

The Maximal Energy Attainable in a Betatron

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1 Problem

A betatron is a circular device of radius R designed to accelerate electrons (charge e , mass m) via a changing magnetic flux $\dot{\Phi} = \pi R^2 \dot{B}_{\text{ave}}$ through the circle.

Deduce the relation between the magnetic field B at radius R and the magnetic field B_{ave} averaged over the area of the circle needed for a betatron to function. Also deduce the maximum energy \mathcal{E} to which an electron could be accelerated by a betatron in terms of B , \dot{B}_{ave} and R .

Hints: The electrons in this problem are relativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2$ where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm . Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.

2 Solution

This problem is due to Iwanenko and Pomeranchuk [1].

The electron is held in its circular orbit by the Lorentz force due to the field B . Newton's law, $F = ma$, for this circular motion can be written (in Gaussian units)

$$F = \gamma m a = \frac{\gamma m v^2}{R} = e \frac{v}{c} B. \quad (1)$$

For a relativistic electron, $v \approx c$, so we have

$$\gamma \approx \frac{eRB}{mc^2}. \quad (2)$$

The electron is being accelerated by the electric field that is induced by the changing magnetic flux. Applying the integral form of Faraday's law to the circle of radius R , we have (ignoring the sign)

$$2\pi R E_\phi = \frac{\dot{\Phi}}{c} = \frac{\pi R^2 \dot{B}_{\text{ave}}}{c}, \quad (3)$$

and hence,

$$E_\phi = \frac{R \dot{B}_{\text{ave}}}{2c}, \quad (4)$$

The rate of change of the electron's energy \mathcal{E} due to E_ϕ is

$$\frac{d\mathcal{E}}{dt} = \mathbf{F} \cdot \mathbf{v} \approx ecE_\phi = \frac{eR \dot{B}_{\text{ave}}}{2}, \quad (5)$$

Since $\mathcal{E} = \gamma mc^2$, we can write

$$\dot{\gamma} mc^2 = \frac{eR\dot{B}_{\text{ave}}}{2}, \quad (6)$$

which integrates to

$$\gamma = \frac{eRB_{\text{ave}}}{2mc^2}. \quad (7)$$

Comparing with eq. (2), we find the required condition on the magnetic field:

$$B = \frac{B_{\text{ave}}}{2}. \quad (8)$$

As the electron accelerates it radiates energy at rate given by the Larmor formula in the rest frame of the electron,

$$\frac{d\mathcal{E}^*}{dt^*} = -\frac{2e^2\dot{p}^{*2}}{3c^3} = -\frac{2e^2a^{*2}}{3c^3} \quad (9)$$

Because \mathcal{E} and t are both the time components of 4-vectors their transforms from the rest frame to the lab frame have the same form, and the rate $d\mathcal{E}/dt$ is invariant. However, acceleration at right angles to velocity transforms according to $a^* = \gamma^2 a$. Hence, the rate of radiation in the lab frame is

$$\frac{d\mathcal{E}}{dt} = -\frac{2e^2\gamma^4 a^2}{3c^3} = -\frac{2e^4\gamma^2 B^2}{3m^2 c^3}, \quad (10)$$

using eq. (1) for the acceleration a .

The maximal energy of the electrons in the betatron obtains when the energy loss (10) cancels the energy gain (5), *i.e.*, when

$$\frac{eR\dot{B}_{\text{ave}}}{2} = \frac{2e^4\gamma_{\text{max}}^2 B^2}{3m^2 c^3}, \quad (11)$$

and

$$\gamma_{\text{max}} = \sqrt{\frac{3m^2 c^3 R \dot{B}_{\text{ave}}}{4e^3 B^2}} = \sqrt{\frac{3R \dot{B}_{\text{ave}} B_{\text{crit}}}{4\alpha c B}} \approx \sqrt{\frac{3R B_{\text{crit}}}{4\alpha c \tau B}}, \quad (12)$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, $B_{\text{crit}} = m^2 c^3/e\hbar = 4.4 \times 10^{13}$ G is the so-called QED critical field strength, and τ is the characteristic cycle time of the betatron such that $\dot{B}_{\text{ave}} = B/\tau$. For example, with $R = 1$ m, $\tau = 0.03$ sec (30 Hz), and $B = 10^4$ G, we find that $\gamma_{\text{max}} \approx 200$, or $\mathcal{E}_{\text{max}} \approx 100$ MeV.

We have ignored the radiation due to the longitudinal acceleration of the electron, since in the limiting case this acceleration ceases.

References

- [1] D. Iwanenko and I. Pomeranchuk, *On the Maximal Energy Attainable in a Betatron*, Phys. Rev. **65**, 343 (1944),

http://puhep1.princeton.edu/~mcdonald/examples/accel/iwanenko_pr_65_343_44.pdf