Green's Function for a Conducting Plane with a Hemispherical Boss

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1 Problem

What is the electric potential in cylindrical coordinates $\phi(r, \theta, z)$ when a charge q is located at $(r_0, z_0 > 0)$ and there is a grounded conducting plane at $z = 0$ that has a (conducting) hemispherical boss of radius $a < b = \sqrt{r_0^2 + z_0^2}$ whose center is at the origin. What is the electrostatic force on the charge q for the case that $r_0 = 0$?

2 Solution

We use the image method.

First, we bring the hemispherical boss to zero potential by imagining that a charge $q' = -qa/b$ is placed at distance a^2/b along the line from the origin to charge q. The cylindrical coordinates of charge q' are

$$\frac{a^2}{b^2}(r_0, 0, z_0). \tag{1}$$

Next, to bring the plane $z = 0$ to zero potential, we add images charges for both q and q'. Namely, we imagine charge $q'' = -q$ at

$$(r_0, 0, -z_0), \tag{2}$$

and $q''' = -q' = qa/b$ at

$$\frac{a^2}{b^2}(r_0, 0, -z_0). \tag{3}$$

Then, both the plane $z = 0$ and the spherical shell of radius a about the origin are at zero potential.

The potential at an arbitrary point (r, θ, z) outside the conductor is therefore

$$\phi(r, \theta, z) = \frac{q}{r_1} - \frac{q}{r_2} - \frac{qab}{r_3} + \frac{qab}{r_4}\,, \tag{4}$$

where

$$r_{1,2} = \sqrt{r^2 - 2rr_0\cos\theta + r_0^2 + (z \mp z_0)^2}, \tag{5}$$

and

$$r_{3,4} = \sqrt{b^4r^2 - 2a^2b^2rr_0\cos\theta + a^4r_0^2 + (b^2z \mp a^2z_0)^2}. \tag{6}$$

When $r_0 = 0$, the force on charge q is in the $-z$ direction, with magnitude

$$F = \frac{q^2}{4z_0^2} + \frac{q^2 a/z_0}{(z_0 - a^2/z_0)^2} - \frac{q^2 a/z_0}{(z_0 + a^2/z_0)^2} = \frac{q^2}{4z_0^2} + \frac{4q^2 a^3 z_0^3}{(z_0^4 - a^4)^2}\,. \tag{7}$$

The electric field at the origin in the absence of the boss would be $E_0 = 2q/z_0^2$. With the boss present, the electric potential along the z-axis is

$$\phi(z) = \frac{q}{z_0 - z} - \frac{q}{z_0 + z} - \frac{qaz_0}{z_0 z - a^2} + \frac{qaz_0}{z_0 z + a^2}\,, \tag{8}$$

so the electric field at the pole of the boss, $(0, 0, a)$ is

$$E_z(a) = -\frac{d\phi(a)}{dz} = \frac{2q(2z_0^2 + a^2)}{(z_0^2 - a^2)^2} \approx \frac{4q}{z_0^2} = 2E_0. \tag{9}$$

The charge density at the pole of the boss is twice that at the origin in its absence.