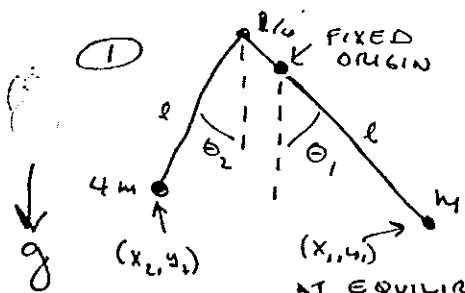


# Ph 205 SOLUTIONS TO FINAL EXAM



WE SHOULD FIRST NOTE THAT IF  $\theta_2 = 0$ , THEN THE SYSTEM IS IN EQUILIBRIUM FOR ANY  $\theta_1$ . THIS SUGGESTS THAT ONE NORMAL MODE WILL HAVE ZERO FREQUENCY. FURTHERMORE, WE CANNOT ASSUME  $\theta_1 = 0$  AT EQUILIBRIUM FOR THE OTHER MODE.

WE NOW TURN TO LAGRANGE, USE  $\theta_1$  &  $\theta_2$  AS 2 COORDS ( $\Leftrightarrow$  2 MODES)

$$T_1 = \frac{m l^2 \dot{\theta}_1^2}{2} \quad V_1 = -m g l \cos \theta_1$$

$$x_2 = -\frac{l}{4} \sin \theta_1 - l \sin \theta_2 \quad y_2 = +\frac{l}{4} \cos \theta_1 - l \cos \theta_2$$

$$\dot{x}_2 = -\frac{l}{4} \cos \theta_1 \dot{\theta}_1 - l \cos \theta_2 \dot{\theta}_2 \quad \dot{y}_2 = -\frac{l}{4} \sin \theta_1 \dot{\theta}_1 + l \sin \theta_2 \dot{\theta}_2$$

$$T_2 = \frac{4m l^2}{2} \left[ \frac{\dot{\theta}_1^2}{16} + \dot{\theta}_2^2 - \frac{1}{2} \cos(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right], \quad V_2 = 4m g l \left( \frac{\cos \theta_1}{4} - \cos \theta_2 \right)$$

$$L = T - V = \frac{m l^2}{2} \left( \frac{5}{4} \dot{\theta}_1^2 + 4 \dot{\theta}_2^2 - 2 \cos(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right) + 4m g l \cos \theta_2$$

$$\theta_1: \quad \frac{\partial L}{\partial \dot{\theta}_1} = m l^2 \left( \frac{5}{4} \dot{\theta}_1 - \cos(\theta_1 + \theta_2) \dot{\theta}_2 \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = m l^2 \left( \frac{5}{4} \ddot{\theta}_1 - \cos(\theta_1 + \theta_2) \ddot{\theta}_2 + \sin(\theta_1 + \theta_2) (\dot{\theta}_1 + \dot{\theta}_2) \dot{\theta}_2 \right)$$

$$= \frac{\partial L}{\partial \theta_1} = m l^2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

$$\text{so } \underline{\frac{5}{4} \ddot{\theta}_1 - \cos(\theta_1 + \theta_2) \ddot{\theta}_2 + \sin(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2 = 0}$$

$$\theta_2: \quad \frac{\partial L}{\partial \dot{\theta}_2} = m l^2 (4 \dot{\theta}_2 - \cos(\theta_1 + \theta_2) \dot{\theta}_1)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} = m l^2 (4 \ddot{\theta}_2 - \cos(\theta_1 + \theta_2) \ddot{\theta}_1 + \sin(\theta_1 + \theta_2) (\dot{\theta}_1 + \dot{\theta}_2) \dot{\theta}_1)$$

$$= \frac{\partial L}{\partial \theta_2} = m l^2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2 - 4m g l \sin \theta_2$$

$$\text{so } \underline{-\cos(\theta_1 + \theta_2) \ddot{\theta}_1 + 4 \ddot{\theta}_2 + \sin(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2 = -4 \omega_0^2 \sin \theta_2} \quad \text{WHERE } \underline{\omega_0^2 \equiv \frac{g}{l}}$$

AT EQUILIBRIUM,  $\dot{\theta}_1 = \dot{\theta}_2 = \ddot{\theta}_1 = \ddot{\theta}_2 = 0 \Rightarrow \sin \theta_2 = 0 \Rightarrow \theta_2 = 0, \pi$

AS EXPECTED, THERE IS NO RESTRICTION ON  $\theta_1$  FOR EQUILIBRIUM!

"CLEARLY" ONLY  $\theta_2 = 0$  IS STABLE.

LET  $\left. \begin{array}{l} \theta_1 = \alpha + A \cos \omega t \\ \theta_2 = B \cos \omega t \end{array} \right\}$  BE OUR TRIAL SOLUTION, WITH  $A, B$  SMALL

$\alpha$  IS FIXED, BUT ARBITRARY!

$\dot{\theta}_1^2 \sim A^2$  AND  $\dot{\theta}_2^2 \sim B^2$ , SO WE IGNORE THESE TERMS AS HIGHER ORDER.

$\omega_0 (\theta_1 + \theta_2) \approx \omega_0 \alpha$        $\sin \theta_2 \approx B \omega_0 t$

AND WE CAN DIVIDE BY  $\omega_0 t$  IN THE EQUATIONS OF MOTION TO GET

$\theta_1: -\frac{5}{4} \omega^2 A + \omega_0 \alpha \omega^2 B = 0$

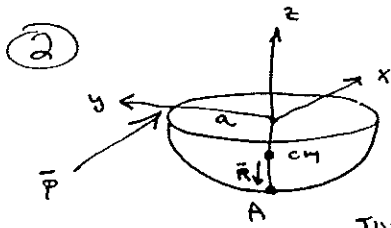
$\theta_2: +\omega_0 \alpha \omega^2 A - 4 \omega^2 B = -4 \omega_0^2 B$

THE 1ST EQ. TELLS US THAT  $\left\{ \begin{array}{l} A = \frac{4}{5} \omega_0 \alpha B \quad (\text{NOT } A=B!) \\ \text{OR } \underline{\omega = 0} \quad \underline{\text{THE TRIVIAL MODE}} \end{array} \right.$

THEN FOR  $\omega \neq 0$ , THE SECOND EQ BECOMES

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{1}{5} \omega_0^2 \alpha}}$$

WHEN  $\alpha = 0$  OR  $\bar{v}$ ,  $\omega = \sqrt{\frac{5g}{4l}}$ .



A 'THIN BOWL' IS A HALF SHELL NOT HALF OF A SOLID SPHERE.

IN GENERAL,  $\bar{v}_A = \bar{v}_{cm} + \bar{\omega} \times \bar{R}$  WHERE  $\bar{R}$  POINTS FROM THE CM TO A.

DUE TO THE IMPULSE  $\bar{P} = P\hat{x}$ ,  $\bar{v}_{cm} = \frac{\bar{P}}{M}$

TO FIND  $\bar{\omega}$ , WE NOTE THAT ANGULAR MOMENTUM ABOUT THE CM IS CONSERVED

$L_x = 0, \quad L_y = P(a-R) = I_y \omega_y, \quad L_z = -Pa = I_z \omega_z$

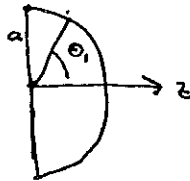
SO  $\bar{\omega} = (0, \frac{P(a-R)}{I_y}, -\frac{Pa}{I_z})$

AND  $\bar{\omega} \times \bar{R} = (-\frac{PR(a-R)}{I_y}, 0, 0)$

$\bar{v}_A = (\frac{P}{M} - \frac{PR(a-R)}{I_y}, 0, 0)$

WE NEED R AND  $I_y$  TO FINISH

FIRST, WHERE IS THE C.M.?



SO  $a - R = \frac{a}{2} \Rightarrow R = \frac{a}{2}$  ALSO.

NEXT, THE MOMENTS OF INERTIA.

$$M z_{cm} = \int_0^{\pi/2} \underbrace{\rho 2\pi a^2 d\theta}_{dm} \cdot \underbrace{a \cos \theta}_z = M \frac{a}{2} \quad \text{NOTING } M = 2\pi a^2 \rho$$

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"CLEARLY"  $I_z$  (HALF SHELL OF MASS  $M$ ) =  $I_z$  (SHELL OF MASS  $M$ )  
 $= M (\langle r_x^2 \rangle + \langle r_y^2 \rangle)$

NOW FOR A SHELL,  $\langle r_x^2 \rangle + \langle r_y^2 \rangle + \langle r_z^2 \rangle = \langle r^2 \rangle = r^2$   
 AND  $\langle r_x^2 \rangle = \langle r_y^2 \rangle = \langle r_z^2 \rangle$  }  $\Rightarrow \langle r_x^2 \rangle = \frac{r^2}{3}$

AND  $I_z = \frac{2M}{3} a^2$  FOR SHELL OF RADIUS  $a$ .

ALSO,  $I_y$  (HALF SHELL, ABOUT DIAMETER THRU CENTER OF SHELL) =  $I_z = \frac{2}{3} M a^2$

BY PARALLEL-AXIS THEOREM:  $I_y = I_{y,cm} + M(\frac{a}{2})^2$  [CAN'T USE PERPENDICULAR-AXIS THEOREM HERE!!]

SO  $I_{y,cm} = M a^2 (\frac{2}{3} - \frac{1}{4}) = \frac{5}{12} M a^2$  (=  $I_{x,cm}$ )

FINALLY,  $V_{A,K} = \frac{P}{M} - \frac{P(\frac{a}{2})(\frac{a}{2})}{\frac{5}{12} M a^2} = \frac{2}{5} \frac{P}{M}$

[FOR A SOLID HEMISPHERE,  $a-R = \frac{3}{8} a$ , AND  $I_{y,cm} = \frac{83}{320} M a^2 \Rightarrow V_A = \frac{8}{83} \frac{P}{M}$ ]

"SHOW THAT THE RIM OF THE BOWL NEVER TOUCHES THE HORIZONTAL SURFACE."

WE USE AN ENERGY ARGUMENT TO SEE THIS.

IN THE FRAME WHERE THE CM DOES NOT MOVE HORIZONTALLY, THE KINETIC ENERGY JUST AFTER THE IMPULSE IS  $KE = \frac{1}{2} \vec{\omega} \cdot \vec{I} \cdot \vec{\omega} = \frac{1}{2} (I_y \omega_y^2 + I_z \omega_z^2)$

FROM ABOVE,  $I_x = I_y = \frac{5}{12} M a^2$ ,  $I_z = \frac{2}{3} M a^2$  } JUST AFTER IMPULSE

AND  $\vec{\omega} = (0, \frac{P a z}{I_y}, \frac{P a}{I_z}) = (0, \frac{6}{5} \frac{P}{M a}, \frac{3}{2} \frac{P}{M a})$

SO  $KE = \frac{P^2}{2M} (\frac{5}{12} (\frac{6}{5})^2 + \frac{2}{3} (\frac{3}{2})^2) = \frac{P^2}{2M} (\frac{3}{5} + \frac{3}{2}) = \frac{21}{20} \frac{P^2}{M}$

NOW SUPPOSE THE BOWL TIPPED UP TIL THE RIM TOUCHES

$KE > \frac{1}{2} I_z \omega_z^2$  CERTAINLY HOLDS.

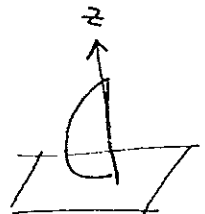
IN THIS CONFIGURATION,  $I_z = \frac{5}{12} M a^2$  WHAT IS  $\omega_z$ ?

AS THE BOWL ROLLS AROUND, THE TORQUE ABOUT THE CM DUE TO THE NORMAL FORCE IS IN THE  $x-y$  PLANE  $\Rightarrow L_z = \text{CONSTANT}$ .

FROM THE FIRST PART OF THE PROBLEM  $|L_z| = P a$ , SO  $\omega_z = \frac{L_z}{I_z} = \frac{12}{5} \frac{P}{M a}$

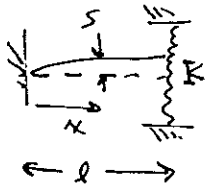
AND  $KE > \frac{1}{2} \frac{5}{12} M a^2 (\frac{12}{5} \frac{P}{M a})^2 = \frac{6}{5} \frac{P^2}{M} > KE_{\text{INITIAL}}$ .

BUT, THE CM RISES AS THE BOWL TIPS, SO WE MUST HAVE  $KE < KE_{\text{INITIAL}}$ ,  $\Rightarrow$  RIM NEVER TOUCHES!



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THE WAVE EQUATION OF THE STRETCHED STRING IS

$$\rho \ddot{s} = T s''$$

THE LEFT BOUNDARY CONDITION IS  $s(0, t) = 0$

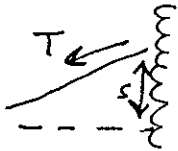
⇒ STANDING WAVES OF FORM  $s = \sin kx \cos \omega t$

WHERE  $\omega = kc = k \sqrt{\frac{T}{\rho}}$  (PLUG IN TO WAVE EQ. TO VERIFY!)

↑  
WAVE NUMBER, NOT SPRING CONSTANT  $K$

AT THE RIGHT END WE HAVE A PECULIAR BOUNDARY CONDITION:

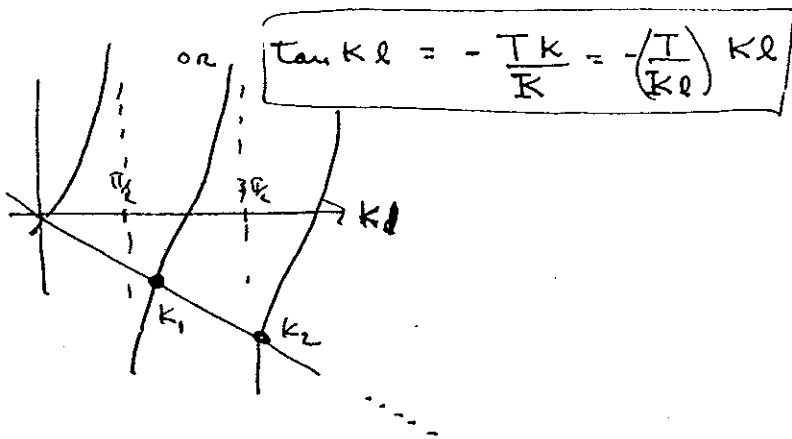
THE TUG OF THE STRING MUST COUNTERACT THE FORCE OF THE SPRING:



$$\sum F = 0 = \underbrace{-T s'}_{\text{STRING}} - \underbrace{Ks}_{\text{SPRING}} \quad (\text{WATCH SIGNS!})$$

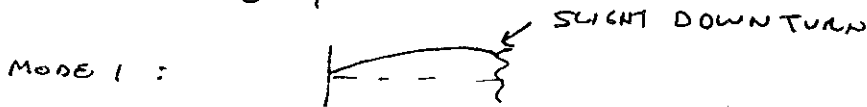
(THE WORDING OF THE PROBLEM WAS AMBIGUOUS; SOME PEOPLE READ THAT THE STRING WAS TIED TO THE MIDDLE OF A SPRING OF CONSTANT  $K$ . THEN CONSIDER THIS AS 2 HALF SPRINGS EACH OF CONSTANT  $2K$ , THE TOTAL FORCE OF THE SPRING IS  $-4Ks$ )

PLUGGING IN  $-Tk \cos kL - K \sin kL = 0$

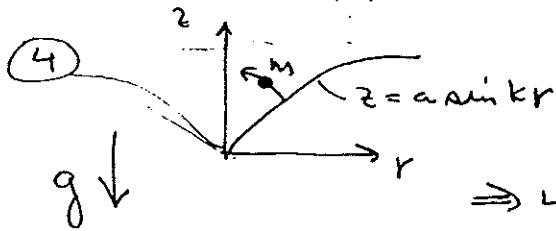


FOR  $\frac{T}{Kl} \gg 1$  WE SEE THAT  $k_n \approx \frac{2n-1}{2} \frac{\pi}{l} - \epsilon$   
 $\omega_n = \sqrt{\frac{T}{\rho}} k_n$

$\omega_2 \approx \frac{3\pi}{2} \sqrt{\frac{T}{\rho}}$



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2 DEGREES OF FREEDOM:  $r, \theta$

BUT  $L_z = m r^2 \dot{\theta} = \text{CONST}$

$\Rightarrow$  LIKELY CAN REDUCE PROBLEM TO 1 DIMENSION.

$$E_{\text{TOTAL}} \text{ IS CONSERVED} = T + V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + m g z$$

$$= \underbrace{\frac{1}{2} m \dot{r}^2 \left(1 + \left(\frac{dz}{dr}\right)^2\right)}_{\frac{1}{2} m_{\text{eff}} \dot{r}^2} + \underbrace{\frac{L^2}{2 m r^2} + m g a \sin kr}_{V_{\text{eff}}(r)}$$

HOWEVER,  $m_{\text{eff}} = m \left(1 + \left(\frac{dz}{dr}\right)^2\right) = m (1 + a^2 k^2 \cos^2 kr) \neq \text{CONSTANT}$

CAN WE USE OUR USUAL METHODS?? FOR SMALL OSCILLATIONS ABOUT AN EQUILIBRIUM RADIUS  $r_0$   $m_{\text{eff}} \approx m (1 + a^2 k^2 \cos^2 kr_0)$  & THINGS SHOULD BE OK. (IF WANTED, WRITE OUT LAGRANGE'S EQ. ...)

$$\frac{dV_{\text{eff}}}{dr} = -\frac{L^2}{m r^3} + m g a k \cos kr = 0 \text{ AT EQUILIBRIUM}$$

$$\Rightarrow \frac{L^2}{m r_0^3} = m g a k \cos kr_0$$

WE ALSO WRITE  $L = m r_0^2 \Omega$  WHERE  $\Omega = \text{EQUILIBRIUM ANGULAR VELOCITY}$

$$\text{SO } \Omega^2 = \frac{g}{r_0} a k \cos kr_0 \text{ ETC.}$$

$$K_{\text{eff}} = \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_0} = \frac{3 L^2}{m r_0^4} - m g a k^2 \sin kr_0 = 3 m g a \frac{k}{r_0} \cos kr_0 - m g a k^2 \sin kr_0$$

FOR STABLE OSCILLATIONS, NEED  $K_{\text{eff}} > 0 \Rightarrow \underline{\underline{\tan kr_0 < \frac{3}{kr_0}}}$  ( $kr_0 \leq 1.2$ )

THE FREQUENCY OF OSCILLATIONS IS

$$\omega^2 = \frac{K_{\text{eff}}}{m_{\text{eff}}} = \frac{3 g a \frac{k}{r_0} \cos kr_0 - g a k^2 \sin kr_0}{1 + a^2 k^2 \cos^2 kr_0}$$

$$= \frac{3 \Omega^2 \left(1 - \frac{kr_0 \tan kr_0}{3}\right)}{1 + \left(\frac{r_0 \Omega^2}{g}\right)^2}$$

ETC