Black Hole Collisions

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Outline

• Motivation: why simulate black hole collisions?
  – expected to be among the strongest and most promising sources of gravitational waves that could be observed by gravitational wave detectors
  – understand the strong-field regime of general relativity
  – in the final stages of coalescence no exact or perturbative analytic techniques known to solve the problem

• Methodology
  – brief overview of numerical relativity, the difficulties in discretizing the field equations, and an evolution scheme based on generalized harmonic coordinates

• Results
  – brief synopsis of recent results in binary black hole merger simulations
    • most detailed studies carried out for equal mass, quasi-circular inspirals
    • much excitement about large recoil velocities for certain cases with spin
      – can understand this as a "frame dragging" effect
  – fine-tuned eccentric orbits
    • evidence that, to a certain degree, some of the interesting phenomenology of test particle motion persists in the equal mass merger regime, including unstable circular orbits and corresponding zoom-whirl like behavior
    • speculative application to the high-energy scattering problem
Numerical Relativity

- Numerical relativity is concerned with solving the field equations of general relativity
  \[ G_{\alpha\beta} = 8\pi T_{\alpha\beta} \]
  using computers.

- When written in terms of the spacetime metric, defined by the usual line element
  \[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \]
  the field equations form a system of 10 coupled, non-linear, second order partial differential equations, each depending on the 4 spacetime coordinates
  - it is this system of equations that we need to solve for the 10 metric elements (plus whatever matter we want to couple to gravity)
  - for many problems this has turned out to be quite an undertaking, due in part to the mathematical complexity of the equations, and also the heavy computational resources required to solve them

- The field equations may be complicated, but they are the equations that we believe govern the structure of space and time (barring quantum effects and ignoring matter). That they can, in principle, be solved in many “real-universe” scenarios is a remarkable and unique situation in physics.
Minimal requirements for a formulation of the field equations that might form the basis of a successful numerical integration scheme

- Choose coordinates/system-of-variables that fix the character of the equations
  - three common choices
    - free evolution — system of hyperbolic equations
    - constrained evolution — system of hyperbolic and elliptic equations
    - characteristic or null evolution — integration along the lightcones of the spacetime

- For free evolution, need a system of equations that is well behaved off the “constraint manifold”
  - analytically, if satisfied at the initial time the constraint equations of GR will be satisfied for all time
  - numerically the constraints can only be satisfied to within the truncation error of the numerical scheme, hence we do not want a formulation that is “unstable” when the evolution proceeds slightly off the constraint manifold

- Need well behaved coordinates (or gauges) that do not develop pathologies when the spacetime is evolved
  - typically need dynamical coordinate conditions that can adapt to unfolding features of the spacetime

- Boundary conditions also historically a source of headaches
  - naive BC’s don’t preserve the constraint nor are representative of the physics
  - fancy BC’s can preserve the constraints, but again miss the physics
  - solution ... compactify to infinity

- Geometric singularities in black hole spacetimes need to be dealt with
Generalized Harmonic Evolution Scheme

- Einstein equations in *generalized harmonic* form with *constraint damping*:

\[
\begin{align*}
g^{\gamma \delta} g_{\alpha \beta, \gamma \delta} + 2g^{\gamma \delta} (g_{\alpha \beta})_{\gamma \delta} + 2H_{(\alpha, \beta)} - 2H_{\delta} \Gamma^\delta_{\alpha \beta} + 2\Gamma^\gamma_{\delta \beta} \Gamma^\delta_{\gamma \alpha} + 8\pi (2T_{\alpha \beta} - g_{\alpha \beta} T) \\
+ \kappa (n_\mu C_\nu + n_\nu C_\mu - g_{\mu \nu} n^\alpha C_\alpha) = 0
\end{align*}
\]

where the generalized harmonic constraints are \( C^\mu \equiv H^\mu - \nabla^\alpha \nabla_\alpha \chi^\mu \), \( \kappa \) is a constant, \( n^\nu \) is a unit time-like vector

and \( \Gamma \) are the Christoffel symbols

\[
\Gamma^\delta_{\alpha \beta} \equiv \frac{1}{2} g^{\delta \epsilon} \left( g_{\alpha \epsilon, \beta} + g_{\beta \epsilon, \alpha} - g_{\alpha \beta, \epsilon} \right)
\]

- Need *gauge evolution equations* to close the system; use the following with \( \xi_1, \xi_2 \) and \( n \) constants, and \( \alpha \) is the so-called lapse function:

\[
\begin{align*}
\nabla^\mu \nabla_\mu H_t &= -\xi_1 \frac{\alpha - 1}{\alpha^n} + \xi_2 \partial_\mu H_t \cdot n^\mu \\
H_x &= H_y = H_z = 0
\end{align*}
\]

- Matter stress energy supplied by a massless scalar field \( \Phi \):

\[
\nabla^\mu \nabla_\mu \Phi = 0
\]

\[
T_{\alpha \beta} = 2\Phi_{, \alpha} \Phi_{, \beta} - g_{\alpha \beta} \Phi_{, \mu} \Phi_{, \mu}
\]
Computational issues in solving numerical solution of the field equations

• Each equation contains tens to hundreds of individual terms, requiring on the order of several thousand floating point operations per grid point with any evolution scheme.

• Problems of interest often have several orders of magnitude of relevant physical length scales that need to be well resolved. In an equal mass binary black hole merger for example:
  
  • radius of each black hole R~2M
  • orbital radius ~ 20M (which is also the dominant wavelength of radiation emitted)
  • outer boundary ~ 200M, as the waves must be measured in the weak-field regime to coincide with what detectors will see

  – Can solve these problems with a combination of hardware technology — supercomputers — and software algorithms, in particular adaptive mesh refinement (AMR)

  • vast majority of numerical relativity codes today use finite difference techniques (predominantly 2nd to 4th order), notable exception is the Caltech/Cornell pseudo-spectral code

• How to deal with the true geometric singularities that exist inside all black holes?

• excision
Brief Synopsis of Recent Progress in the Simulation of Binary Coalescence I

- Two quite different, stable methods of integrating the Einstein field equations for this problem
  - *generalized harmonic coordinates with constraint damping*, F. Pretorius, PRL 95, 121101 (2005)
    - Caltech/Cornell, L. Lindblom et al., Class. Quant. Grav. 23 (2006) S447-S462
    - PITT/AEI/LSU, B. Szilagyi et al., gr-qc/0612150, & L. Lehner et al.
    - Pennstate, F. Herrmann et al., gr-qc/0601026
    - Jena, J. A. Gonzalez et al., gr-qc/06010154
    - LSU/AEI/UNAM, J. Thornburg et al., gr-qc/0701038
    - U. Sperhake, gr-qc/0606079
Brief Synopsis of Recent Progress in the Simulation of Binary Coalescence II

- Several groups have now evolved approximations to quasi-circular inspiral, and are getting similar results.
- Waveforms remarkably simple ... brief merger regime connects inspiral and ringdown phases well fit by perturbative models.
- For the equal mass, non-spinning mergers \( \sim 3-4\% \) of the total energy of the system is radiated in the final couple of orbits, collision and ringdown; the final black hole has a spin of \( \sim 0.7 \).
**Brief Synopsis of Recent Progress in the Simulation of Binary Coalescence III**

- Waveform utility for early gravitational wave detection efforts
  - simplicity of waveforms suggest hybrid (or even completely analytic) templates banks could be constructed
  - for higher-mass mergers events present waveforms seem to be accurate enough to be used as templates
  - current approximate and phenomenological templates families for binary black holes (such as BCV) are again good for detection
    - get very high overlaps between these families (>0.97) and hybrid PN/Numerical waveforms
    - parameters of highest overlap templates are often far off

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*T. Baumgarte et al., qr/qc 0612100*

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*Y. Pan et al. qr/qc 0704.1964*
This animation shows the \textit{lapse function} in the orbital plane.

The lapse function represents the relative time dilation between a hypothetical observer at the given location on the grid, and an observer situated very far from the system.

The redder the color, the slower local clocks are running relative to clocks at infinity.

If this were in “real-time” it would correspond to the merger of two \(\sim 5000\) solar mass black holes.
Gravitational waves from the simulation

The real component of $\psi_4$ in the orbital plane, which is proportional to the second time derivative of the plus-polarization component of the wave.
Recoil velocities in binary mergers

- “kick” velocity of remnant black hole due to asymmetric beaming of radiation
  - typical values for spinning black holes of 100’s km/s, but can be as large as 4000km/s for equal mass black holes with spins vectors anti-aligned and in the orbital plane [F. Herrmann et al., gr-qc/0701143; M. Koppitz et al., gr-qc/0701163; M. Campanelli et al. gr-qc/0701164 & gr-qc/gr-qc/0702133]

- uniform sampling over spin vector orientations and mass ratios for two a=.9 black holes with m1/m2 between 1 and 10 suggested only around 2% of parameter space has kicks larger than 1000km/s, and 10% larger than 500km/s [J. Schnittman & A. Buonanno, astro-ph/0702641]

- astrophysical population is most likely highly non-uniform, e.g. torques from accreting gas in supermassive merger scenarios tend to align the spin and orbital angular momenta, which will result in more modest kick velocities <~200km/s [T.Bogdanovic et al, astro-ph/0703054]
Large recoil velocities in binary mergers

- scenario giving rise to very large kick velocities is, at a first glance, quite bizarre:
  - equal masses with equal but opposite spin vectors in the orbital plane
  - kick direction is normal to the orbital plane, but magnitude depends sinusoidally on the initial phase
  - is linearly dependent on the magnitude of the spin
  - still “small” from a dimensional analysis point of view, but enormous in an astrophysical setting
Frame dragging induced kicks

• Can intuitively understand this phenomena as a *frame dragging effect*

  – think of BH 1 (2) moving in the background space time of BH 2 (1)

  – relative to BH 1 (2)’s spin vector, BH 2 (1) has zero orbital angular momentum ... such a “particle” in a BH background is dragged by the rotation of the BH with a velocity

\[ v \sim m a \sin(\theta) / r^2 \]
Frame dragging induced kicks

- The result of this in the equal mass problem with the particular configuration for maximum kicks is that the *entire orbital plane* will oscillate sinusoidally in the direction normal to the orbital plane with the orbital frequency, and maximum velocity of \( \frac{ma}{r^2} \)

- The binary is emitting gravitational radiation, with the largest flux normal to the orbital plane ... the frame dragging induced oscillations will cause a periodic doppler shift of the radiation along the axis, alternating between red and blue shift in one direction with the opposite in the other

- Averaged over an orbit, and without inspiral, the net momentum radiated is zero. However, the merger event stops this process, and depending on where it’s stopped there will be linear momentum radiated normal to the orbital plane ... to conserve momentum the final black hole is given a kick in the opposite direction
Frame dragging induced kicks

- To estimate the maximum kick:
  
  - assume radiation stops once the black hole separation is within $r \sim 2M$ ($M=2m$)

  - energy of a gravitational wave $\sim$ frequency-squared ... doppler shift by maximum frame dragging velocity at $r$

  - let the net energy radiated over the last $1/2$ orbit be $\varepsilon M$, then:

    $$v_{\text{recoil}} \sim \frac{\varepsilon (a/M)}{4}$$

  - with typical $\varepsilon \sim 1\text{-}2\%$, get $v_{\text{recoil}} \sim (a/M) \text{ 1000 km/s}$
Frame-dragging induced kicks

- this seems to be consistent with what’s happening in simulations
  - however, none of this is gauge invariant, so take this in the same spirit as the order-of-magnitude calculation

*equal mass merger, “scattering” initial conditions with impact parameter ~14m, initial velocities ~0.12 in +/- y direction, initial spins a/m=0.5 anti-aligned in +/- x direction*
Fine-tuned eccentric orbits

- Use scalar field collapse to construct binaries
  
  - initial, equal mass scalar field pulses separated a coordinate (proper) distance 8.9M (10.8M) on the x-axis, one boosted with boost parameter k in the +y direction, the other with k in the -y direction
  
  - note, resultant black hole velocities are related to, but not equal to k

- To find interesting orbital dynamics, tune the parameter k to get as many orbits as possible
  
  - in the limit as k goes to 0, get head-on collisions
  
  - in the large k limit, black holes are deflected but fly apart

- Generically these black hole binaries will have some eccentricity (not easy to define given how close they are initially), and so arguably of less astrophysical significance
  
  - want to explore the non-linear interaction of BH’s in full general relativity
Scalar field $\phi_r$, compactified (code) coordinates

$$\bar{x} = \tan(x\pi/2), \quad \bar{y} = \tan(y\pi/2), \quad \bar{z} = \tan(z\pi/2)$$
Sample Orbit

![Sample Orbit Diagram](image-url)
Lapse and Gravitational Waves

6/8h resolution, $v=0.21909$ merger example

Lapse function $\alpha$, orbital plane

Real component of the Newman-Penrose scalar $\Psi_4$ (times $rM$), orbital plane
The threshold of “immediate” merger

- Tuning the parameter $k$ between merger and deflection, one approaches a very interesting dynamical regime

- the black holes move into a state of near-circular evolution before either merging ($k<k^*$), or moving apart again ($k<k^*$)---at least temporarily

- for $k$ near $k^*$, there is exponential sensitivity of the resultant evolution to the initial conditions. In fact, the number of orbits $n$ spent in this phase scales approximately like

$$e^n \propto \left| k - k^* \right|^{-\gamma}$$

where for this particular set of initial conditions $\gamma \sim 0.35$

- In all cases, to within numerical error the spin parameter of the final black hole for the cases that merge is $a \sim 0.7$
The threshold of “immediate” merger

- The binary is still radiating significant amounts of energy (on the order of 1-1.5% per orbit), yet the black holes do not spiral in.

- Technical comment: the coordinates seem to be very well adapted to the physics of the situation, as the coordinate motion plugged verbatim into the quadrupole formula for two point masses gives a very good approximation to the actual numerical waveform measured in the far-field regime of the simulation.

- Taking these coordinates at face-value, they imply the binaries are orbiting well within the innermost stable circular orbit of the equivalent Kerr spacetime of the final black hole.

![Graphs showing orbital motion and waveform](image-url)
Kerr equatorial geodesic analogue

- We can play the same fine-tuning game with equatorial geodesics on a black hole background
  - here, we tune between capture or escape of the geodesic
  - regardless of the initial conditions, at threshold one tunes to one of the unstable circular orbits of the Kerr geometry (for equatorial geodesics)
    - I.e. any smooth, one parameter family of geodesics that has the property that at one extreme of the parameter the geodesic falls into the black hole, while at the other extreme it escapes, exhibits this behavior

![un-bound orbit example](image1)

![bound orbit example; the threshold orbit in this case is sometimes referred to as a homoclinic orbit](image2)
Kerr geodesic analogue cont.

- Can quantify the unstable behavior by calculating the Lyapunov exponent $\lambda$ of the orbit (or in this case calling it an “instability” exponent may be more accurate)

  - easy way: measure $n(k - k^*)$, and find the slope $\gamma$ of $n$ versus $-\ln|k - k^*|$ ... $\gamma = \omega / 2\pi \lambda$.

  - easier(?) way: do a perturbation theory calculation (following N. Cornish and J. Levin, CQG 20, 1649 (2003)) ...

\[
\gamma = \frac{r^2}{2\pi} \left[ 3r^2 \Delta + \frac{4m}{\omega^2} \left( rR_0^2 \omega^2 - 4ma\omega - r + 2m \right) \right]^{-1/2} , \\
\Delta = r^2 + a^2 - 2mr , \quad R_0^2 = r^2 + a^2 \left( 1 + 2m / r \right) \\
m\omega = \left[ \left( r / m \right)^{3/2} \pm a / m \right]^{-1} 
\]
Comparison of $\gamma$

- The dashed black lines are the previous formula evaluated for various ranges of $a$ & $r$
- The colored dots are from calculations finding the capture-threshold using numerical integration of geodesics
  - Technical note: analytic expression derived using Boyer-Lindquist coordinates, numerical integration done in Kerr-Schild coordinates, though neither expect nor see significant differences
- The red-dashed ellipse is the single “dot” from the full numerical experiment performed. The size of the ellipse is an indication of the numerical uncertainty, though the trend suggests that $r$ is slowly decreasing with the approach to threshold, so the ellipse may move a bit to the left if one could tune closer to threshold
  - interestingly, the final spin parameter of the black hole that forms in the merger case is $\sim 0.7$
How far can this go in the non-linear case?

- System is losing energy, and quite rapidly, so there must be a limit to the number of orbits we can get

- **Hawking’s area theorem**: assume cosmic censorship and “reasonable” forms of matter, then net area of all black holes in the universe can *not* decrease with time
  - the area of a single, isolated black hole is:
    \[
    A = 8\pi M^2 \left( 1 + \sqrt{1 - \frac{J^2}{M^4}} \right)
    \]
  - initially, we have two non-rotating \((J=0)\) black holes, each with mass \(M/2\):
    \[
    \sum A_i = 8\pi M^2
    \]
  - maximum energy that can be extracted from the system is if the final black hole is also non-rotating:
    \[
    A_f = 16\pi M_f^2 \geq 8\pi M^2
    \]
    in otherwords, the maximum energy that can be lost is a factor \(1-1/\sqrt{2} \approx 29\%\)
  - If the trend in the simulations continues, and the final \(J\sim 0.7M^2\), we still get close to 24\% energy that could be radiated
    - the simulations show around 1-1.5\% energy is lost per whirl, so we may get close to 15-30 orbits at the threshold of this fine-tuning process!
Can we go even further?

- The preceding back-of-the-envelope calculation assumed the energy in the system was dominated by the rest mass of the black holes.

- What about the black hole scattering problem?
  - give the black holes sizeable boosts, such that the net energy of the system is dominated by the kinetic energy of the black holes.
  - set up initial conditions to have a one-parameter family of solutions that smoothly interpolate between coalescence and scatter.
    - “natural” choice is the impact parameter.
    - it is plausible that at threshold, all of the kinetic energy is converted to gravitational radiation (think of what happens to a “failed” merger, and what the resultant orbit must look like in the limit).
      - this can be an arbitrarily large fraction of the total energy of the system (scale the rest mass to zero as the boost goes to 1).
An application to the LHC?

• The Large Hadron Collider (LHC) is a particle accelerator currently under construction near Lake Geneva, Switzerland
  – it will be able to collide beams of protons with center of mass energies up to $14 \text{ TeV}$

  – we (ordinary particles) live on a 4-dimensional brane of a higher dimensional spacetime
    • “large” extra dimensions are sub-mm in size, but large compared to the 4D Planck length of $10^{-33} \text{ cm}$
    • gravity propagates in all dimensions

• The 4D Planck Energy, where we expect quantum gravity effects to become important, is $10^{19} \text{ GeV}$; however the presence of extra dimensions can change the “true” Planck energy

• A Planck scale in the TeV range is preferred as this solves the hierarchy problem
  • current experiments rule out Planck energies $\sim 1 \text{ TeV}$

• Collisions of particles with super-Planck energies in these scenarios would cause black holes to be produced at the LHC!

• can “detect” black holes by observing energy loss (from gravitational radiation or newly formed black holes escaping the detector) and/or measuring the particles that should be produced as the black holes decay via Hawking radiation
The black hole scattering problem

- Consider the high speed collision of two black holes with impact parameter $b$
  - good approximation to the collision of two partons if energy is beyond the Planck regime
  - for sufficiently high velocities charge and spin of the parton will be irrelevant (though both will probably be important at LHC energies)
- threshold of immediate merger *must* exist
- if similar scaling behavior is seen as with geodesics and full simulations of the equal mass/low velocity regime in general, can use the geodesic analogue to obtain an approximate idea of the cross section and energy loss to radiation vs. impact parameter ... Ingredients:
  - map geodesic motion on a Kerr back ground with $(M,a)$ to the scattering problem with total initial energy $E=M$ and angular momentum $a$ of the black hole that’s formed near threshold
  - find $\gamma$ and $b^*$ using geodesic motion
  - assume a constant fraction $\varepsilon$ of the remaining energy of the system is radiated per orbit near threshold (estimate using quadrupole formula)
  - Integrate near-threshold scaling relation to find $E(b)$ with the above parameters and the following “boundary” conditions: $E(0)$, $E(b^*)$ and $E(\text{infinity})$
    - $E(b^*)$ must be $\sim 1$ in kinetic energy dominated regime
    - $E(\text{infinity})=0$
    - $E(0)$ ... need some other input, either perturbative calculations, or full numerical simulations.
The black hole scattering problem

- What value of the Kerr spin parameter to use?
  - in the ultra-relativistic limit the geodesic asymptotes to the light-ring at threshold
  - it also seems “natural” that in this limit the final spin of the black hole at threshold is $a=1$. This is consistent with simple estimates of energy/angular momentum radiated

- quadrupole physics gives the following for the relative rates at which energy vs. angular momentum is radiated in a circular orbit with orbital frequency $\omega$:

\[
\frac{d(J/E^2)}{dn} = \frac{1}{E} \frac{dE}{dn} \left( \frac{1}{E\omega} - 2 \frac{J}{E^2} \right)
\]

- for the scattering problem with the same impact parameter as a threshold geodesic on an extremal Kerr background, the initial $J/E^2=1$. The Boyer-Lindquist value of $E\omega$ is $\sqrt{2}$ for a geodesic on the light ring of an extremal Kerr BH, in that regime $d(J)/E^2=0$

- But now we have a bit of a dilemma, as the extremal Kerr background has no unstable circular geodesics, and hence $\gamma$ tends to infinity in this limit
  - will use $a$ close to but not exactly 1 to find out what $E(b)$ might look like
Sample energy radiated vs. impact parameter curves (normalized)

- An estimate of $E(0)$ from *Cardoso et al., Class.Quant.Grav. 22 (2005) L61-R84*

- Cross section for black hole formation ($b<\sim 1$) would thus be $2\pi E^2$

- In higher dimensions for equatorial geodesics of Myers-Perry black holes $\gamma$ becomes quite small regardless of the spin (*C. Merrick*)
  - probably related to the fact that there are no stable circular geodesics for $d>4$
  - implies $E(b)$ is well approximated by the function
    \[ E(b) \sim E_0 \Theta(b^* - b) \]

- $dE/dn \sim \pi 40$ in this limit, so expect all the energy to be radiated away in around a dozen orbits.
Conclusions

- the next few of decades are going to be a very exciting time for gravitational physics
  - numerical simulations are finally beginning to reveal the fascinating landscape of binary coalescence with Einstein’s theory of general relativity
    - most of parameter space still left to explore
      - the “extreme” regions, though perhaps not astrophysically relevant, will be the most challenging to simulate, and may reveal some of the more interesting aspects of the theory
  - gravitational wave detectors should allow us to see the universe in gravitational radiation for the first time
    - even if we only see what we expect to see we can learn a lot about the universe, though history tells us that each time a new window into the universe has been opened, surprising things have been discovered
    - if we don’t see anything, something is “broken” ... unless it’s the detectors even that will be a remarkable discovery