the second term on the right is a Weyl-invariant polynomial and the only non-polynomial piece is $\frac{8ic^2r^3}{(1+r^2\Phi^2)^2}$, implying that only $c = 0$ is consistent with polynomiality. However, if $N_f = 0$, then one finds that the poles at $\Phi = \pm i/r$ vanish when

$$c^2 = (2r)^{-4N_a} \implies c = \pm(2r)^{-2N_a}. \tag{3.93}$$

The sign of c remains undetermined, and indeed, a choice of sign is simply a choice of basis for the algebra $\mathcal{A}_C$. We find it convenient to fix the sign so that

$$c = (-4r^2)^{-N_a}. \tag{3.94}$$

This value of c agrees with that for $SO(3)$ gauge theory with adjoint matter, which admits the minuscule monopole $\mathcal{M}^1$ and in which we can define $\mathcal{M}^2 = \mathcal{M}^1 \star \mathcal{M}^1$ and $[\Phi\mathcal{M}^2] = [\Phi\mathcal{M}^1] \star \mathcal{M}^1$. Alternatively, we can obtain $SU(N)$ gauge theory by gauging the topological $U(1)$ symmetry of a $U(N)$ theory with the same matter content, whose minuscule monopoles and their dressed versions do not bubble. Proceeding along these lines gives the same value for c as in (3.94).

To summarize, we find that the bubbling coefficient in $SU(2)$ gauge theory with fundamental and adjoint matter, up to the operator mixing ambiguity, takes the form

$$Z(\Phi) = \begin{cases} 0 & \text{if } N_f > 0, \\ \frac{(-4r^2)^{-N_a}}{\Phi(\Phi - \frac{i}{r})} & \text{if } N_f = 0, \end{cases} \tag{3.95}$$

which determines $\widetilde{M^2} = M^2 + Z(\Phi)$.

3.6 Application: $SU(2)$ SQCD

Let us apply the results of the previous section in some simple examples involving $SU(2)$ gauge theory with matter.

3.6.1 (Quantized) Chiral Rings

To see how our formalism can be used to derive the chiral ring relations obeyed by Coulomb branch operators as well as their quantizations, consider again the example of $SU(2)$ gauge theory with N_f fundamental and N_a adjoint hypermultiplets (the following discussion generalizes easily to arbitrary matter content [119]). We restrict to the case $N_f > 0$. In this case, we showed in Section 3.5.3 that the abelianized bubbling coefficient $Z^{\text{ab}}_{2\to 0}(\Phi)$ is a polynomial. Hence, up to operator mixing, we can set all bubbling coefficients to zero.

The Coulomb branch chiral ring is generated by the two primitive monopoles of minimal charge $b = 2$, as well as the Casimir operator Φ^2 (since the Cartan is one-dimensional, we regard Φ as a complex number). We write them as

$$\mathcal{U} = 2^{N_f-1}(M^2 + M^{-2}), \qquad \mathcal{V} = -i2^{N_f-1}\Phi(M^2 - M^{-2}), \qquad \mathcal{W} = \Phi^2 \tag{3.96}$$

where $\Delta_{\mathcal{U}} = N_f + 2N_a - 2$, $\Delta_{\mathcal{V}} = N_f + 2N_a - 1$, and $\Delta_{\mathcal{W}} = 2$. Using the corresponding shift operators obtained from (3.30), we find that

$$\mathcal{V}^2 + \mathcal{U} \star \mathcal{W} \star \mathcal{U} = P(\mathcal{W}) + \frac{2}{r}\mathcal{U} \star \mathcal{V}, \tag{3.97}$$

where all products are understood to be star products and $P(\mathcal{W})$ is a polynomial in $\mathcal{W}$:

$$P(\mathcal{W}) \equiv \frac{\left(\sqrt{\mathcal{W}} + \frac{2i}{r}\right)\left(\sqrt{\mathcal{W}} + \frac{i}{r}\right)^{2(N_f-1)}\left[\left(\sqrt{\mathcal{W}} + \frac{i}{2r}\right)\left(\sqrt{\mathcal{W}} + \frac{3i}{2r}\right)\right]^{2N_a}}{2\sqrt{\mathcal{W}}} + (i \leftrightarrow -i). \tag{3.98}$$

To leading order in $1/r$, we reproduce

$$\mathcal{V}^2 + \mathcal{U}^2\mathcal{W} = \mathcal{W}^{N_f+2N_a-1}, \tag{3.99}$$

which is the defining equation of a $D_{N_f+2N_a}$ singularity ((3.99) could have been obtained directly using the commutative limit of shift operators). When $N_f + 2N_a > 2$, the theory is good and the Coulomb branch is a hyperkähler cone. The generators (3.96) also satisfy the relations

$$[\mathcal{U},\mathcal{W}]_\star = \frac{4}{r}\mathcal{V} - \frac{4}{r^2}\mathcal{U}, \quad [\mathcal{V},\mathcal{W}]_\star = -\frac{4}{r}\mathcal{W}\star\mathcal{U} - \frac{4}{r^2}\mathcal{V}, \quad [\mathcal{U},\mathcal{V}]_\star = -\frac{2}{r}\mathcal{U}^2 + Q(\mathcal{W}), \tag{3.100}$$

where $Q(\mathcal{W})$ is a polynomial in $\mathcal{W}$ given by

$$Q(\mathcal{W}) \equiv \frac{i\left(\sqrt{\mathcal{W}} + \frac{i}{r}\right)^{2(N_f-1)}\left[\left(\sqrt{\mathcal{W}} + \frac{i}{2r}\right)\left(\sqrt{\mathcal{W}} + \frac{3i}{2r}\right)\right]^{2N_a}}{2\sqrt{\mathcal{W}}} + (i \leftrightarrow -i). \tag{3.101}$$

We see that (3.97) and (3.100) do not depend only on the combination $N_f + 2N_a$ that determines the Coulomb branch, so the theories under consideration provide examples of different quantizations of the same chiral ring. Note, however, that the $1/r$ terms in (3.100), like the chiral ring relation (3.99), do depend only on $N_f + 2N_a$: thus the Poisson structure on $D_{N_f+2N_a}$ is the same for all of the distinct quantizations.

3.6.2 Correlation Functions and Mirror Symmetry

Our shift operator formalism also makes correlation functions of twisted CBOs eminently computable, and therefore allows for refined tests of 3D mirror symmetry [120, 121, 122].

3D mirror symmetry is an IR duality of $\mathcal{N} = 4$ theories that exchanges the Higgs and Coulomb branches of dual pairs [120]. The duality is nontrivial for several reasons: while the Higgs branch is protected by a non-renormalization theorem and can be fixed classically from the UV Lagrangian [36], the Coulomb branch receives quantum corrections; the duality exchanges order operators and disorder operators; and nonabelian flavor symmetries visible in one theory may be accidental in the dual. At the same time, $\mathcal{N} = 4$ supersymmetry allows for calculations of protected observables that led to the discovery of the duality and to various tests thereof, such as the match between the IR metrics of the Coulomb and Higgs branches [122], scaling dimensions of monopole operators [84], various curved-space partition functions [123, 124, 51], expectation values of loop operators [53, 54], and the Hilbert series [125].

Consider the following simple example, again involving $SU(2)$ SQCD. It is known that $SU(2)$ SQCD with $N_f = 3$ is dual to $U(1)$ SQED with $N_f = 4$, because both theories are mirror dual to the $U(1)^4$ necklace quiver gauge theory.[12] Their Coulomb branch is given by $\mathbb{C}^2/\mathbb{Z}_4$: it has three holomorphic generators $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ subject to the chiral ring relation $\mathcal{X}\mathcal{Y} = \mathcal{Z}^4$. The generators have dimensions $\Delta_{\mathcal{Z}} = 1$ and $\Delta_{\mathcal{X}} = \Delta_{\mathcal{Y}} = 2$. Let us identify $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ in the SQCD theory.

As before, we work in conventions where $B \in 2\mathbb{Z}$. To compute correlation functions, we use that the vacuum wavefunction (3.32) is

$$\Psi_0(\sigma, B) = \delta_{B,0} \frac{[\frac{1}{2\pi}\Gamma(\frac{1-i\sigma}{2})\Gamma(\frac{1+i\sigma}{2})]^3}{\frac{1}{2\pi}\Gamma(1-i\sigma)\Gamma(1+i\sigma)} = \delta_{B,0} \frac{\sinh(\pi\sigma/2)}{2\sigma\cosh^2(\pi\sigma/2)} \tag{3.102}$$

and that the gluing measure is

$$\mu(\sigma, B) = \frac{(-1)^{3B/2}}{2r^2}\left(\sigma^2 + \frac{B^2}{4}\right). \tag{3.103}$$

Using $|\mathcal{W}| = 2$, this gives the S^3 partition function

$$Z = \frac{1}{2}\int d\sigma\, \mu(\sigma, 0)\Psi_0(\sigma, 0)^2 = \frac{1}{12\pi r^2}, \tag{3.104}$$

in agreement with the S^3 partition function of the four-node quiver theory and SQED with four flavors [2].

The Coulomb branch chiral ring operators are gauge-invariant products of Φ and GNO monopoles with $b \in 2\mathbb{Z}$. The smallest-dimension such operator is the monopole $\mathcal{M}^2$. This operator has $\Delta = 1$, so it should correspond to $\mathcal{Z}$ in the four-node quiver theory. Matching

[12]Indeed, these are two of the simplest mirror symmetries of ADE type [120]: $D_3 \cong A_3$.

the normalization of the two-point function gives

$$\mathcal{Z} = \frac{1}{4\pi}\mathcal{M}^2. \tag{3.105}$$

There are three operators with $\Delta = 2$: $\mathcal{M}^2 \star \mathcal{M}^2$, $\operatorname{tr}\Phi^2$ (represented by $\frac{1}{2r^2}(\sigma + iB/2)^2$ in the North picture), and the dressed monopole $\Phi\mathcal{M}^2$. Clearly, $\mathcal{M}^2 \star \mathcal{M}^2 = (4\pi)^2 \mathcal{Z} \star \mathcal{Z}$, so we expect to obtain $\mathcal{X}$ and $\mathcal{Y}$ as linear combinations of $\operatorname{tr}\Phi^2$ and $\Phi\mathcal{M}^2$. We find that

$$\mathcal{X} = \frac{1}{64\pi^2}\left(\operatorname{tr}\Phi^2 - 4\mathcal{M}^2 \star \mathcal{M}^2 - \frac{1}{2r^2} + 2i\left(\Phi\mathcal{M}^2 - \frac{i}{r}\mathcal{M}^2\right)\right), \tag{3.106}$$

$$\mathcal{Y} = \frac{1}{64\pi^2}\left(\operatorname{tr}\Phi^2 - 4\mathcal{M}^2 \star \mathcal{M}^2 - \frac{1}{2r^2} - 2i\left(\Phi\mathcal{M}^2 - \frac{i}{r}\mathcal{M}^2\right)\right) \tag{3.107}$$

obey the following relations:

$$[\mathcal{X}, \mathcal{Z}]_\star = \frac{1}{4\pi r}\mathcal{X}, \qquad [\mathcal{Y}, \mathcal{Z}]_\star = -\frac{1}{4\pi r}\mathcal{Y}, \qquad \mathcal{X} \star \mathcal{Y} = \left(\mathcal{Z} + \frac{1}{8\pi r}\right)^4_\star. \tag{3.108}$$

These are precisely the relations obeyed in the four-node quiver theory. In addition, one can check that $\langle\mathcal{X}\rangle = \langle\mathcal{Y}\rangle = \langle\mathcal{Z}\rangle = 0$, just as in the four-node quiver theory. The last relation in (3.108) shows that the Coulomb branch is indeed $\mathbb{C}^2/\mathbb{Z}_4$.

3.7 Application: Abelian Mirror Symmetry

At a formal level, 3D mirror symmetry for abelian gauge theories was derived in [126], and a concrete map between the operators of a given such theory and its mirror dual can be found, e.g., in [21]. Our construction allows us to go beyond the operator map and show that the correlation functions, or equivalently the star product, match precisely across the duality.

Here, we outline a strategy for matching all twisted HBO/CBO correlators in arbitrary abelian mirror pairs, amounting to a proof of abelian 3D mirror symmetry at the level of two- and three-point functions of half-BPS local operators. Throughout this section, we leave all correlators unnormalized (i.e., we omit an overall factor of $1/Z$) and set $r = 1$.

Any abelian 3D $\mathcal{N} = 4$ gauge theory consisting of only ordinary or twisted multiplets has a known abelian mirror dual: therefore, the 1D topological theory for twisted HBOs in such a theory gives a completely general prescription for computing correlators of twisted CBOs in its mirror dual. On the other hand, shift operators provide a completely general prescription for computing correlators of twisted CBOs in any such theory directly. To show that these two prescriptions give identical results for all correlators consists of two steps:

1. Prove this statement for the fundamental abelian mirror symmetry: namely, an arbitrary twisted HBO correlator in the free massive hyper is equal to the corresponding twisted CBO correlator in SQED_1 with matching FI parameter.

2. Show how to obtain twisted CBO correlators in a general abelian theory from those of the free hyper/SQED_1, namely as sums of products of two-point functions, integrated over appropriate subsets of mass/FI parameters.[13]

We carry out the first step in Section 3.7.3 by proving that all twisted correlators match across the basic duality between a free hyper with mass m and SQED_1 with FI parameter m. An illustration of the second step can be found in Appendix F of [2], which contains a proof that all twisted CBO correlators in SQED_N match the corresponding twisted HBO correlators in the N-node abelian necklace quiver (in this case, the map between CBOs and HBOs is very simple, and we derive explicit formulas for all correlators). In principle, our arguments can be extended to match correlators of twisted HBOs and CBOs in arbitrary abelian mirror pairs.

3.7.1 Mass and FI Parameters

Before diving into calculations, we first review how the shift operator prescription works in the presence of mass and FI deformations, which we have so far set to zero. Real masses modify the vacuum wavefunctions, the gluing measure, and the multiplicative factors in the

[13] All abelian mirror pairs can be deduced from the fundamental one by gauging global symmetries [126] (see also [127]).

monopole shift operators via $\sigma \to \sigma + m$. On the other hand, FI parameters modify the gluing measure by a factor of $e^{-8\pi^2 i\zeta\sigma}$ for each $U(1)$ factor in the gauge group. Moreover, in the non-conformal case, correlators take the form of topological correlators dressed with simple position-dependent factors. The latter are fixed by symmetry (see (3.15) and (3.16)), and the shift operator prescription allows us to compute the topological parts, which we denote by $\langle\rangle_{\text{top}}$. In particular, mass (FI) parameters leave the topological nature of CBO (HBO) correlators unchanged while making HBO (CBO) correlators non-topological. For an n-point function of twisted Higgs/Coulomb branch operators, each global (flavor/topological) $U(1)$ symmetry contributes a factor of $e^{-\zeta \sum_{i=1}^{n} q_i \varphi_i}$ where q_i is the charge of the i^{th} operator in the correlation function and ζ is the associated mass/FI parameter.[14]

3.7.2 Example: Abelian A-Series

Let us illustrate how these rules work in one of the simplest examples of mirror symmetry, namely the duality between SQED_N and the abelian necklace (affine A_{N-1}) quiver gauge theory with gauge group $U(1)^N/U(1)$ [120]. The latter theory has N $U(1)$ gauge nodes, N bifundamental hypermultiplets, and no matter charged under the diagonal $U(1)$. Each of these theories has a Higgs branch that is mapped to the Coulomb branch of the other. We focus on the map between the Higgs branch of the N-node quiver theory and the Coulomb branch of SQED_N. Namely, we match the three-point function of a monopole $\mathcal{X}^q$, anti-monopole $\mathcal{Y}^q$, and (composite) product of twisted scalars $(\mathcal{Z}^p)_\star$ in SQED_N to its mirror. This correlator will be a useful base case in the arguments to follow.

To set up the notation, note that the partition functions of the two theories can be seen to agree as follows. For SQED_N (whose hemisphere wavefunctions $\Psi(\sigma, B)$ are functions of $\sigma \in \mathbb{R}$ and $B \in \mathbb{Z}$), we have

$$Z = \int \frac{d\sigma}{[2\cosh(\pi\sigma)]^N} = \frac{\Gamma\left(\frac{N}{2}\right)}{2^N \sqrt{\pi}\,\Gamma\left(\frac{N+1}{2}\right)}. \tag{3.109}$$

[14]Strictly speaking, our conventions require an extra factor in the map between mass and FI parameters: $m \leftrightarrow -4\pi\zeta$.

On the necklace quiver side, we have

$$Z = \int d\mu(\sigma) = \int \left(\prod_{j=1}^{N} d\sigma_j\right) \delta\left(\frac{1}{N}\sum_{j=1}^{N}\sigma_j\right) Z_\sigma, \tag{3.110}$$

$$Z_\sigma \equiv \int \left(\prod_{j=1}^{N} D\widetilde{Q}_j\, DQ_j\right) \exp\left(4\pi r \int d\varphi \sum_{j=1}^{N} \tilde{Q}_j(\partial_\varphi + \sigma_{j-1,j})Q_j\right) = \prod_{j=1}^{N} \frac{1}{2\cosh(\pi\sigma_{j-1,j})},$$

where $\sigma_{j-1,1} \equiv \sigma_{j-1} - \sigma_j$ and $\sigma_0 \equiv \sigma_N$. To evaluate this integral, we appeal to the following trick. If $F_j(\sigma)$ are arbitrary functions whose Fourier transforms $\widetilde{F}_j(\tau)$ are defined by

$$F_j(\sigma) = \int d\tau\, e^{-2\pi i\sigma\tau} \widetilde{F}_j(\tau), \qquad \widetilde{F}_j(\tau) = \int d\sigma\, e^{2\pi i\sigma\tau} F_j(\sigma), \tag{3.111}$$

then the following cyclic convolution identity holds:

$$\int \left(\prod_{j=1}^{N} d\sigma_j\right) \delta\left(\frac{1}{N}\sum_{j=1}^{N}\sigma_j\right) \prod_{j=1}^{N} F_j(\sigma_{j-1,j}) = \int d\tau \prod_{j=1}^{N} \widetilde{F}_j(\tau). \tag{3.112}$$

Using (3.112) with

$$F_j(\sigma) = \frac{1}{2\cosh(\pi\sigma)}, \qquad \widetilde{F}_j(\tau) = \frac{1}{2\cosh(\pi\tau)} \tag{3.113}$$

for all j shows that (3.110) is precisely equal to (3.109).

We denote by $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ both the generators of the Higgs branch chiral ring of the N-node quiver theory and those of the Coulomb branch chiral ring of SQED_N, to emphasize that their correlation functions match. These operators satisfy $\mathcal{X} \star \mathcal{Y} = \mathcal{Z}^N + O(1/r)$, corresponding to a quantization of the $\mathbb{C}^2/\mathbb{Z}_N$ singularity. Having mapped the chiral ring generators between the two theories, one can construct the mapping of composite operators using the OPE.

Masses in SQED_N/FI Parameters in N-Node Quiver

FI parameters in the abelian necklace quiver correspond to real masses for the Cartan of the $SU(N)$ flavor symmetry in SQED_N. For massive SQED_N, we use

$$\mu(\sigma, 0) = \prod_{I=1}^{N} \frac{\Gamma(1/2 + i(\sigma + m_I))}{\Gamma(1/2 - i(\sigma + m_I))}, \quad \Psi_0(\sigma, B) = \delta_{B,0} \prod_{I=1}^{N} \frac{\Gamma(1/2 - i(\sigma + m_I))}{\sqrt{2\pi}}, \tag{3.114}$$

with the mass parameters m_I satisfying $\sum_{I=1}^{N} m_I = 0$. We take the Coulomb branch chiral ring generators to be

$$\mathcal{X} = \frac{1}{(-4\pi)^{N/2}}\mathcal{M}^1, \quad \mathcal{Y} = \frac{1}{(-4\pi)^{N/2}}\mathcal{M}^{-1}, \quad \mathcal{Z} = \frac{i}{4\pi}\Phi. \tag{3.115}$$

The corresponding North shift operators (appropriately modified by m_I) are

$$\mathcal{M}_N^1 = \left[\prod_{I=1}^{N}\left(\frac{B-1}{2} - i(\sigma + m_I)\right)\right] e^{-\frac{i}{2}\partial_\sigma - \partial_B}, \quad \mathcal{M}_N^{-1} = e^{\frac{i}{2}\partial_\sigma + \partial_B}, \quad \Phi_N = \sigma + \frac{iB}{2}. \tag{3.116}$$

Using (3.115) and (3.116), we compute that for $\varphi_1 > \varphi_2 > \varphi_3$,

$$\langle (\mathcal{Z}^p)_\star(\varphi_1)\mathcal{X}^q(\varphi_2)\mathcal{Y}^q(\varphi_3)\rangle = \int \frac{d\sigma\,(i\sigma)^p}{(4\pi)^{qN+p}} \prod_{I=1}^{N}\left[\frac{\prod_{\ell=1}^{q}(i(\sigma+m_I) - \ell + 1/2)}{2\cosh(\pi(\sigma+m_I))}\right] \tag{3.117}$$

in SQED$_N$. On the necklace quiver side, we write the N FI parameters (of which $N-1$ are independent) as $\zeta_j = \omega_{j-1} - \omega_j$ subject to the condition $\sum_j \omega_j = 0$. We now define

$$\mathcal{X} = Q_1 \cdots Q_N, \quad \mathcal{Y} = \tilde{Q}_1 \cdots \tilde{Q}_N, \quad (\mathcal{Z}^p)_\star = \prod_{j=1}^{p}(\tilde{Q}_j Q_j + i\omega_j), \tag{3.118}$$

assuming for simplicity that $p \leq N$. The definition of $(\mathcal{Z}^p)_\star$ is the natural one from the point of view of the D-term relations (the parameters ω_j resolve the geometry of the Higgs branch). The integration measure in (3.110) is modified as

$$Z_\sigma = \prod_{j=1}^{N} \frac{e^{8\pi^2 i\omega_j \sigma_{j,j+1}}}{2\cosh(\pi\sigma_{j,j+1})}, \tag{3.119}$$

while the 1D propagator (3.25) (which is sensitive to mass parameters) remains unchanged. Counting Wick contractions carefully yields the basic three-point function

$$\begin{aligned} \langle (\mathcal{Z}^p)_\star(\varphi_1)\mathcal{X}^q(\varphi_2)\mathcal{Y}^q(\varphi_3)\rangle &= (q!)^N \int d\mu(\sigma_j) \prod_{j=p+1}^{N} G_{\sigma_{j,j+1}}(\varphi_{23})^q \\ \times \prod_{a=1}^{p}(G_{\sigma_{a,a+1}}(0)G_{\sigma_{a,a+1}}(\varphi_{23}) &+ qG_{\sigma_{a,a+1}}(\varphi_{21})G_{\sigma_{a,a+1}}(\varphi_{13}))G_{\sigma_{a,a+1}}(\varphi_{23})^{q-1}. \end{aligned} \tag{3.120}$$

Assuming that $\varphi_1 > \varphi_2 > \varphi_3$, we may use (3.112), the identity

$$\frac{(\operatorname{sgn}\varphi_{12} + \tanh(\pi\sigma))^m}{2\cosh(\pi\sigma)} = \frac{1}{m!}\left[\prod_{j=1}^{m}\left((2j-1)\operatorname{sgn}\varphi_{12} - \frac{1}{\pi}\frac{d}{d\sigma}\right)\right]\frac{1}{2\cosh(\pi\sigma)}, \tag{3.121}$$

integration by parts, and $\frac{1}{2\cosh(\pi\sigma)} = \int d\tau\, \frac{e^{2\pi i\sigma\tau}}{2\cosh(\pi\tau)}$ to simplify (3.120) to

$$\langle (\mathcal{Z}^p)_\star(\varphi_1)\mathcal{X}^q(\varphi_2)\mathcal{Y}^q(\varphi_3)\rangle = \int \frac{d\tau\,(i\tau)^p}{(4\pi)^{qN+p}}\prod_{I=1}^{N}\left[\frac{\prod_{j=1}^{q}(i(\tau-4\pi\omega_I)-j+1/2)}{2\cosh(\pi(\tau-4\pi\omega_I))}\right]. \tag{3.122}$$

This matches the SQED_N result if we identify $m_I \leftrightarrow -4\pi\omega_I$.

FI Parameters in SQED_N/Masses in N-Node Quiver

Mass parameters in the abelian necklace quiver correspond to FI parameters in SQED_N. Consider adding a real mass associated to the $U(1)$ flavor symmetry of the necklace quiver under which $Q_i, \tilde{Q}_i$ carry charge $\pm 1/N$. In practice, this means replacing all instances of $\sigma_{j,j+1}$ by $\sigma_{j,j+1} + m/N$ in the 1D theory computations. Using the identity

$$\int \left(\prod_{j=1}^{N} d\sigma_j\right)\delta\left(\frac{1}{N}\sum_{j=1}^{N}\sigma_j\right)\prod_{j=1}^{N}F_j(\sigma_{j,j+1}+m/N) = \int d\tau\, e^{2\pi i m\tau}\prod_{j=1}^{N}\widetilde{F}_j(\tau), \tag{3.123}$$

which is the appropriate modification of (3.112), we obtain (with $\varphi_1 > \varphi_2 > \varphi_3$)

$$\langle (\mathcal{Z}^p)_\star(\varphi_1)\mathcal{X}^q(\varphi_2)\mathcal{Y}^q(\varphi_3)\rangle_{\text{top}} = \int \frac{d\tau\, e^{2\pi i m\tau}}{(4\pi)^{qN+p}}\frac{(i\tau)^p}{(2\cosh(\pi\tau))^N}\prod_{j=1}^{q}(i\tau - j + 1/2)^N. \tag{3.124}$$

This matches the expression

$$\langle (\mathcal{Z}^p)_\star(\varphi_1)\mathcal{X}^q(\varphi_2)\mathcal{Y}^q(\varphi_3)\rangle_{\text{top}} = \int \frac{(-i)^p\, d\sigma\, e^{2\pi i m\sigma}}{(-4\pi)^{qN+p}}\mu(\sigma,0)\Psi_0(\sigma,0)[\mathcal{M}_N^{-q}\mathcal{M}_N^{q}\Phi_N^{p}\Psi_0(\sigma,B)]|_{B=0} \tag{3.125}$$

on the SQED_N side.

3.7.3 Proof: Basic Mirror Duality

With this warmup complete, we now match all twisted correlators in SQED_1 with FI parameter ζ and a free hyper of mass $m = -4\pi\zeta$. In the free hyper theory, correlation functions of $\mathcal{X} = Q$, $\mathcal{Y} = \tilde{Q}$, $\mathcal{Z} = Q\tilde{Q}$ are computed using the measure

$$d\mu(\sigma) = \frac{d\sigma\,\delta(\sigma)}{2\cosh(\pi m)}, \tag{3.126}$$

and Wick contractions are performed using the σ-independent Green's function

$$G(\varphi_{12}) = \langle Q(\varphi_1)\tilde{Q}(\varphi_2)\rangle = -\frac{\operatorname{sgn}\varphi_{12} + \tanh(\pi m)}{8\pi}e^{-m\varphi_{12}}. \tag{3.127}$$

Correlators are no longer topological due to the factor of $e^{-m\varphi_{12}}$.

In matching all correlators, let us focus only on the topological parts (as the position-dependent parts match trivially). We wish to show that

$$\langle \mathcal{S}\rangle_{\text{top, SQED}_1} \overset{!}{=} \langle \mathcal{S}\rangle_{\text{top, free hyper}} \tag{3.128}$$

where $\mathcal{S}$ is some operator string in $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and operators appearing in correlation functions are understood to be in *descending* order by insertion point (i.e., $\varphi_1 > \cdots > \varphi_n$). Shift operators in SQED_1 with FI parameter ζ give

$$\langle \mathcal{O}_1^{p_1} \cdots \mathcal{O}_n^{p_n}\rangle_{\text{top}} = \int d\sigma\, e^{-8\pi^2 i\zeta\sigma}\mu(\sigma,0)\Psi_0(\sigma,0)[(\mathcal{O}_n)_N^{p_n}\cdots(\mathcal{O}_1)_N^{p_1}\Psi_0(\sigma,B)]|_{B=0} \tag{3.129}$$

where $\mathcal{O}_i \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and

$$\mathcal{X}_N = \left(\frac{B-1}{2} - i\sigma\right)\frac{e^{-\frac{i}{2}\partial_\sigma - \partial_B}}{(-4\pi)^{1/2}}, \quad \mathcal{Y}_N = \frac{e^{\frac{i}{2}\partial_\sigma + \partial_B}}{(-4\pi)^{1/2}}, \quad \mathcal{Z}_N = \frac{i}{4\pi}\left(\sigma + \frac{iB}{2}\right). \tag{3.130}$$

Here, the notation $\mathcal{Z}^p$ is understood to mean p adjacent insertions of $\mathcal{Z}$ at separated points, which is equivalent to a single insertion of the composite operator $(\mathcal{Z}^p)_\star$. On the other hand, the 1D theory for the free hyper with mass m gives

$$\langle \mathcal{O}_1^{p_1} \cdots \mathcal{O}_n^{p_n}\rangle_{\text{top}} = \int d\tau\, e^{2\pi i m\tau}\int d\sigma\,\frac{e^{-2\pi i\tau\sigma}}{2\cosh(\pi\sigma)}w(\mathcal{O}_1^{p_1}\cdots\mathcal{O}_n^{p_n}) \tag{3.131}$$

where $w(s)$ denotes the sum of all full Wick contractions of the operator string s and Wick contractions are performed using the "topological" propagators

$$G_\pm = -\frac{\pm 1 + \tanh(\pi\sigma)}{8\pi}, \quad G_0 = -\frac{\tanh(\pi\sigma)}{8\pi}. \tag{3.132}$$

We proceed by induction. In the previous subsection, we established the base case

$$\langle \mathcal{Z}^p \mathcal{X}^q \mathcal{Y}^q \rangle_{\text{top, SQED}_1} = \langle \mathcal{Z}^p \mathcal{X}^q \mathcal{Y}^q \rangle_{\text{top, free hyper}}. \tag{3.133}$$

Now fix some $\mathcal{S}$ and suppose we have established that $\langle \mathcal{S} \rangle_{\text{top, SQED}_1} = \langle \mathcal{S} \rangle_{\text{top, free hyper}}$, as well as a similar statement for all operator strings containing fewer operators than $\mathcal{S}$. Consider swapping two adjacent operators in $\mathcal{S}$ to form a new operator string $\mathcal{S}'$. Starting from the basic string $\mathcal{Z}^p \mathcal{X}^q \mathcal{Y}^q$, one can obtain any other string by performing three types of swaps (below, let $\mathcal{S}_{L,R}$ denote substrings of $\mathcal{S}$):

1. Let $\mathcal{S} \equiv \mathcal{S}_L \mathcal{X} \mathcal{Y} \mathcal{S}_R$, $\mathcal{S}' \equiv \mathcal{S}_L \mathcal{Y} \mathcal{X} \mathcal{S}_R$, and $\mathcal{S}_0 \equiv \mathcal{S}_L \mathcal{S}_R$.

2. Let $\mathcal{S} \equiv \mathcal{S}_L \mathcal{Z} \mathcal{X} \mathcal{S}_R$, $\mathcal{S}' \equiv \mathcal{S}_L \mathcal{X} \mathcal{Z} \mathcal{S}_R$, and $\mathcal{S}_0 \equiv \mathcal{S}_L \mathcal{X} \mathcal{S}_R$.

3. Let $\mathcal{S} \equiv \mathcal{S}_L \mathcal{Z} \mathcal{Y} \mathcal{S}_R$, $\mathcal{S}' \equiv \mathcal{S}_L \mathcal{Y} \mathcal{Z} \mathcal{S}_R$, and $\mathcal{S}_0 \equiv \mathcal{S}_L \mathcal{Y} \mathcal{S}_R$.

In all three cases, the Wick contractions of the strings so defined are related in a simple way, implying relations between the corresponding correlators (3.131) in the free hyper theory:

1. $w(\mathcal{S}') = w(\mathcal{S}) + (G_- - G_+) w(\mathcal{S}_0) = w(\mathcal{S}) + \frac{1}{4\pi} w(\mathcal{S}_0) \implies \langle \mathcal{S}' \rangle_{\text{top}} = \langle \mathcal{S} \rangle_{\text{top}} + \frac{1}{4\pi} \langle \mathcal{S}_0 \rangle_{\text{top}}$.

2. $w(\mathcal{S}') = w(\mathcal{S}) + (G_+ - G_-) w(\mathcal{S}_0) = w(\mathcal{S}) - \frac{1}{4\pi} w(\mathcal{S}_0) \implies \langle \mathcal{S}' \rangle_{\text{top}} = \langle \mathcal{S} \rangle_{\text{top}} - \frac{1}{4\pi} \langle \mathcal{S}_0 \rangle_{\text{top}}$.

3. Same as in case (1).

On the other hand, the shift operators (3.130) for SQED_1 satisfy the commutation relations

$$[\mathcal{X}_N, \mathcal{Y}_N] = \frac{1}{4\pi}, \quad [\mathcal{X}_N, \mathcal{Z}_N] = \frac{1}{4\pi} \mathcal{X}_N, \quad [\mathcal{Y}_N, \mathcal{Z}_N] = -\frac{1}{4\pi} \mathcal{Y}_N, \tag{3.134}$$

implying that the correlators (3.129) in SQED_1 satisfy identical relations in the three cases:

1. $\langle \mathcal{S}' \rangle_{\text{top}} = \langle \mathcal{S} \rangle_{\text{top}} + \frac{1}{4\pi} \langle \mathcal{S}_0 \rangle_{\text{top}}$.

2. $\langle \mathcal{S}' \rangle_{\text{top}} = \langle \mathcal{S} \rangle_{\text{top}} - \frac{1}{4\pi} \langle \mathcal{S}_0 \rangle_{\text{top}}$.

3. Same as in case (1).

By the induction hypothesis, $\langle \mathcal{S} \rangle_{\text{top}}$ and $\langle \mathcal{S}_0 \rangle_{\text{top}}$ both match in SQED_1 and the free hyper, which immediately implies that $\langle \mathcal{S}' \rangle_{\text{top, SQED}_1} = \langle \mathcal{S}' \rangle_{\text{top, free hyper}}$, as desired.

3.8 Summary

We have presented a shift operator formalism that allows for the computation of correlation functions of Coulomb branch operators in good and ugly 3D $\mathcal{N} = 4$ gauge theories. Such correlation functions can be used to derive how Higgs and Coulomb branch operators map across 3D mirror symmetry. Using our formalism, we are able to derive the precise normalization factors in the mirror map and to distinguish operators that could mix on the basis of symmetries.

At a structural level, our formalism provides an alternative route to the abelianization description of the Coulomb branch and clarifies the meaning of the "abelianization map" [21]. Our approach further provides an algebraic way to determine previously unknown monopole bubbling coefficients, based on symmetries and algebraic consistency of the OPE. It avoids the technicalities of previous analytic bubbling computations [17, 18, 19, 20].

Finally, shift operators provide a powerful way to compute the Coulomb branch chiral rings of 3D $\mathcal{N} = 4$ gauge theories as well as their quantizations. When the moduli space is a hyperkähler cone, previously known techniques for extracting generators and ring relations — such as the Hilbert series [125, 128], abelianization [21, 129], and the type IIB realization [130] of 3D mirror symmetry [131] — work well. In other situations, such as in bad theories, the chiral ring has not been as thoroughly studied. We expect such theories to present good opportunities for applications of our formalism.

Chapter 4

Conclusion and Outlook

We conclude by mentioning some possible extensions of the work described here.

4.1 Knot Polynomials from Matrix Models

As observed in Appendix E of [1], 3D $\mathcal{N} = 2$ supersymmetry provides an efficient analytic (as opposed to surgery-based) method to compute certain knot polynomials. However, this approach is restricted to very special classes of links, such as Hopf links and torus links in S^3 (depending on the Seifert fibration that one chooses).

On the other hand, 3D $\mathcal{N} = 4$ supersymmetry allows for the construction of 1/4-BPS Wilson loops supported on arbitrary curves in $\mathbb{R}^3$ by appropriate twisting [132]. It should also be possible to construct such loops on S^3, preserving two supercharges. Thus it may be possible to derive a matrix model for expectation values of 1/4-BPS Wilson loops of arbitrary shape in 3D $\mathcal{N} = 4$ theories on S^3. In the case of $\mathcal{N} = 4$ Chern-Simons theory, such a matrix model would compute knot polynomials of arbitrary links.

The major caveat with this approach is that such loops may have a more complicated localization locus than in the 1/2-BPS case. Indeed, they may not localize to simple matrix integrals.

4.2 Complete Solution of 1D Topological Sectors

Aside from further applications of the TQM techniques in [14, 2, 3] to 3D $\mathcal{N} = 4$ gauge theories,[1] a number of structural questions present interesting avenues for future work. For one, it would be interesting to determine the relation between the monopole bubbling terms obtained using our method and those obtained by dimensional reduction of 4D results [18]. While preliminary comparisons performed in [3] found that the two agree up to operator mixing and normalization factors, a more systematic understanding is needed.

But perhaps the most outstanding open problem from [2, 3] is to extend our construction to $\mathcal{N} = 4$ gauge theories with Chern-Simons couplings, such as those of Gaiotto-Witten [98], ABJ(M) [133, 134], and others [135, 136]. Such theories contain both ordinary and twisted hypermultiplets and vector multiplets. One class of examples for which the generalization is straightforward is that of abelian gauge theories with BF couplings [137]: see [2].

A potential outcome of a solution to this problem — or perhaps even a solution strategy — is the formulation of a unified algebraic treatment of the Higgs and Coulomb branch topological sectors. For abelian theories, some hints as to this additional structure come from the fact that one can use an integral transform (which in the rank-one case, takes the form of a Fourier-Mellin transform [107]) to translate between shift operators for the Coulomb branch and shift operators for the Higgs branch. One might speculate that such a "mirror transform" generalizes to nonabelian theories with the aid of harmonic analysis.

[1] For example, see [119] for precision studies of nonabelian ADE mirror symmetry along these lines.

Appendix A

Conventions

A.1 1D $\mathcal{N} = 2$

We work in Lorentzian[1] 1D $\mathcal{N} = 2$ superspace with coordinates $(t, \theta, \theta^\dagger)$ and let ϵ be a complex spinor parameter. The representations of the supercharges as differential operators on superspace and the supercovariant derivatives are

$$\hat{Q} = \partial_\theta + i\theta^\dagger \partial_t, \qquad \hat{Q}^\dagger = \partial_{\theta^\dagger} + i\theta\partial_t, \tag{A.1}$$

$$D = \partial_\theta - i\theta^\dagger \partial_t, \qquad D^\dagger = \partial_{\theta^\dagger} - i\theta\partial_t. \tag{A.2}$$

The nonvanishing anticommutators are

$$\{\hat{Q}, \hat{Q}^\dagger\} = -\{D, D^\dagger\} = 2i\partial_t. \tag{A.3}$$

A general superfield takes the form

$$\Xi(t, \theta) = \phi(t) + \theta\psi(t) + \theta^\dagger \chi(t) + \theta\theta^\dagger F(t), \tag{A.4}$$

with SUSY acting as $\delta\Xi = (\epsilon\hat{Q} + \epsilon^\dagger \hat{Q}^\dagger)\Xi$. The vector multiplet satisfies

$$V = V^\dagger \implies V(t, \theta) = \phi(t) + \theta\psi(t) - \theta^\dagger \psi^\dagger(t) + \theta\theta^\dagger F(t). \tag{A.5}$$

[1] To Euclideanize the following, take $\tau = it$ and $iS = -S_E$.

The SUSY transformations of its components are

$$(\delta\phi, \delta\psi, \delta F) = (\epsilon\psi - \epsilon^\dagger\psi^\dagger, -i\epsilon^\dagger\dot{\phi} + \epsilon^\dagger F, -i\epsilon\dot{\psi} - i\epsilon^\dagger\dot{\psi}^\dagger). \tag{A.6}$$

The chiral multiplet satisfies

$$D^\dagger\Phi = 0 \implies \Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\theta\theta^\dagger\dot{\phi}(t). \tag{A.7}$$

The SUSY transformations of its components are

$$(\delta\phi, \delta\psi) = (\epsilon\psi, -2i\epsilon^\dagger\dot{\phi}). \tag{A.8}$$

To integrate over superspace, we use $d^2\theta \equiv d\theta^\dagger d\theta$.

The 1D SUSY' transformations are derived as follows. In Wess-Zumino gauge, we have $V|_{\text{WZ}} = \theta\theta^\dagger F$, which transforms under SUSY to $V' = V|_{\text{WZ}} + \delta V|_{\text{WZ}}$. To preserve Wess-Zumino gauge, we choose the compensatory super gauge transformation parameter $\Phi = i\Lambda$ such that $\delta V|_{\text{WZ}} + \frac{1}{2}(\Phi + \Phi^\dagger)$ is $O(\theta\theta^\dagger)$:

$$e^{2V'} \to e^{\Phi}e^{2V'}e^{\Phi^\dagger} \Longleftrightarrow V' \to V|_{\text{WZ}}, \tag{A.9}$$

which means that δ' acts trivially on the vector multiplet. For the chiral multiplet, only the transformation rule for ψ is modified by taking $\partial_0 \to D_0$:

$$(\delta'\phi, \delta'\psi) = (\epsilon\psi, -2i\epsilon^\dagger D_0\phi), \tag{A.10}$$

where the gauge field appearing in D_0 is the single nonzero component of $V|_{\text{WZ}}$.

A.2 3D $\mathcal{N} = 2$

We raise and lower spinor indices on the left by $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$ ($\epsilon^{12} \equiv 1$), with $\psi\chi \equiv \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta$ (as in [138]). This convention requires that we distinguish matrix multiplication ("·") from spinor contraction (no symbol), which differ by a sign: $\gamma^a\xi = -\gamma^a \cdot \xi$.[2] By default, our spinors are anticommuting; the notation "$|_0$" applied to a Grassmann-odd spinor denotes its Grassmann-even version. Spinors that would be conjugate (e.g., $\lambda, \bar{\lambda}$) in Lorentzian signature are independent in Euclidean signature (e.g., $\lambda, \tilde{\lambda}$). In Lorentzian signature, as in 1D, we use $x^*, \bar{x}, x^\dagger$ interchangeably to denote the complex conjugate of x. In Euclidean signature, we use bars and stars interchangeably to denote complex conjugation, while daggers denote Hermitian conjugation: $(\psi^\dagger)^\alpha = (\psi_\alpha)^*$.

In $\mathbb{R}^{1,2}$ (with signature $-++$), the 3D gamma matrices are

$$(\gamma^{\mu=0,1,2})_{\alpha\beta} = (-1, \sigma^1, \sigma^3)_{\alpha\beta}, \quad \epsilon^{\alpha\beta}\gamma^\mu_{\gamma\alpha}\gamma^\nu_{\beta\delta} = \eta^{\mu\nu}\epsilon_{\gamma\delta} - \epsilon^{\mu\nu\rho}(\gamma_\rho)_{\gamma\delta}. \tag{A.11}$$

With lowered indices, these matrices are real and symmetric, so that $\theta\gamma^\mu\bar{\theta}$ is real. We take $\epsilon^{012} = 1$. The 3D $\mathcal{N} = 2$ algebra is

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta}P_\mu + 2i\epsilon_{\alpha\beta}Z, \quad \{Q_\alpha, Q_\beta\} = 0. \tag{A.12}$$

The representations of the supercharges as differential operators on superspace and the supercovariant derivatives are

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu, \qquad \bar{\mathcal{Q}}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} + i\theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \tag{A.13}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu, \qquad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu. \tag{A.14}$$

We abbreviate $\bar{\partial}_\alpha \equiv \partial/\partial\bar{\theta}^\alpha$, $\partial_\beta \equiv \partial/\partial\theta^\beta$ and define $\int d^4\theta\, \theta^2\bar{\theta}^2 = 1$.

The SUSY transformations of the 3D $\mathcal{N} = 2$ vector and chiral multiplets (2.95) and (2.96) follow from applying $\delta_\xi = \xi\mathcal{Q} - \bar{\xi}\bar{\mathcal{Q}}$ to the corresponding superfields. The 3D SUSY' transformations are derived as follows. Under SUSY, $V|_{\text{WZ}}$ transforms into $V' \equiv V|_{\text{WZ}} + \delta V|_{\text{WZ}}$. To preserve Wess-Zumino gauge, we choose Λ such that $\delta V|_{\text{WZ}} + \frac{1}{2}(\Lambda + \bar{\Lambda})$ is $O(\theta\bar{\theta})$ and set the lowest component of Λ to zero. With these choices, and to first order in $\xi, \bar{\xi}$, the

[2]We also sometimes use "$\cdot$" to denote multiplication in the appropriate representation of the gauge group: for example, $[D_\mu, D_\nu](\cdots) = -iF_{\mu\nu} \cdot (\cdots)$.

super gauge transformation $e^{2V'} \to e^{\bar{\Lambda}} e^{2V'} e^{\Lambda}$ truncates to

$$V' \to V' + \frac{1}{2}(\Lambda + \bar{\Lambda}) + \frac{1}{2}[V', \Lambda - \bar{\Lambda}], \tag{A.15}$$

from which we read off (2.99). SUSY' also modifies the chiral multiplet transformation laws by terms involving vector multiplet fields, so that $\Phi + \delta\Phi \to e^{-\Lambda}(\Phi + \delta\Phi)$ and $\bar{\Phi} + \delta\bar{\Phi} \to (\bar{\Phi} + \delta\bar{\Phi})e^{-\bar{\Lambda}}$, from which (2.100) follows.

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