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Open inflation without false vacua

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Abstract

We show that within the framework of a definite proposal for the initial conditions for the universe, the Hartle–Hawking “no boundary” proposal, open inflation is generic and does not require any special properties of the inflaton potential. In the simplest inflationary models, the semiclassical approximation to the Euclidean path integral and a minimal anthropic condition lead to $\Omega_0 \approx 0.01$. This number may be increased in models with more fields or extra dimensions. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The inflationary universe scenario provides an appealing explanation for the size, flatness and smoothness of the present universe, as well as a mechanism for the origin of fluctuations. But whether inflation actually occurs within a given inflationary model is known to depend very strongly on the pre-inflationary initial conditions. In the absence of a measure on the set of initial conditions inflationary theory inevitably rests on ill-defined foundations. One such measure is provided by continuing the path integral to imaginary time and demanding that the Euclidean four manifold so obtained be compact [1]. This is the Hartle–Hawking “no boundary” proposal. In this Letter we show that the no boundary prescription, coupled to a minimal anthropic condi-

tion, actually predicts open inflationary universes for generic scalar potentials. The simplest inflationary potentials with a minimal anthropic requirement favour values of $\Omega_0 \sim 0.01$, but generalisations including extra fields favour more reasonable values. At the very least these calculations demonstrate that the measure for the pre-inflationary initial conditions *does* matter. More importantly, we believe the implication is that inflation itself is now seen to be perfectly compatible with an open universe.

Until recently it was believed that all inflationary models predicted $\Omega_0 = 1$ to high accuracy. This view was overturned by the discovery that a special class of inflaton potentials produce nearly homogeneous open universes with interesting values of $\Omega_0 < 1$ today [2,3]. The potentials were required to have a metastable minimum (a “false vacuum”) followed by a gently sloping region allowing slow roll inflation. The idea was that the inflaton could become trapped in the “false vacuum”, driving a period of inflation and creating a near-perfect De Sitter space

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with minimal quantum fluctuations. The field would then quantum tunnel, nucleating bubbles within which it would roll slowly down to the true minimum. The key observation, due to Coleman and De Luccia [4] is that the interior of such a bubble is actually an infinite open universe. By adjusting the duration of the slow roll epoch one can arrange that the spatial curvature today is of order the Hubble radius [3].

All inflationary models must be fine tuned to keep the quantum fluctuations small. This requires that the potentials be very flat. In open inflation this must be reconciled with the requirement that the potential have a false vacuum. Furthermore, a classical bubble solution of the Coleman De Luccia form only exists if the mass of the scalar field in the false vacuum is large, so that the bubble “fits inside” the De Sitter Hubble radius. Taken together these requirements meant that the scalar potentials needed for open inflation were very contrived for single field models. Two-field models were proposed, but even these required a false vacuum [5] and the pre-bubble initial conditions were imposed essentially by hand.

Within the Hartle–Hawking framework, the period of “false vacuum” inflation is no longer required. The quantum fluctuations are computed by continuing the field and metric perturbation modes from the Euclidean region where they are governed by a positive definite measure. The Hartle–Hawking prescription in effect starts the universe in a state where the fluctuations are at a minimal level in the first place.

2. Instantons

We consider the path integral for Einstein gravity coupled to a scalar field ϕ , with potential $V(\phi)$, which we assume has a true minimum with $V = 0$. As usual, we approximate the path integral by seeking saddle points i.e. solutions of the classical equations of motion, and expanding about them to determine the fluctuation measure. We begin with the Euclidean instanton. If $V(\phi)$ has a stationary point at some nonzero value then there is an $O(5)$ invariant solution where ϕ is constant and the Euclidean manifold is a four sphere. We shall be interested in

more general solutions possessing only $O(4)$ invariance. The metric takes the form

$$ds^2 = d\sigma^2 + b^2(\sigma)d\Omega_3^2 \\ = d\sigma^2 + b^2(\sigma)(d\psi^2 + \sin(\psi)d\Omega_2^2) \quad (1)$$

with $b(\sigma)$ the radius of the S^3 “latitudes” of the S^4 . For the $O(5)$ invariant solution $b(\sigma) = H^{-1}\sin(H\sigma)$, with $H^2 = 8\pi GV/3$, but in the general case $b(\sigma)$ is a deformed version of the sine function.

Solutions possessing only $O(4)$ invariance are naturally continued to an open universe as follows (Fig. 1). First we continue from Euclidean to Lorentzian space. To obtain a real Lorentzian metric we must continue on a three surface where the metric is stationary (more properly, where the second fundamental form vanishes). One obtains an open universe by continuing ψ , so that ψ runs from 0 to $\pi/2$ in the Euclidean region and then in the imaginary direction in the Lorentzian region. Setting $\psi = \pi/2 + i\tau$ we obtain

$$ds^2 = d\sigma^2 + b^2(\sigma)(-d\tau^2 + \cosh^2(\tau)d\Omega_2^2) \quad (2)$$

which is a spatially inhomogeneous De Sitter-like metric. This metric describes region II of the solution, the exterior of the inflating bubble. The radius

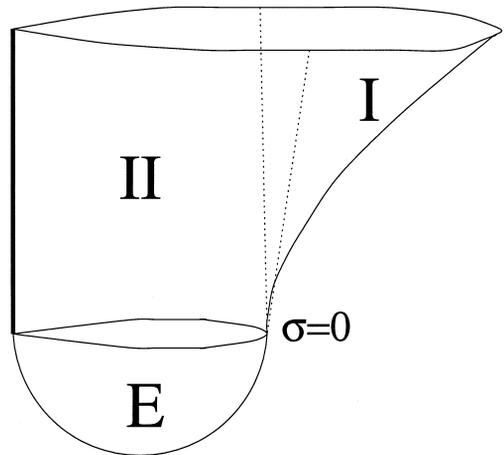


Fig. 1. Global structure of the open instanton and its continuation. The Euclidean region E is half of a deformed four sphere. It continues into a De Sitter like region II, and thence into an open inflating universe, region I. The dotted lines show the null surface (the “bubble wall”) emanating from the point $\sigma = 0$ on the instanton. The heavy line shows the singularity discussed in the text.

$b(\sigma)$ vanishes at two values of σ . Near the first, which we shall call $\sigma=0$, $b(\sigma)$ vanishes linearly with σ . The metric has a unique continuation through the null surface defined by $\sigma=0$. One sets $\sigma = it$ and $\tau = i\pi/2 + \chi$ giving

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sinh^2(\chi)d\Omega_2^2) \quad (3)$$

where $a(t) = -ib(it)$. This is an expanding open universe describing region I of the solution.

There is another inequivalent continuation from the Euclidean instanton which produces a closed universe. This is obtained by continuing the coordinate σ in the imaginary direction beyond the value σ_{\max} at which the radius $b(\sigma)$ is greatest. So σ runs from 0 to σ_{\max} in the Euclidean region, and $\sigma = \sigma_{\max} + iT$ in the Lorentzian region. The latter is a De Sitter-like space with homogeneous but time dependent spatial sections:

$$ds^2 = -dT^2 + b^2(T)(d\psi^2 + \sin^2(\psi)d\Omega_2^2). \quad (4)$$

We shall return to this solution later – it describes a closed inflating universe.

Now let us discuss the properties of the Euclidean instanton in more detail. The field ϕ and the radius b obey the field equations

$$\phi'' + 3\frac{b'}{b}\phi' = V_{,\phi}, \quad b'' = -\frac{8\pi G}{3}b(\phi'^2 + V) \quad (5)$$

where primes denote derivatives with respect to σ . According to the first equation, ϕ rolls in the upside down potential $-V$. The point $\sigma=0$ is assumed to be a nonsingular point so the manifold looks locally like R^4 in spherical polar coordinates. This requires that $b(\sigma) \sim \sigma$ at small σ . The field takes the value ϕ_0 at $\sigma=0$. We assume the potential has a nonzero slope at this field value $V_{,\phi}(\phi_0) \neq 0$ (otherwise we would obtain the $O(5)$ invariant instanton). Analytically and $O(4)$ invariance imply that $\phi'(0) = 0$. Following the solutions forward in σ , $b(\sigma)$ decelerates and its velocity $b'(\sigma)$ changes sign. Thereafter $b(\sigma)$ is driven to zero, at a point we call σ_f . The field ϕ on the other hand is driven up the potential by the forcing term, initially with damping but after the sign change in $b'(\sigma)$ with antidamping. The antidamping diverges as we approach σ_f , and $\phi'(\sigma)$ goes to infinity there. As we approach σ_f the potential terms

become irrelevant in the field equations: the first equation then implies that $\phi' \propto b^{-3}$ and the second yields $b \propto (\sigma_f - \sigma)^{\frac{1}{3}}$. Thus $\phi' \propto (\sigma_f - \sigma)^{-1}$ and ϕ diverges logarithmically as we approach the singularity. The above behaviour is true for any ϕ_0 if the potential increases monotonically away from the true minimum. If there are additional extrema it is possible for the driving term $V_{,\phi}$ to change sign and, if it is large enough to counteract the antidamping term, to actually stop the motion of ϕ . The Coleman–De Luccia instanton is obtained only for potentials where this is possible (see e.g. [6]). It occurs when the value of ϕ_0 is chosen so that ϕ' returns to zero precisely at σ_f . In that case, both ends of the solution are nonsingular and a continuation into a third Lorentzian region becomes possible. The Coleman–De Luccia instanton was employed in previous versions of open inflation because it is unique and nonsingular, in analogy with tunnelling solutions in Minkowski space. But De Sitter space is quite different from Minkowski space, possessing finite closed spatial sections, and the question of which instantons are allowed needs to be separately examined.

The primary criterion for deciding whether an instanton solution is physically allowed is to compute the Euclidean action S_E . The wavefunction for the system is in the leading approximation proportional to e^{-S_E} so configurations of infinite action are suppressed. The Euclidean action is given by

$$S_E = \int d^4x \sqrt{g} \left[-\frac{R}{16\pi G} + \frac{1}{2}(\partial\phi)^2 + V \right]. \quad (6)$$

But in four dimensions the trace of the Einstein equation reads $R = 8\pi G((\partial\phi)^2 + 4V(\phi))$ and so the action is just ³

$$S_E = -\int d^4x \sqrt{g} V = -\pi^2 \int d\sigma b^3(\sigma) V(\phi) \quad (7)$$

where we have integrated over half of the S^3 . Note

³ As Vilenkin [7] has pointed out, there is also a surface term coming from the singular boundary of the instanton. This term is given by $-\pi^2(b^3)M_{\text{pl}}^2$. We have computed the action for an $m^2\phi^2$ potential including all corrections and find it is well approximated by (8) multiplied by $(1 - (1.84/N)^{1/2})$ where $N = \frac{1}{4}\phi_0^2/M_{\text{pl}}^2$ is the number of efoldings. Including the correction changes none of our conclusions.

that the action is *negative*, a result of the well known lack of positivity of the Euclidean gravitational action. The surprising thing however is that even for our singular instantons, at the singularity V diverges only logarithmically. The volume measure $b(\sigma)^3$ vanishes linearly with $(\sigma_f - \sigma)$ so the Euclidean action is perfectly convergent. If one examines more closely how this result emerges, one finds that the scalar field part of the action diverges logarithmically (since ϕ' diverges linearly) but this divergence is precisely cancelled by an opposite divergence in the gravitational action. There are two key differences between the present calculation and that for tunnelling in Minkowski space. First, the instanton is spatially finite and this cuts off the divergence associated with the field not tending to a minimum of the potential. Second, the gravitational action is not positive and is thus able to cancel a divergence in the scalar field action. These two differences have the remarkable consequence that unlike the situation in Minkowski space, there is a one parameter family of allowed instanton solutions.

Let us now comment on the singularity at σ_f , which is timelike. Timelike spacetime singularities are not necessarily fatal in semiclassical descriptions of quantum physics, as the example of the hydrogen atom teaches us. Generic particle trajectories “miss” the singularity, and quantum fluctuations may be enough to smooth out its effect. In the present case we shall see that the singularity is mild enough for the quantum field fluctuations to be well defined. The field and metric fluctuations are defined by continuation from the Euclidean region, singular only at a point on its edge. The mode functions for the field fluctuations are most easily studied by changing coordinates from σ to the conformal coordinate $X = \int_{\sigma}^{\sigma_f} d\sigma/b(\sigma)$. Because the integral converges at σ_f , the range of X is bounded below by zero. After a rescaling $\phi = \chi/b$, the field modes obey a Schrodinger-like equation with a potential given by $b^{-1}(d^2b/dX^2) - V_{,\phi\phi}b^2 \sim -\frac{1}{4}X^{-2}$ at small X . This divergence is precisely critical – for more negative coefficients an inverse square potential has a continuum of negative energy states and the quantum mechanics is pathological. But for $-\frac{1}{4}X^{-2}$ there is a positive continuum and a well defined complete set of modes. The causal structure of region II is easily seen in the same conformal coordi-

nates. Near the singularity the spatial metric of region II is conformal to a tube $R^+ \times S^2$. The singularity is a world line corresponding to the end of the tube.

As mentioned above, there is another instanton describing a closed inflationary universe where one continues σ in the imaginary direction from σ_{\max} . The action of this instanton is given by twice the expression (7) but with the integral taken only over the interval $[0, \sigma_{\max}]$. The functions $b(\sigma)$ and $\phi(\sigma)$ are also somewhat different – analyticity still implies that $\phi'(0)$ must be zero, but since the potential has a nonzero slope, the velocity ϕ' is nonzero on the matching surface. This leads to odd terms in the Taylor expansion for ϕ around $\sigma=0$, so $\phi(T)$ is complex in the Lorentzian region. One would like the solution for ϕ be real at late times. This is impossible to arrange exactly, but one can add a small imaginary part to ϕ_0 in such a way that the imaginary part of ϕ is in the pure decaying mode during inflation. Then both the field and metric are real to exponential accuracy at late times.

The potentials of interest are those whose slope is sufficiently shallow to allow many inflationary efoldings. We have numerically computed the action as a function of the parameter ϕ_0 for various scalar potentials. In the regime where the number of efoldings is large, the result is very simple – to a good approximation one has $\phi(\sigma) \approx \phi_0$ and $b(\sigma) \approx H^{-1} \sin H\sigma$ over most of the range of σ , where $H^2 = 8\pi G V(\phi_0)/3$. The Euclidean action is then just

$$S_E \approx -\frac{12\pi^2 M_{\text{Pl}}^4}{V(\phi_0)} \quad (8)$$

in both open and closed cases, where the reduced Planck mass $M_{\text{Pl}} = (8\pi G)^{-\frac{1}{2}}$.

3. The value of Ω_0

The value of the density parameter today, Ω_0 , is determined by the number of inflationary efoldings. On the relevant matching surface the value of Ω is zero in the open case, infinity in the closed case. It

approaches unity as $\Omega^{-1} - 1 \propto a^{-2}$ during inflation. After reheating it deviates from unity as $\Omega^{-1} - 1 \propto a^2$ in the radiation era and $\Omega^{-1} - 1 \propto a$ in the matter era.

Putting this together, and assuming instantaneous reheating, one finds [3] that

$$\Omega_0 \approx \frac{1}{1 \pm \mathcal{A} e^{-2N(\phi_0)}}, \quad \mathcal{A} \approx 4 \left(\frac{T_{\text{reheat}}}{T_{\text{eq}}} \right)^2 \frac{T_{\text{eq}}}{T_0} \quad (9)$$

where the + and – refer to the open and closed cases respectively. The temperature today is T_0 , that at matter-radiation equality is T_{eq} . We assume that $T_{\text{eq}} > T_0$, otherwise one should set $T_{\text{eq}} = T_0$. The constant \mathcal{A} depends on the reheating temperature – it ranges between 10^{25} and 10^{50} for reheating to the electroweak and GUT scales respectively.

The number of inflationary efoldings is given in the slow roll approximation by

$$N(\phi_0) \approx \int_{\phi_0} d\phi \frac{V(\phi)}{V_{,\phi}(\phi) M_{\text{Pl}}^2} \quad (10)$$

where the lower limit is the value of ϕ where the slow roll condition is first violated. For example, for a quadratic potential $N \sim (\phi_0/2M_{\text{Pl}})^2$. For small ϕ_0 there are few efoldings and Ω_0 is very small in the open case, or the universe collapsed before T_0 in the closed case. For large N , Ω_0 is very close to unity. But for N in the range $\frac{1}{2} \log \mathcal{A} \pm 1$, which is 30 ± 1 or 60 ± 1 for reheating to the electroweak or GUT scales respectively, we have $0.1 < \Omega_0 < 0.9$. So some tuning of ϕ_0 is required to obtain interesting values for $\Omega_0 < 1$ today, but it is only logarithmic and therefore quite mild [3].

The formula (9) involves several unknown parameters, and depending on the context one has to decide which of them to keep fixed. The Einstein equations for matter, radiation and curvature allow three independent constants, which may be taken as H_0 , Ω_0 and T_0 . The temperature at matter-radiation equality T_{eq} is not independent since it is determined by the matter density today, fixed by Ω_0 and H_0 , and the radiation density today, fixed by T_0 . In principle T_{eq} it is determined in terms of the fundamental Lagrangian just as the temperature at decoupling is, but since we do not know the Lagrangian it is better to eliminate T_{eq} using $T_{\text{eq}} = 2.4 \times 10^4 \Omega_0 h^2 T_0$. This in-

troduces Ω_0 dependence into the right hand side of (9), so one solves to obtain

$$\Omega_0 \approx 1 \mp \mathcal{A}' e^{-2N(\phi_0)}, \quad \mathcal{A}' \approx 4 \left(\frac{T_{\text{reheat}}^2}{2.4 \times 10^4 h^2 T_0^2} \right). \quad (11)$$

(For the open case if $\Omega_0 < (2.4 \times 10^4 h^2)^{-1}$ and $T_0 > T_{\text{eq}}$ one should use (9) with T_{eq} replaced by T_0 .) The formula (11) gives us Ω_0 in terms of the presently observed parameters T_0 and H_0 , plus the inflationary parameters namely the initial field ϕ_0 and the reheat temperature T_{reheat} .

Let us summarise the argument so far. We have constructed families of complete background solutions describing open and closed inflationary universes for essentially any inflaton potential. These solutions solve the standard inflationary conundrums, since exponentially large, homogeneous universes are obtained from initial data specified within a single Hubble volume. Each also has a well defined spectrum of fluctuations obtained by analytic continuation from the Euclidean region. It is worthwhile to explore how well these solutions, and their associated perturbations, match the observed universe. We shall do so in future work.

More ambitiously, one can also attempt to understand the theoretical probability distribution for Ω_0 , and it is to this that we turn next.

4. Anthropic estimate of Ω_0

The a priori probability for a universe to have given value of Ω_0 is proportional the square of the wavefunction, given in the leading semiclassical approximation by αe^{-2S_E} . We will work in some fixed theory in which T_{reheat} is determined by the Lagrangian. The initial field ϕ_0 is however still a free parameter labelling the relevant instanton. We consider a generic inflationary potential which increases away from zero. Both closed and open solutions exist for arbitrarily large ϕ_0 , so at least for suitably flat potentials essentially all possible values of Ω_0 are allowed. There are also closed solutions where $\mathcal{A} e^{-2N} > 1$, in which the universe turns round and recollapses before ever reaching the present temperature T_0 .

The Euclidean action (8) is typically *huge* – and

in the simplest theories is likely to be the dominant factor in the probability distribution $P(\Omega_0)$. The most favoured universes are those with the smallest initial field ϕ_0 : these universes are either essentially empty at T_0 in the open case, or recollapsed long before T_0 in the closed case. These universes are quite different from our own, and one might be tempted to discard the theory. Before doing so, we might remind the reader that all other versions of inflation fail *just as badly* in this regard – they are just less mathematically explicit about the problem. According to the heuristic picture of chaotic inflation for example, an exponentially large fraction of the universe is still inflating, and we certainly do not inhabit a typical region. So as in that case (and with some reluctance!) we shall be forced to make an anthropic argument.

If one knew the precise conditions required for the formation of observers it would be reasonable to restrict attention to the subset of universes containing them. The problem is that we do not. The best we can do is to make a *guess* based on our poor knowledge of the requirements for the formation of life, namely the production of heavy elements in stars and a reasonably long time span to allow evolution to take place. Such an invocation of the anthropic principle represents a retreat for theory – we give up on the goal of explaining all the properties of the universe by using some (our existence) to constrain others (e.g. Ω_0). However we don't think it is completely unreasonable, and it may (unfortunately!) turn out to be essential. An alternative attitude is to seek a future theoretical development that will fix the parameter ϕ_0 and the problem of its probability distribution. Both avenues are in our view worth pursuing.

The anthropic condition is naturally implemented within a Bayesian framework where one regards the wavefunction as giving the prior probability for Ω_0 , and then computes the posterior probability for Ω_0 given the fact that our galaxy formed. So one writes

$$\begin{aligned} \mathcal{P}(\Omega_0|\text{gal}) &\propto \mathcal{P}(\text{gal}|\Omega_0)\mathcal{P}(\Omega_0) \\ &\propto \exp\left(-\frac{\delta_c^2}{2\sigma_{\text{gal}}^2} - 2S_E(\phi_0)\right) \end{aligned} \quad (12)$$

where the first factor represents the probability that the galaxy-mass region in our vicinity underwent

gravitational collapse, for given Ω_0 . The rms contrast of the linear density field smoothed on the galaxy mass scale today is σ_{gal} , and $\delta_c \approx 1$ is the threshold set on the linear perturbation amplitude by the requirement that gravitational collapse occurs. We have only included the leading exponential terms in (12), and have assumed Gaussian perturbations as predicted by the simplest inflationary models.

The rms contrast in the density field today σ_{gal} is given by the perturbation amplitude at Hubble radius crossing for the galaxy scale $\Delta(\phi_{\text{gal}})$ multiplied by the growth factor $G(\Omega_0)$. The latter is strongly dependent on Ω_0 both through the redshift of matter-radiation equality and the loss of growth at late times in a low density universe [8]. Roughly one has $G(\Omega_0) \sim 2.4 \times 10^4 h^2 \Omega_0^2 \sim 10^4 \Omega_0^2$ for $h = 0.65$. In the slow roll approximation the linear perturbation amplitude at horizon crossing is

$$\Delta^2(\phi) \equiv \frac{V^3}{M_{\text{pl}}^6 V_{,\phi}^2}. \quad (13)$$

At this point it is interesting to compare and contrast the open and closed inflationary continuations. If we fix ϕ_0 , the Euclidean actions and therefore the prior probabilities are very similar. From (11) one sees that for fixed H_0 and T_0 , an open universe with density parameter Ω_0 is as likely a priori as a closed universe with density parameter $2 - \Omega_0$. Of course the two universes are very different. The first difference is that the closed universe is considerably younger – for $h = 0.65$ the open universe is 15 Gyr old, the closed one is 8 Gyr old. The second and most striking difference is that the open universe is spatially infinite whereas the closed universe is finite. If one accepted the arguments of some other authors [9,10] that the number of observers is the determining factor, one would conclude that open inflation was infinitely more probable because it would produce an infinite number of galaxies. However we do not agree with this line of reasoning because it would be like arguing that we are more likely to be ants because there are more ants than people! For this reason we prefer to use Bayesian statistics and consider the probability of forming a galaxy at fixed H_0 and T_0 rather than the total number of galaxies.

In the open case, the galaxy formation probability produces a peak in the posterior probability for Ω_0 .

At very low Ω_0 the growth factor is so small that galaxies become exponentially rare. From (11)

$$\frac{d\Omega_0}{d\phi_0} = \frac{2V}{M_{\text{pl}}^2 V_{,\phi}} (1 - \Omega_0) \quad (14)$$

and, substituting $\sigma_{\text{gal}} = \Delta(\phi_{\text{gal}})G(\Omega_0)$ into (12) and maximising with respect to ϕ_0 , it follows that the most likely value for Ω_0 is given by

$$\Omega_0 \approx 0.01 \left(\frac{\Delta^2(\phi_0)}{\Delta^2(\phi_{\text{gal}})} \right)^{\frac{1}{5}}. \quad (15)$$

The simplest inflationary models are close to being scale invariant, so the latter factor is close to unity. The result, $\Omega \approx 0.01$, is interestingly close to the baryon density required for primordial nucleosynthesis, but too low to be compatible with current observations.

According to these arguments the most probable open universe is one where matter-radiation equality happened at a redshift of 100, well after decoupling. The horizon scale at that epoch is $\sim 2500h^{-1}$ Mpc (for $h = 0.65$) and for a pure baryonic universe the power spectrum for matter perturbations would be scale invariant from that scale down to the Silk damping scale, an order of magnitude smaller. The nonlinear collapsed region around us would be somewhere between these scales in size. It would be an isolated, many sigma high density peak surrounded by a very low density universe. Interestingly, the value of Ω_0 we would measure would be much higher than the global average. However even though such a region would be large, it is hard to see how the universe would appear as isotropic as it does to us (in the distribution of radio galaxies and X-rays for example) unless we lived in the centre of the collapsed region, and it was nearly spherical.

In the closed case, the prior probability distribution favours universes which recollapsed before the temperature ever reached T_0 . If we fix T_0 and H_0 (i.e. demanding the universe be expanding) a peak in the posterior probability for Ω_0 is produced by imposing the anthropic condition that the universe should be old enough to allow the evolution of life, say 5 billion years. For a Hubble constant $h = 0.65$, this requires that $\Omega_0 < 10$, and the peak in the posterior probability would be at $\Omega_0 = 10$. If we raised this age requirement to 10 billion years, the

most likely value for Ω_0 would be just above unity. The most likely closed universe would be more probable than the most likely open universe in the first case, but less probable in the second.

Even though these most likely universes (i.e. very closed or very open) are probably not an acceptable fit to our own, we nevertheless find it striking that such simple arguments lead to a value of Ω_0 not very far from the real one. The simple inflationary models we have discussed here are certainly not final theories of quantum gravity, and it is quite possible that a more complete theory would lead to a modified distribution for ϕ_0 giving a more acceptable values of Ω_0 . In particular it seems possible that the prior distribution for Ω_0 would favour values closer to unity while disfavouring intermediate values, but one would still need to invoke anthropic arguments to exclude very high or very low values.

One possible mechanism for increasing the probability of a high initial field value ϕ_0 and therefore a value for Ω_0 nearer unity might just be phase space. In a realistic theory, with many more fields, there are an infinite number of instanton solutions of the type we have discussed. Each starts at some point in field space, with the fields rolling up the potential in the instanton and down the potential in the open universe. If we assume one field ϕ provides most of the inflation, it is possible that as ϕ_0 increases, the other fields it couples to become massless. This would increase the phase space available at given Euclidean action. For example, ϕ could be the scalar field parametrising the radius of an extra dimension: in this case the radius R would be proportional to $e^{(\phi/M_{\text{pl}})}$. Then ϕ getting large would mean that the tower of Kaluza Klein modes became exponentially light and there would be a corresponding exponential growth in the phase space available at fixed Euclidean action. This exponential growth in phase space would cease when the extra dimension became so large that the extra dimensional gauge coupling became of order unity, and one entered the strong coupling regime.

In summary, we have proposed a new framework for inflation in which values for the density parameter $\Omega_0 \neq 1$ are allowed for generic inflaton potentials, whilst retaining the usual successes of inflation including a predictive pattern of density perturbations. The generic prediction of this framework is a

very open or very closed universe, but it is possible that including other fields and extra dimensions could result in more acceptable values of Ω_0 closer to unity.

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