Expansion of an Axially Symmetric, Static Magnetic Field in Terms of Its Axial Field

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1 Problem

Deduce a series expansion of an axially symmetric, static magnetic field in terms of its axial field $B_z(0,0,z)$ in cylindrical coordinates (r,ϕ,z). Also give an expansion for the vector potential of this field. The azimuthal currents that produce this field are at very large radius r.

2 Solution

This problem is a peculiar kind of boundary value problem in which a field is specified only along a line. In case the on-axis field has transverse components there is no a unique solution, as discussed in [1]. Here we obtain a unique solution under the assumption that the field off-axis is azimuthally symmetric. See sec. 13.4.2 of [2] for a multipole expansion for fields without azimuthal symmetry.[1]

2.1 Expansion of the Field

Suppose a magnetic field in a current-free region is rotationally symmetric about the z-axis. Then,

$$\mathbf{B} = B_r(r,z)\hat{\mathbf{r}} + B_z(r,z)\hat{\mathbf{z}} \tag{1}$$

in cylindrical coordinates (r,ϕ,z). If we write

$$B_z(r,z) = \sum_{n=0}^{\infty} a_n(z) r^n, \qquad \text{and} \qquad B_r(r,z) = \sum_{n=0}^{\infty} b_n(z) r^n \ , \tag{2}$$

then $a_0(z) = B_z(0,z)$. Since the divergence of the magnetic field vanishes, the proposed expansions (2) obey

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{r}\frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} = \sum_n \left[(n+1)b_n r^{n-1} + a_n^{(1)} r^n\right] = 0, \tag{3}$$

where $a^{(m)}(z) \equiv d^m a/dz^m$. For this to be true at all r, the coefficients of r^n must separately vanish for all n. Hence,

$$b_0 = 0, \tag{4}$$

$$b_n = -\frac{a_{n-1}^{(1)}}{n+1} \ . \tag{5}$$

[1]The function $a_0(z) = a_0^{(0)}(z)$ used here is the same as $C_0^{[1]}(z)$ in [2].

Since the curl of the magnetic field also vanishes (outside the source currents),

$$(\nabla \times \mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \sum_n \left(b_n^{(1)} r^n - n a_n r^{n-1} \right) = 0 \ , \tag{6}$$

Again, the coefficient of r^n must vanish for all n, so that

$$b_n^{(1)} = (n+1) a_{n+1}. \tag{7}$$

Using eq. (7) in eq. (5), we find

$$b_n = -\frac{b_{n-2}^{(2)}}{(n+1)(n+3)} \ . \tag{8}$$

Since b_0 vanishes, b_{2n} vanishes for all n, and from eq. (7), a_{2n+1} vanishes for all n. Then, using eq. (8) in eq. (7), we find

$$a_{2n} = -\frac{a_{2n-2}^{(2)}}{4n^2} \ . \tag{9}$$

Repeatedly applying this to itself gives

$$a_{2n} = (-1)^n \frac{a_0^{(2n)}}{2^{2n}(n!)^2} \ . \tag{10}$$

Inserting this in eq. (5), we get

$$b_{2n+1} = (-1)^{n+1} \frac{a_0^{(2n+1)}}{2^{2n+1}(n+1)(n!)^2} \ . \tag{11}$$

Combining eqs. (10)-(11) with eq. (2), we arrive at the desired forms,

$$B_z(r,z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n} , \tag{12}$$

and

$$B_r(r,z) = \sum_n (-1)^{n+1} \frac{a_0^{(2n+1)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1} , \tag{13}$$

for the field components, where

$$a_0^{(n)} = \frac{d^n a_0}{dz^n} \ . \tag{14}$$

These results are overly detailed for some purposes. If one is interested only in the leading behavior at small r, then eqs. (12)-(13) simplify to

$$B_z(r,z) \approx B_z(0,z), \qquad B_r(r,z) \approx -\frac{r}{2} \frac{\partial B_z(0,z)}{\partial z} \ . \tag{15}$$

The result for B_r also follows quickly from $\nabla \cdot \mathbf{B} = 0$, according to eq. (3),

$$B_r(r,z) = -\int_0^r r \frac{\partial B_z(r,z)}{\partial z} dr \approx -\int_0^r r \frac{\partial B_z(0,z)}{\partial z} dr = -\frac{r}{2} \frac{\partial B_z(0,z)}{\partial z} \ . \tag{16}$$

It is also instructive that the approximation (16) can be deduced quickly from the integral form of Gauss' law (without the need to recall the form of $\nabla \cdot \mathbf{B}$ in cylindrical coordinates). Consider a Gaussian pillbox of radius r and thickness dz centered on $(r = 0, z)$. Then,

$$
\begin{aligned}
0 &= \int \mathbf{B} \cdot d\mathbf{S} \approx \pi r^2 [B_z(0, z + dz) - B_z(0, z)] + 2\pi r \; dz \; B_r(r, z) \\
&\approx \pi r^2 \; dz \; \frac{\partial B_z(0, z)}{\partial z} + 2\pi r \; dz \; B_r(r, z)\,, \qquad (17)
\end{aligned}
$$

which again implies eqs. (15).

2.2 Expansion of the Vector Potential

The magnetic field can be generated by (distant) currents that are purely azimuthal, so that a purely azimuthal vector potential A_ϕ suffices. Then,

$$
B_r = -\frac{\partial A_\phi}{\partial z}\,, \qquad \text{and} \qquad B_z = \frac{1}{r}\frac{\partial (r A_\phi)}{\partial r}\,. \qquad (18)
$$

Hence, $A_\phi = -\int B_r \, dz$, are recalling eq. (13) we find

$$
A_\phi(r, z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n + 1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1}\,. \qquad (19)
$$

This result also follows from $A_\phi = (1/r) \int r B_z \, dr$ and eq. (12).

3 B Deduced from Its Value on a Surface

A more typical boundary value problem is to determine the field $\mathbf{B}$ from its value on a bounding surface.

One approach for this is to recall the results of vector diffraction theory, particularly as formulated by Kottler [3, 4] for fields with time dependence $e^{-i\omega t}$ in vacuum,[2]

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \int_V \left(\frac{ik}{c} \mathbf{J}(\mathbf{x}') \frac{e^{ikr}}{r} + \rho(\mathbf{x}') \nabla' \frac{e^{ikr}}{r} \right) d\text{Vol}' + \frac{i}{\omega} \oint_S (\mathbf{J} \cdot \hat{\mathbf{n}}') \nabla' \frac{e^{ikr}}{r} \, d\text{Area}' \\
&\quad - \frac{1}{4\pi} \nabla \times \oint_S \left\{ [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \frac{e^{ikr}}{r} + \frac{i}{k} \nabla \times [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \frac{e^{ikr}}{r} \right\} d\text{Area}', \qquad (20) \\
\mathbf{B}(\mathbf{x}) &= \frac{1}{c} \int_V \mathbf{J}(\mathbf{x}') \times \nabla' \frac{e^{ikr}}{r} \, d\text{Vol}' \\
&\quad - \frac{1}{4\pi} \nabla \times \oint_S \left\{ [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \frac{e^{ikr}}{r} - \frac{i}{k} \nabla \times [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \frac{e^{ikr}}{r} \right\} d\text{Area}', \qquad (21)
\end{aligned}
$$

where $\hat{\mathbf{n}}'$ is the outward unit vector normal to surface S, $r = |\mathbf{x} - \mathbf{x}'|$, c is the speed of light in vacuum, $k = \omega/c$, and Gaussian units are employed. See the Appendix of [5] for derivations and discussion of these forms.

[2]The operations involving ∇, which act only on the factor r, should be performed before the surface integrations in eqs. (20)-(21).

For a region with no currents the magnetic field can be related to a vector potential that follows from eq. (21) as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\oint_S \left\{ \mathbf{B}(\mathbf{x}') \times \hat{\mathbf{n}}' \frac{e^{ikr}}{r} - \frac{i}{k}\boldsymbol{\nabla} \times [\mathbf{E}(\mathbf{x}') \times \hat{\mathbf{n}}'] \frac{e^{ikr}}{r} \right\} d\text{Area}', \tag{22}$$

assuming that we can take the curl after performing the integrations. If $\mathbf{E}$ and $\mathbf{B}$ are zero everywhere on the surface of a region then $\mathbf{A}$ is zero in its interior, according to eq. (22). The prescription of eq. (22) cannot be extended to all of space since there must be currents somewhere if $\mathbf{B}$ is nonzero somewhere.

In the static limit, $\omega = 0 = k$, the electric field does not depend the current density $\mathbf{J}$ or the magnetic field, and the magnetic field does not depend on the electric field. Noting that $\boldsymbol{\nabla}'(1/r) = \hat{\mathbf{r}}/r^2 = -\boldsymbol{\nabla}(1/r)$, we obtain

$$\mathbf{E}(\mathbf{x}) = \int_V \rho(\mathbf{x}') \frac{\hat{\mathbf{r}}}{r^2}\, d\text{Vol}' + \frac{1}{4\pi}\oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')]}{r^2}\, d\text{Area}', \tag{23}$$

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2}\, d\text{Vol}' + \frac{1}{4\pi}\oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')]}{r^2}\, d\text{Area}', \tag{24}$$

If there are no currents within the volume of integration, the static magnetic field there can be deduced from the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\oint_S \frac{\mathbf{B}(\mathbf{x}') \times \hat{\mathbf{n}}'}{r}\, d\text{Area}' \qquad \text{(static limit)}, \tag{25}$$

recalling eq. (21). The example of a static, toroidal magnetic field (for which $\mathbf{B} = 0$ outside the torus but $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area}$ is nonzero for loops that link the torus) suggests that eqs. (22) and (25) are restricted to simply connected regions.

3.1 Uniform Axial Field

As an example, consider a uniform axial field, $\mathbf{B} = B_0\,\hat{\mathbf{z}}$ that is generated by azimuthal currents about the z-axis. The associated vector potential has only the azimuthal component

$$A_\phi = \frac{\rho B_0}{2}\,. \tag{26}$$

in a cylindrical coordinate system (ρ, ϕ, z).

We take the point of observation to be $(\rho, 0, 0)$. As the surface of integration for eq. (25) we consider a cylinder of radius $a > \rho$ with faces at $-z_1$ and z_2. Then, $\mathbf{B} \times \hat{\mathbf{n}}' = B_0\,\hat{\boldsymbol{\phi}}$ and

$$\begin{aligned}
A_\phi &= A_y = \frac{1}{4\pi}\int_0^{2\pi} a\, d\phi \int_{-z_1}^{z_2} dz\, \frac{B_0 \cos\phi}{\sqrt{z^2 + a^2 + \rho^2 - 2a\rho\cos\phi}} \\
&= \frac{aB_0}{4\pi}\int_0^{2\pi} \cos\phi\, d\phi\, \ln \frac{z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho\cos\phi}}{-z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho\cos\phi}} \\
&= \frac{aB_0}{4\pi}\int_0^{2\pi} \cos\phi\, d\phi \left[\ln\left(z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho\cos\phi}\right) \right. \\
&\qquad \left. + \ln\left(z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho\cos\phi}\right) - \ln\left(a^2 + \rho^2 - 2a\rho\cos\phi\right) \right] \\
&= -\frac{aB_0}{4\pi}\int_0^{2\pi} \cos\phi\, d\phi\, \ln\left(1 + \frac{\rho^2}{a^2} - 2\frac{\rho}{a}\cos\phi\right) = \frac{\rho B_0}{2}\,,
\end{aligned} \tag{27}$$

using 4.397.6 of [6]. A delicacy is our assumption that

$$\int_0^{2\pi} \cos\phi \, d\phi \, \ln\left(z + \sqrt{z^2 + a^2 + \rho^2 - 2a\rho\cos\phi}\right) = 0, \tag{28}$$

for nonzero values of z. This integral clearly goes to zero for large z, and the calculation (27) of A_ϕ must be independent of the values of z_1 and z_2.

3.2 Other Formulations

Section 14.3-4 of [2] gives a formalism by which $\mathbf{B}$ can be computed from knowledge of its normal component, $\mathbf{B} \cdot \hat{\mathbf{n}}$, on elliptical cylindrical surfaces, and sec. 18.2 describes the use of the tangential component $\mathbf{B} \times \hat{\mathbf{n}}$ on circular cylinders.

References

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