The Rolling Motion of a Half-Full Beer Can

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(Nov. 14, 1996)

1 Problem

Discuss the motion of a half-full (or half-empty) beer can as it rolls down an incline of angle α to the horizontal.

You may simplify your discussion to a solid half cylinder that moves without friction down the inclined plane, with the axis of the cylinder always perpendicular to the gradient of the incline. A further simplification is to consider only motion that departs from the 'trivial' solution by small oscillations.

2 Solution

We give two solutions, following some preliminary remarks.

What is the 'trivial' solution? The can rolls down the hill with the half cylinder inside always at the same azimuth about the axis of the can. Also, the acceleration of the center of mass of the half cylinder is $g \sin \alpha$ where g is the acceleration due to gravity, since by hypothesis there is no friction. That is, defining s to be the distance the center of mass of the half cylinder has moved down the incline, we expect

$$s = \frac{1}{2} g \sin \alpha t^2. \tag{1}$$

The surface of the half cylinder is not, however, horizontal but is tilted at some angle. To deduce this quickly, consider the accelerated frame in which the c.m. of the half cylinder is at rest. In this frame there is an apparent force $mg \sin \alpha$ on any mass m. This force points up the incline at angle α. With the aid of the diagram below we deduce that $g_{\rm eff} = g \cos \alpha$ and this makes angle α to the vertical as shown in Fig. 1. That is, $\mathbf{g}_{\rm eff}$ is perpendicular to the incline.

The equilibrium surface of the half cylinder is perpendicular to $\mathbf{g}_{\rm eff}$, and hence parallel to the incline.

2.1 Solution via the Accelerated Frame

If the half cylinder oscillates about equilibrium, there is both rotation of the half cylinder and motion of the c.m. perpendicular to the incline, as viewed in the accelerated frame.

The oscillatory motion is similar to, but not exactly, that of the half cylinder oscillating about its axis under the influence of $\mathbf{g}_{\rm eff}$. In that case, the frequency ω of oscillation would be

$$\omega^2 = \frac{mbg_{\rm eff}}{I}, \tag{2}$$

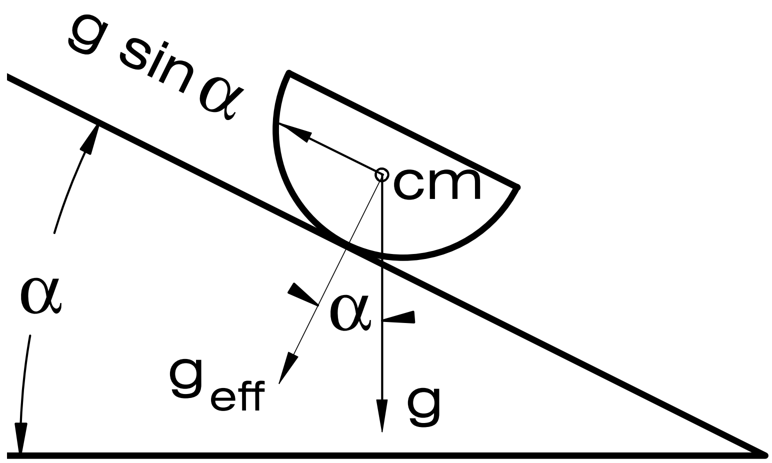

Figure 1: A half cylinder will slide down a frictionless incline of angle α without oscillation if its flat side is parallel to the incline. In the accelerated frame, the effective gravity vector, $\mathbf{g}_{\rm eff}$ is perpendicular to the incline.

where b is the distance from the axis to the center of mass of the half cylinder, and I is the moment of inertia about the axis of the cylinder.

The solution to the present problem can be found using a little-known trick described by Tiersten [1]. If a rigid body has an instantaneous center of motion, then its instantaneous kinetic energy can be written as $I_C\omega^2/2$ where I_C is the moment of inertia about the instantaneous center. The rate of change of energy is $\tau_C\omega$ where τ_C is the torque about the instantaneous center. (If it's not obvious, this follows from $\mathbf{F}\cdot\mathbf{v} = \mathbf{F}\cdot\boldsymbol{\omega}\times\mathbf{r} = \mathbf{r}\times\mathbf{F}\cdot\boldsymbol{\omega}$ where $\mathbf{r}$ runs from the instantaneous center to the point of application of force $\mathbf{F}$.) Taking the time derivative of the instantaneous kinetic energy, we have

$$\tau_C = \frac{1}{\omega}\frac{d(I_C\omega^2/2}{dt} = I_C\dot\omega + \frac{\dot I_C\omega}{2}. \qquad (3)$$

as the equation of motion.

In the accelerated frame, the half cylinder has an instantaneous center of motion. It can be located by noting that the velocity of a point is at right angles to the line joining it to the center of motion. In particular, the center of mass moves only perpendicular to the incline. Hence, the instantaneous center is on a line parallel to the incline through the center of mass. Also, the points on the half cylinder that are instantaneously in contact with the incline have instantaneous velocity parallel to the incline. Hence, the instantaneous center of rotation is on the line perpendicular to the incline through the point of contact. The instantaneous center of rotation is at the intersection of these two lines, shown as point C in Fig. 2.

We define ϕ as the angle between the flat surface of the cylinder and the incline, which is also the angle between the perpendicular to the incline through the point of contact and the line joining the center of the cylinder to the c.m. Then, by the parallel axis theorem,

$$I_C = I_{\rm cm} + b^2\sin^2\phi \approx I_{\rm cm} \qquad (4)$$

for small ϕ. In the same approximation, $\dot I_C = 0$. The torque about the instantaneous center

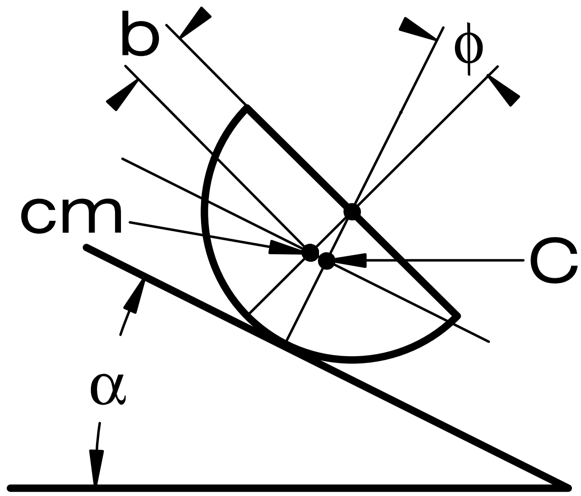

Figure 2: The center of mass of the half cylinder is distance b from the flat surface. When the half cylinder has tilted by angle ϕ from equilibrium in the accelerated frame, the center of rotation C is at the intersection of the normal to the incline through the point of contact with the line parallel to the flat surface that passes through the center of mass.

(calculated in the accelerated frame) is

$$\tau_C = -mg_{\rm eff} b \sin\phi \approx mbg\cos\alpha\,\phi. \tag{5}$$

Since $\dot\phi = \omega$, the equation of motion becomes

$$I_{\rm cm}\ddot\phi \approx -mbg\cos\alpha\,\phi, \tag{6}$$

so the frequency of small oscillations is related by

$$\omega^2 = \frac{mbg\cos\alpha}{I_{\rm cm}}. \tag{7}$$

To calculate b and I, we use polar coordinates in which the half cylinder occupies the region $r < a$ and $-\pi/2 < \theta < \pi/2$. The cross-sectional area of the half cylinder is, of course, $\pi a^2/2$. Then,

$$b \cdot {\rm Area} = \int x \cdot d{\rm Area} = \int_0^a dr \int_{-\pi/2}^{\pi/2} r d\theta\, r\cos\theta = \frac{2a^3}{3}, \qquad \text{so} \qquad b = \frac{4a}{3\pi}\,. \tag{8}$$

Similarly,

$$I = \frac{m}{\rm Area}\int_0^a dr \int_{-\pi/2}^{\pi/2} r d\theta\, r^2 = \frac{ma^2}{2}\,, \tag{9}$$

as might have been "guessed". The moment of inertia of the half cylinder about its center of mass is then (by the parallel-axis theorem)

$$I_{\rm cm} = I - mb^2 = m(a^2/2 - b^2). \tag{10}$$

Returning to the oscillations of the half cylinder, the frequency is now expected to obey

$$\omega^2 = \frac{2bg\cos\alpha}{a^2 - 2b^2}. \tag{11}$$

Since the trick using the instantaneous center of motion is obscure, we consider solutions using Lagrange's method.

2.2 Solution via Lagrange's Method

We use Lagrange's method with generalized coordinates s and ϕ, where s is the distance parallel to the incline that the c.m. has moved, and ϕ is the angle between the flat surface of the half cylinder and the incline.

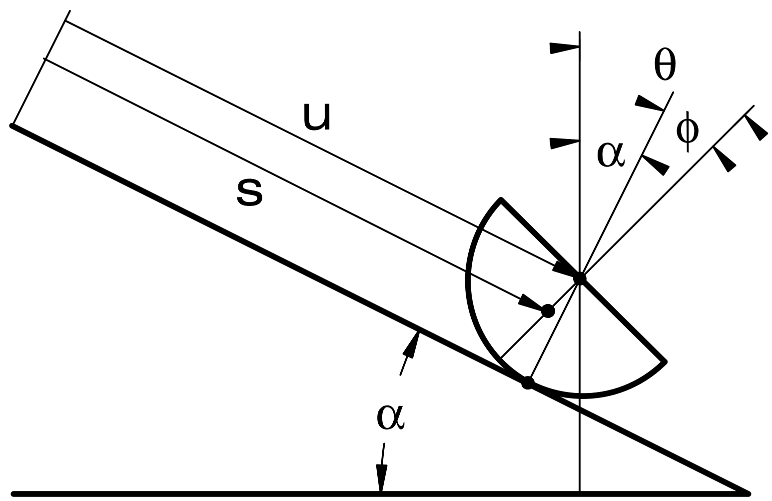

Figure 3: Illustrating the coordinates s, u, ϕ and θ.

The perpendicular distance between the incline and the c.m. is then $h = a - b\cos\phi$. The kinetic energy is the sum of the kinetic energy of the motion of the c.m. plus the energy of rotation about the c.m.:

$$T = \frac{m}{2}(\dot{s}^2 + \dot{h}^2) + \frac{I_{\rm cm}}{2}\dot{\phi}^2 = \frac{m}{2}(\dot{s}^2 + b^2\sin^2\phi\dot{\phi}^2) + \frac{I_{\rm cm}}{2}\dot{\phi}^2. \tag{12}$$

The potential is just $V = mgy$ where y is the vertical coordinate of the c.m. Some care is required to relate y to s and ϕ. I found it useful to introduce another set of coordinates for this: u = distance that the axis of the cylinder has moved parallel to the incline, and θ = angle between the vertical and the line joining the axis of the half cylinder to its c.m.

We see from Fig. 3 that

$$u = s + b\sin\phi, \qquad \text{and} \qquad \theta = \alpha + \phi. \tag{13}$$

Then,

$$y = -u\sin\alpha - b\cos\theta = -s\sin\alpha - b\cos\alpha\cos\phi. \tag{14}$$

Finally, the potential energy is

$$V = -mg(s\sin\alpha + b\cos\alpha\cos\phi). \tag{15}$$

The Lagrange equation of motion deduced from coordinate s is

$$\ddot{s} = g\sin\alpha, \tag{16}$$

as was expected. The equation from coordinate ϕ is

$$(mb^2\sin^2\phi + I_{\rm cm})\ddot{\phi} + mb^2\sin\phi\cos\phi\dot{\phi}^2 = -mbg\cos\alpha\sin\phi. \tag{17}$$

For small oscillations about the equilibrium $\phi = 0$, the above equation simplifies to

$$I_{\rm cm}\ddot{\phi} = -mbg\cos\alpha\,\phi, \tag{18}$$

as found previously in eq. (6) of method 1. Again, the frequency ω of small oscillations is related by

$$\omega^2 = \frac{mbg\cos\alpha}{I_{\rm cm}} = \frac{2bg\cos\alpha}{a^2 - 2b^2}, \tag{19}$$

where $b = 4a/3\pi$.

If the half cylinder started from rest with its surface horizontal, then $\phi_0 = \alpha$ and the oscillations are not small unless α is small.

An analysis can be made using u and θ as the generalized coordinates. Although it is simpler to express the Lagrangian in terms of (u,θ) than (s,ϕ), the latter results in simpler equations of motion since s is the coordinate of the center of mass.

Other aspects of a rolling can have been discussed in [2].

References

[1] M.J. Tiersten, *Moments not to forget — The conditions for equating torque and rate of change of angular momentum around the instantaneous center*, Am. J. Phys. **59**, 733 (1991); **60**, 187(E) (1992),
http://puhep1.princeton.edu/~mcdonald/examples/mechanics/tiersten_ajp_59_733_91.pdf

[2] K.A. Jackson *et al.*, *Viscous and nonviscous models of the partially filled rolling can*, Am. J. Phys. **64**, 277-282 (1996),
http://puhep1.princeton.edu/~mcdonald/examples/mechanics/jackson_ajp_64_277_96.pdf