To Construct a Square with Edges on Any Four Points

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(February 11, 2001)

1 Problem

Given a set of four points on a plane, construct a square with one of the four points on each of its four edges, as shown in Fig. 1.

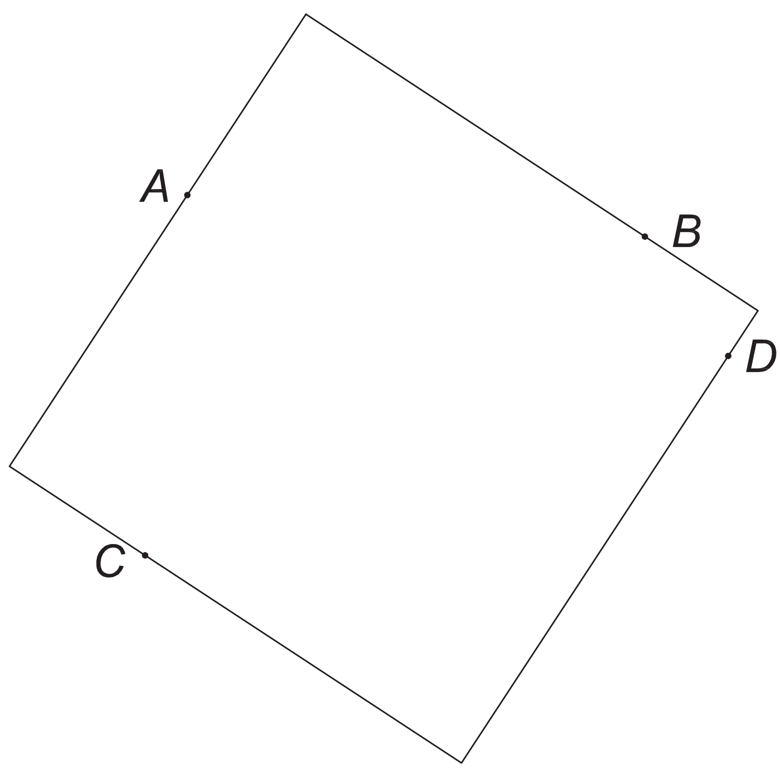

Figure 1: Construct a square with a given set of four points A, B, C and D on its edges.

2 Solution

This problem, and its solution, was suggested to us by Konstantin Shmakov.

In the figures, points are labeled by upper case letters, and angles are labeled by lower case letters.

We first prove two theorems.

2.1 Theorem 1

The diameter of a circle subtends a 90° angle from any point on the circumference.

In Fig. 2, the diameter is line AB, the point is P, and angle a is claimed to be 90°.

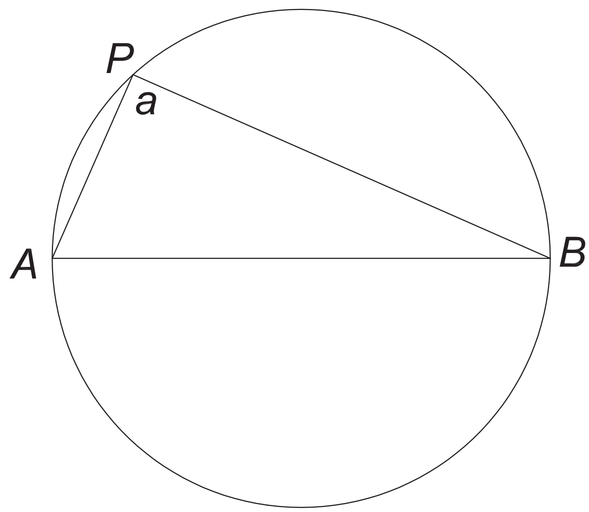

Figure 2: To show that angle $a = 90°$.

Proof: Draw line OP from the center of the circle to point P, as shown in Fig. 3.

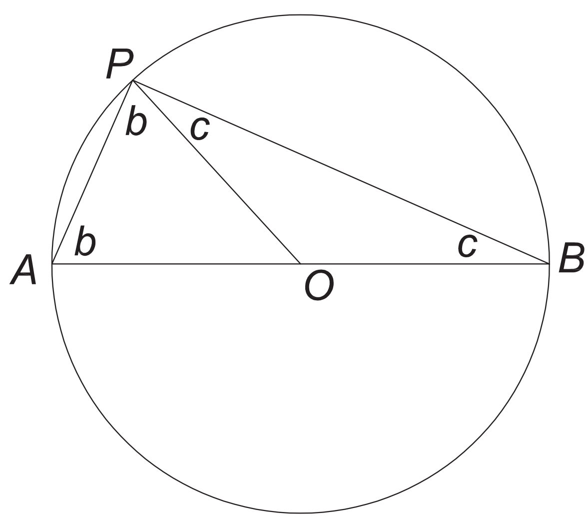

Figure 3: Line OP aids in the proof of theorem 1.

We see that ΔAOP and ΔBOP are both isosceles, and hence $\angle PAO = \angle APO \equiv b$, and $\angle PBO = \angle BPO \equiv c$.

The sum of the interior angles of ΔABP is $2b + 2c$, which is also equal to 180°.

Hence, $b + c = 90°$.

Comparing Figs. 2 and 3, we see that angle $a = b + c$, and hence angle a is 90°. *QED.*

2.2 Theorem 2

The angle subtended from any point on the circumference of a circle to the endpoints of two orthogonal diameters is $45°$*, if the point does not lie along the* $90°$ *arc between the two endpoints of the diameters. If the point does lie along that arc, the angle subtended is* $135°$*.*

In Fig. 4, the point is P, and the orthogonal diameters are AB and CD. Then, $\angle AOD = \angle AOC = \angle COB = \angle BOD = 90°$.

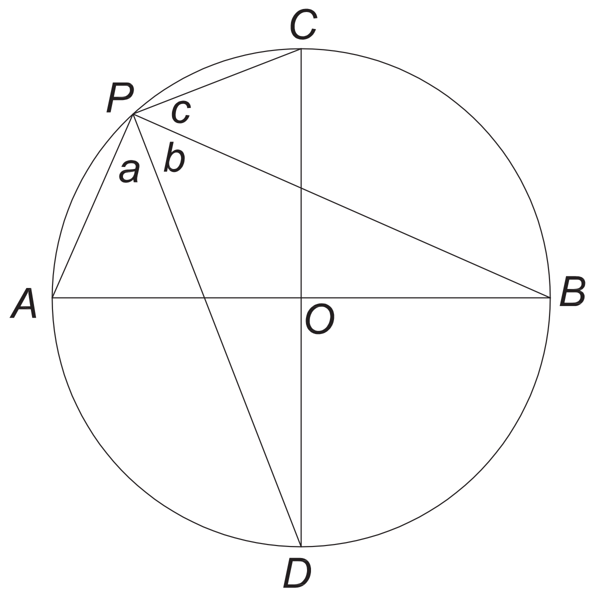

Figure 4: To show that angles $a = b = c = 45°$ when AB and CD are two orthogonal diameters of a circle.

The claim is that $b = \angle BPD = 45°$.

Angles a and c in Fig. 4 also satisfy the conditions of the theorem, and so are also claimed to be $45°$.

Angle APC also satisfies the conditions of the theorem for the case that the point lies within the $90°$ arc defined by the endpoints of the two orthogonal diameters, and so is claimed to be $135°$.

Since $\angle APC = a + b + c$, if $a = b = c = 45°$ as claimed, then $\angle APC = 135°$.

Proof: Theorem 1 applies to ΔABP and ΔCDP. Hence, $a + b = 90°$, and $b + c = 90°$.

These two relations imply that $a = c$.

Again, draw line OP from the center of the circle to point P, as shown in Fig. 5.

We see that ΔBOP and ΔDOP are both isosceles, and hence $\angle DPO = \angle PDO \equiv d$, and $\angle BPO = \angle PBO \equiv e$.

ΔDEO is a right triangle, since diameters AB and CD are orthogonal. Hence, $\angle DEO \equiv f = 90° - d$.

$\angle AEP = f = 90° - d$ also, since $\angle AEP$ and $\angle DEO$ are opposite angles between a pair of straight lines.

According to theorem 1, ΔABP is a right triangle, so $\angle BAP \equiv g = 90° - e$.

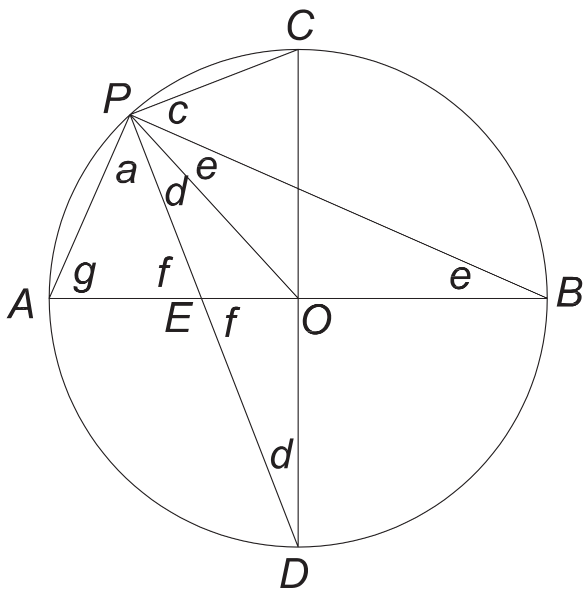

Figure 5: Line OP aids in the proof of theorem 2.

The sum of the interior angles of ΔAEP is $a + f + g = a + 90° - d + 90° - e$, which is also equal to 180°.

Hence, $a = d + e$.

Comparing Figs. 4 and 5, we see that $b = d + e$.

Hence, $a = b$.

Since we showed earlier that $a+b = 90°$ and that $a = c$, we conclude that $a = b = c = 45°$. *QED*.

2.3 The Four Point Construction

2.3.1 Rectangle Through Four Points Using Theorem 1

An infinite set of rectangles can be fit through four points using the result of Theorem 1, as shown in Fig. 6.

Construct four circles which have diameters as the four sides of the quadrilateral $ABCD$. Pick an arbitrary point I on the arc AB, and extend a line from point I through point B until it intersects arc BC at point J. Extend a line from point J through point C until it intersects arc CD at point K. Then $\angle BJC = 90°$ according to Theorem 1.

Extend a line from point K through point D until it intersects arc DA at point L, forming right angle CKD. Finally, extend a line from point L through point A until it intersects arc AB at point I, forming right angles LDA and AIB. Then, quadrilateral $IJKL$ is a rectangle with points A, B, C, and D on its four sides.

However, in general the rectangles $IJKL$ are not squares.

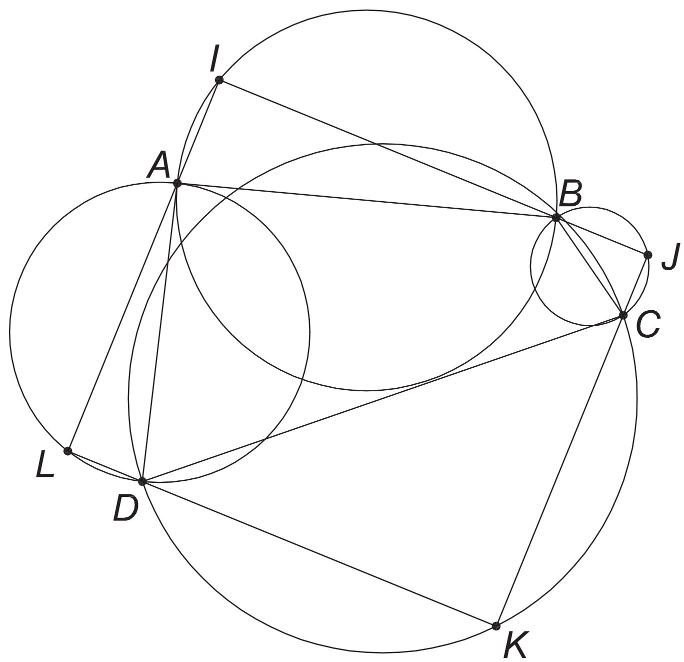

Figure 6: Construction of a rectangle $IJKL$ whose sides contain the four points A, B, C, and D, using Theorem 1.

2.3.2 Square Through Four Points Using Theorem 2

We seek a method to choose among the infinite set of rectangles $IJKL$ that contain points A, B, C, and D those that are squares.

In general, the diagonal IK, shown in Fig. 7, of rectangle $IJKL$ does not bisect the right angles AIB and CDK, but if it did the rectangle would be a square. That is, we desire that $\angle AIK$, $\angle BIK$, $\angle CKI$, and $\angle DKI$ all be 45°. This suggests use of Theorem 2.

For example, consider the two circles whose diameters are the lines AB and CD, as shown in Fig. 8. Construct diameters EF perpendicular to AB and GH perpendicular to CD. Then, extend line FH until it intersects arc AB at point I and arc CD at point K. Theorem 2 tells us that $\angle AIK$, $\angle BIK$, $\angle CKI$, and $\angle DKI$ are all 45° as desired.

We complete the construction of a square by extending lines IB and CD to intersect at point J, and lines AI and DK to intersect at point L. Since $\angle JIK$ and $\angle JKI$ of ΔIJK are both 45°, $\angle IJL = 90°$. Similarly, $\angle KLI = 90°$, and so the quadrilateral $IJKL$ is a square which contains points A, B, C, and D on its sides. *QED*

2.3.3 The Eightfold Way

We note that the prescription given in sec. 2.3.2 can be implemented eight ways: one may chose to start with circles on sides AB and CD or on sides BC and AD (circles on adjacent sides do not work); then for each of the pair of circles, either end of the orthogonal diameter may be used to define the diagonal of the square.

The eight constructions corresponding to the four points of Fig. 1 are shown in Fig. 9.

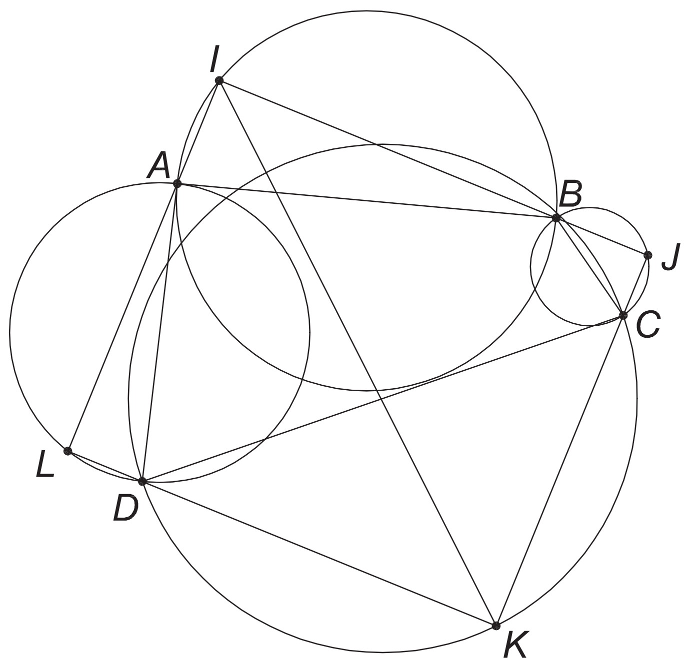

Figure 7: The rectangle $IJKL$ would be a square if the diagonal IK bisected the right angles AIB and CKD.

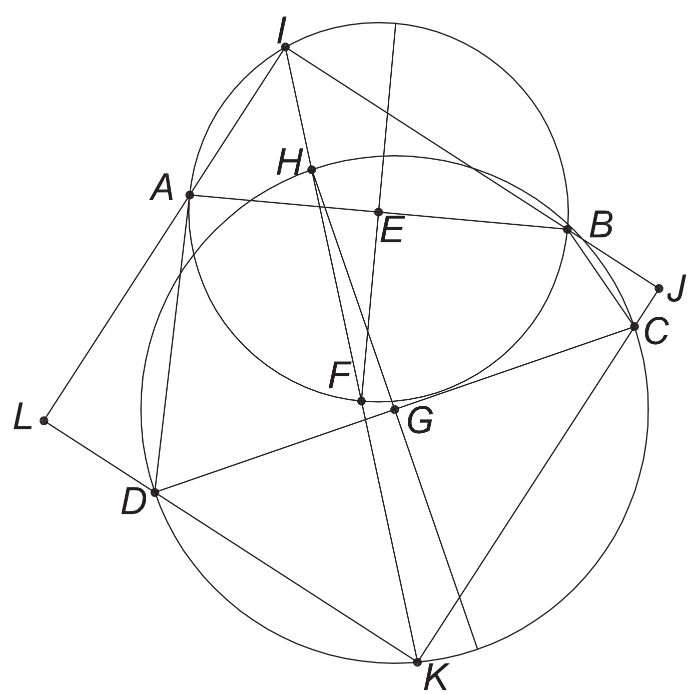

Figure 8: Construction of a square $IJKL$ whose sides contain the four points A, B, C, and D, using Theorem 2.

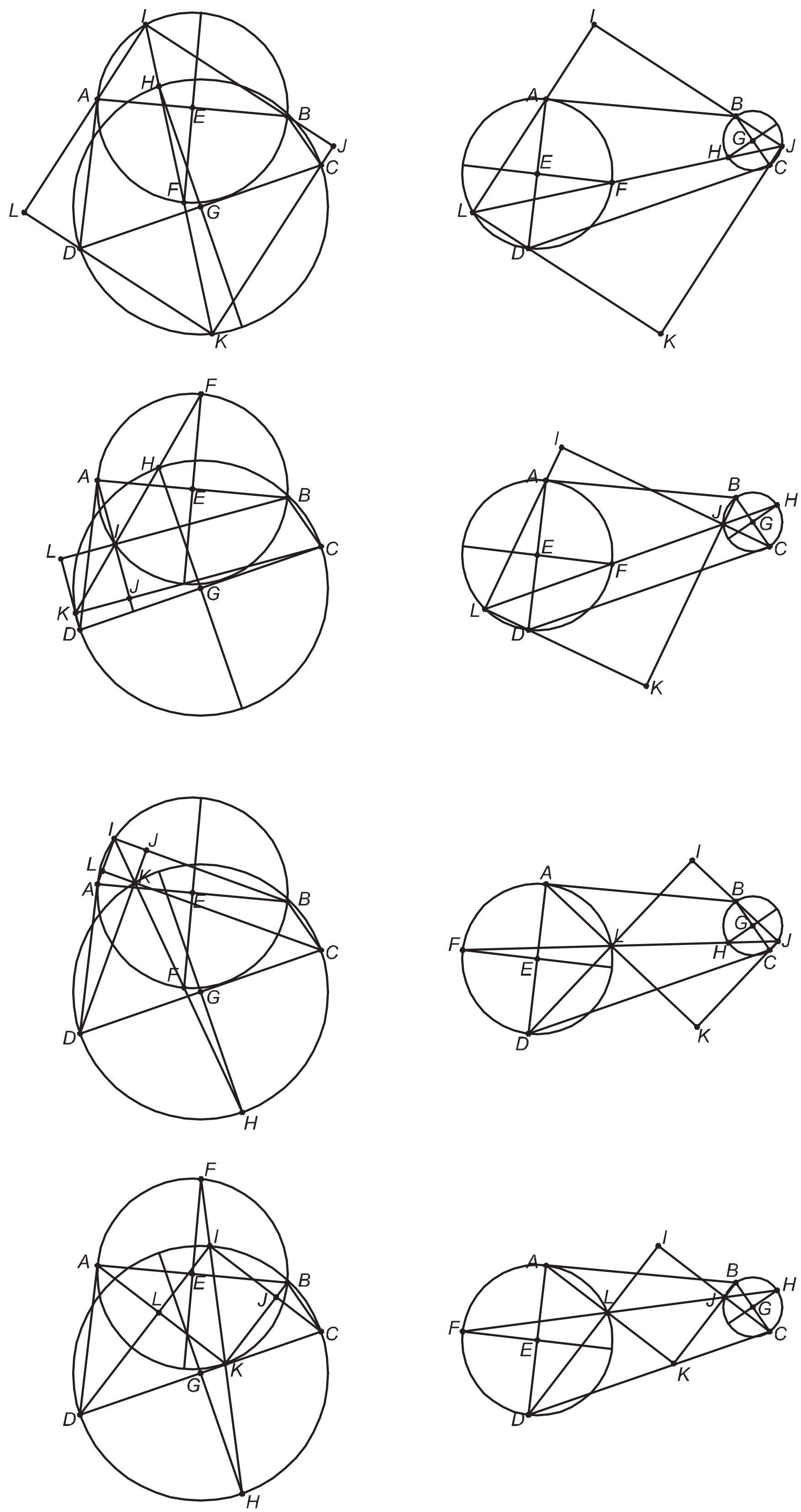

Figure 9: The 8 variants of the construction of a square $IJKL$ whose (extended) sides contain the four points A, B, C, and D.

We see that for most of the constructed squares, one or more of the four original points A, B, C, and D lie on an extension of one of the sides of the square. Indeed, at most 2 of the 8 constructions can lead to squares the actually contain the four points on the sides of the square, and those 2 constructions yield the same square.

If we accept the generalization that an extended side of a square can contain one of the original four points, then we can construct squares that contain any four points. For example, Fig. 10 illustrates the construction of one of the eight squares associated with four points on a common line.

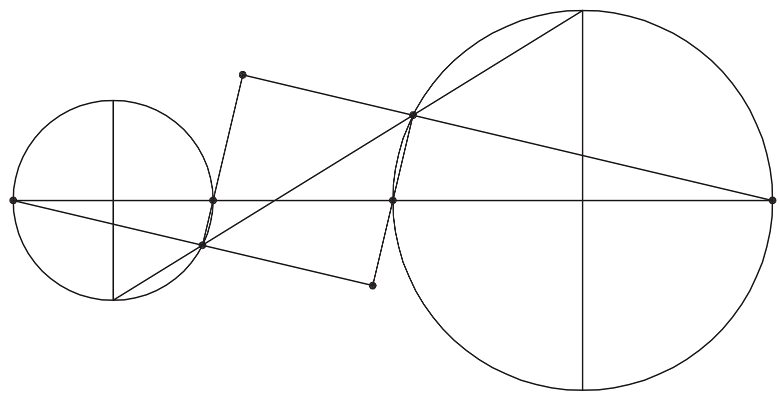

Figure 10: One of the 8 squares whose extended sides contain four points on a common line.