Bunching of Photons When Two Beams Pass Through a Beam Splitter

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

Lijun J. Wang

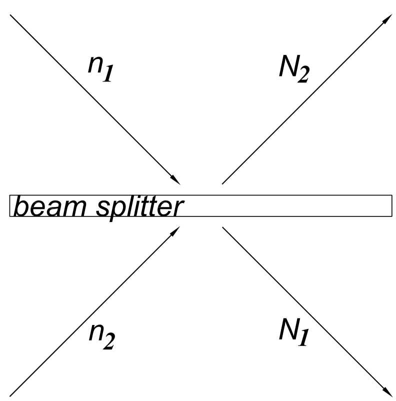
NEC Research Institute, Inc., Princeton, NJ 08540

(Aug. 17, 2003)

1 Problem

Dirac has written [1] "Each photon then interferes only with itself. Interference between two different photons never occurs." Indeed, a practical definition is that "classical" optics consists of phenomena due to the interference of photons only with themselves. However, photons obey Bose statistics which implies a "nonclassical" tendency for them to "bunch".

For a simple example of nonclassical optical behavior, consider two pulses containing n_1 and n_2 photons of a single frequency that are simultaneously incident on two sides of a lossless, 50:50 beam splitter, as shown in the figure. Deduce the probability that N_1 photons are observed in the direction of beam 1, where $0 \le N_1 \le n_1 + n_2$ for a lossless splitter.



Hint: a relatively elementary argument can be given by recalling that the phase of a reflected photon (*i.e.*, of the reflected wave from a single input beam) is 90^0 different from that of a transmitted photon [2]. Consider first the cases that n_1 or n_2 are zero.

2 Solution

An elegant solution can be given by noting the the creation and annihilation operators relevant to a beam splitter obey an SU(2) symmetry [3, 4]. Here, we give a more elementary solution, in the spirit of Feynman [5].

Experimental demonstration of the case where $n_1 = n_2 = 1$ was first given in [6], and the case of $n_1 = n_2 = 2$ has been studied in [7].

2.1 A Single Input Beam

We first consider the case of a single input beam with $n_1 > 0$. Then, of course, $n_2 = 0$.

In a classical view, the input beam would have energy $u_1 = n_1 \hbar\omega$, where ω is the angular frequency of the photons. Then, the effect of the 50:50 beam splitter would be to create output beams of equal energies, $U_1 = U_2 = u_1/2$. In terms of photon numbers, the classical view would imply that the only possibility for the output beams is $N_1 = N_2 = n_1/2$.

But in fact, the transmitted beam can contain any number N_1 of photons between 0 and n_1, while the reflected beam contains $N_2 = n_1 - N_1$ photons.

If the photons were distinguishable, we would assign a probability of $(1/2)^{n_1}$ to each configuration of transmitted and reflected photons in the 50:50 splitter. But the photons are indistinguishable, so that the probability that N_1 out of n_1 photons are transmitted is larger than $(1/2)^{n_1}$ by the number of ways the n_1 photons can be arranged into a group of N_1 transmitted and $n_1 - N_1$ reflected photons without regard to their order, *i.e.*, by the binomial coefficient,

$$C^{n_1}_{N_1} = \frac{n_1!}{N_1!(n_1 - N_1)!}. \tag{1}$$

Thus, the probability $P(N_1, n_1 - N_1|n_1, 0)$ that N_1 out of n_1 photons (in a single input beam) are transmitted by the beam splitter is

$$P(N_1, n_1 - N_1|n_1, 0) = C^{n_1}_{N_1} \left(\frac{1}{2}\right)^{n_1}. \tag{2}$$

The result (2) is already very nonclassical, in that there is a small, but nonzero probability that the entire input beam is transmitted, or reflected. However, in the limit of large n_1 the largest probability is that the numbers of photons in the reflected and transmitted beams are very nearly equal. We confirm this by use of Stirling's approximation for large n,

$$n! \approx e^{-n} n^n \sqrt{2\pi n}. \tag{3}$$

For large n, and $k = (1 + \epsilon)n/2$, we have

$$\begin{aligned} C^n_k &\approx \frac{1}{\sqrt{2\pi n}\left(\frac{k}{n}\right)^{k+1/2}\left(1 - \frac{k}{n}\right)^{n-k+1/2}} = \frac{2^{n+1}}{\sqrt{2\pi n}(1-\epsilon^2)^{(n+1)/2}\left(\frac{1+\epsilon}{1-\epsilon}\right)^{n\epsilon/2}} \\ &\approx \frac{2^{n+1}}{\sqrt{2\pi n}(1 + n\epsilon^2/2)}. \end{aligned} \tag{4}$$

The probability of k photons out of n being transmitted drops to 1/2 the peak probability when $\epsilon \approx \sqrt{2/n}$. Hence, for large n the number distribution of photons in the transmitted (and reflected) beam is essentially a delta function centered at $n/2$, in agreement with the classical view.

The most dramatic difference between the classical and quantum behavior of a single beam in a 50:50 beam splitter occurs when $n_1 = 2$,

$$P(0, 2|2, 0) = \frac{1}{4}, \qquad P(1, 1|2, 0) = \frac{1}{2}, \qquad P(2, 0|2, 0) = \frac{1}{4}. \tag{5}$$

In the subsequent analysis we shall need to consider interference effects, so we note that the magnitude of the probability amplitude that k out of n photons in a single beam are transmitted by a 50:50 beam splitter can obtained by taking the square root of eq. (2),

$$|A(k, n-k|n, 0)| = \sqrt{C_k^n}\left(\frac{1}{2}\right)^{n/2}. \tag{6}$$

These amplitudes have the obvious symmetries,

$$|A(k, n-k|n, 0)| = |A(n-k, k|n, 0)| = |A(k, n-k|0, n)| = |A(n-k, k|0, n)|. \tag{7}$$

We must also consider the phases of these amplitudes, or at least the relative phases. The hint is that we may consider the phase of a reflected photon to be shifted with respect to that of a transmitted photon by 90°, as follows from a classical analysis of waves in a 50:50 beam splitter [2] (see also the Appendix). In this problem, we define the phase of a transmitted photon to be zero, so that the probability amplitude should include a factor of $i = \sqrt{-1}$ for each reflected photon. Thus, we have

$$A(k, n-k|n, 0) = i^{n-k}\sqrt{C_k^n}\left(\frac{1}{2}\right)^{n/2}, \tag{8}$$

$$A(n-k, k|n, 0) = i^{k}\sqrt{C_k^n}\left(\frac{1}{2}\right)^{n/2}, \tag{9}$$

$$A(k, n-k|0, n) = i^{k}\sqrt{C_k^n}\left(\frac{1}{2}\right)^{n/2}, \tag{10}$$

$$A(n-k, k|0, n) = i^{n-k}\sqrt{C_k^n}\left(\frac{1}{2}\right)^{n/2}. \tag{11}$$

2.2 Two Input Beams

We now calculate the general probability $P(N_1, n_1+n_2-N_1|n_1, n_2)$ that N_1 output photons are observed along the direction of input beam 1 when the number of photons in the input beams in n_1 and n_2.

We first give a classical wave analysis. The input waves have amplitudes $a_{1,2} = \sqrt{n_{1,2}\hbar\omega}$, and are in phase at the center of the beam splitter. The output amplitudes are the sums of the reflected and transmitted parts of the input amplitudes. A reflected amplitude has a phase shift of 90° relative to its corresponding transmitted amplitude, as discussed in sec. 2.1. In the 50:50 beam splitter, the magnitude of both the reflected and transmitted amplitudes from a single input beam are $1/\sqrt{2}$ times the magnitude of the amplitude of that beam. Hence, the output amplitudes are

$$A_1 = \frac{1}{\sqrt{2}}(a_1 + ia_2), \tag{12}$$

$$A_2 = \frac{1}{\sqrt{2}}(ia_1 + a_2). \tag{13}$$

Taking the absolute square of eqs. (12)-(13), we find the output beams to be described by

$$N_{1,2} = \frac{|A_{1,2}|^2}{\hbar\omega} = \frac{a_1^2 + a_2^2}{2\hbar\omega} = \frac{n_1 + n_2}{2}. \tag{14}$$

The classical view is that a 50:50 beam splitter simply splits both input beams, when they are in phase.

For a quantum analysis, we proceed by noting that of the N_1 photons in output beam 1, k of these could have come by transmission from input beam 1, and $N_1 - k$ by reflection from input beam 2 (so long as $N_1 - k \leq n_2$). The probability amplitude that k out of N_1 photons are transmitted from beam 1 while $N_1 - k$ photons are reflected from beam 2 is, to within a phase factor, the product of the amplitudes for each of these configurations resulting from a single input beam:

$$A(k, N_1 - k|n_1, 0)A(N_1 - k, n_2 - N_1 + k|0, n_2) = (-1)^{n_1 - k}\sqrt{C_k^{n_1} C_{N_1-k}^{n_2}} \left(\frac{1}{2}\right)^{(n_1+n_2)/2}, \quad (15)$$

referring to eqs. (8)-(11). The most dramatic nonclassical features to be found below can be attributed to the presence of the factor $(-1)^{n_1-k}$ that arises from the 90° phase shift between reflected and transmitted photons.

Since photons obey Bose statistics, we sum the sub-amplitudes (15), weighting each one by the square root of the number of ways that k out of the N_1 photons in the first output beam can be assigned to input beam 1, namely $C_k^{N_1}$, time the square root of the number of ways that the remaining $n_1 - k$ photons from input beam 1 can be assigned to the N_2 photons in output beam 2, namely $C_{n_1-k}^{N_2}$ to obtain[1]

$$\begin{aligned} A(N_1, n_1 + n_2 - N_1|n_1, n_2) &= \sum_k \sqrt{C_k^{N_1} C_{n_1-k}^{N_2}} A(k, N_1 - k|n_1, 0)A(N_1 - k, n_2 - N_1 + k|0, n_2) \\ &= (-1)^{n_1} \left(\frac{1}{2}\right)^{(n_1+n_2)/2} \sum_k (-1)^k \sqrt{C_k^{n_1} C_{N_1-k}^{n_2} C_k^{N_1} C_{n_1-k}^{N_2}}. \quad (16) \end{aligned}$$

When evaluating this expression, any binomial coefficient C_m^n in which m is negative, or greater than n, should be set to zero.

The desired probability is, of course,

$$P(N_1, n_1 + n_2 - N_1|n_1, n_2) = |A(N_1, n_1 + n_2 - N_1|n_1, n_2)|^2 \quad (17)$$

Some examples of the probability distributions for small numbers of input photons are given below.

2.2.1 Two Input Photons

Input $\|n_1, n_2\rangle$	Output $\langle N_1, N_2\|$ $\langle 0,2\|$	$\langle 1,1\|$	$\langle 2,0\|$
$\|2,0\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\|1,1\rangle$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\|0,2\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

[1] Delicate to justify not also including factors $C_{N_1-k}^{N_1}$, and $C_{N_2-(n_1-k)}^{N_2}$, these being the ways of assigning photons to output beam 2 – but these factors are the same as those already included, and so should not be counted twice...

When n_1 or n_2 is zero, the probability distribution is binomial, as found in sec. 2.1. When $n_1 = n_2 = 1$ there is complete destructive interference between the cases where both photons are reflected (combined phase shift = 180°) and when both are transmitted (combined phase shift = 0). This quantum result is strikingly different from the classical expectation that there would be one photon in each output beam.

2.2.2 Three Input Photons

Input	Output $(N_1, N_2\|$			
$\|n_1, n_2)$	$(0,3\|$	$(1,2\|$	$(2,1\|$	$(3,0\|$
$\|3,0)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$\|2,1)$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\|1,2)$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\|0,3)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

2.2.3 Four Input Photons

Input	Output $(N_1, N_2\|$				
$\|n_1, n_2)$	$(0,4\|$	$(1,4\|$	$(2,2\|$	$(3,1\|$	$(4,0\|$
$\|4,0)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
$\|3,1)$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$\|2,2)$	$\frac{3}{8}$	0	$\frac{1}{4}$	0	$\frac{3}{8}$
$\|1,3)$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$\|0,4)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

2.2.4 Symmetric Input Beams: $n_1 = n_2 \equiv n$

There is zero probability of observing an odd number of photons in either output beam.

To see this, note that when $n_1 = n_2 = n$, the magnitudes of the subamplitudes are equal for having k photons appearing in output beam 1 from either input beam 1 or input beam 2. However, the phases of these two subamplitudes are 180° apart, so that they cancel. In particular, when k photons are transmitted into output beam 1 from input beam 1, then $N_1 - k$ photons are reflected from input beam 2 into output beam 1; meanwhile, $n - k$ photons are reflected from input beam 1 into output beam 2. So the overall phase factor of this subamplitude is $i^{N_1-k+n-k} = (-1)^k\, i^{n+N_1}$. Whereas, if k photons are reflected from input beam 2 into output beam 1, then $N_1 - k$ photons are transmitted from input beam 1 into output beam 1, and so $n - N_1 + k$ photons are reflected from input beam 1 into output beam 2. So the overall phase factor of this subamplitude is $i^{k+n-N_1+k} = (-1)^k\, i^{n-N_1}$. The phase factor between these two subamplitudes (whose magnitudes are equal) is $i^{2N_1} = (-1)^{N_1}$, which is -1 for odd N_1, as claimed.

For the case of observing an even number of photons in the output beams, a remarkable simplification of eq. (16) holds [3]. *I have not been able to show this by elementary means. It does follow by inspection when $m = 0$ or n, in which case eq. (16) contains only a single nonzero term. In general, the index k in eq. (16) for $A(2m, 2n-2m|n, n)$ runs from 0 to $2m$ if $2m \leq n$, or from $2m - n$ to n if $2m \geq n$. There are an odd number of terms, the central one having index $k = m$. By a strange miracle of combinatorics, the sum collapses to a simplified version of the central term of the series....* Namely,

$$A(2m, 2n-2m|n, n) = (-1)^{n-m} \left(\frac{1}{2}\right)^n \sqrt{C_m^{2m} C_{n-m}^{2n-2m}}. \tag{18}$$

Therefore, the $n+1$ nonvanishing probabilities for symmetric input beams are

$$P(2m, 2n-2m|n, n) = \left(\frac{1}{2}\right)^{2n} C_m^{2m} C_{n-m}^{2n-2m} \approx \frac{1}{n\pi\sqrt{\frac{m}{n}(1-\frac{m}{n})}}, \tag{19}$$

where the approximation holds for large m and large n. Note that $\int_0^1 dx/\sqrt{x(1-x)} = \pi$. This probability distribution peaks for $m = 0$ or n, *i.e.*, for all photons in one or the other output beam, with value

$$P(0, 2n|n, n) = P(2n, 0|n, n) = \left(\frac{1}{2}\right)^{2n} C_n^{2n}. \tag{20}$$

The probability of finding all output photons in a single beam when the input beams are symmetric is larger by a factor C_n^{2n} than when there is only a single input beam (of the same total number of photons), because there are C_n^{2n} ways of assigning the n photons from input beam 1 to the $2n$ photons in the output beam. This is an extreme example of photon bunching caused by the beam splitter.

It is noteworthy that the result (19) does not agree with the classical prediction (14) in the large n limit.

Of course, as pointed out by Glauber [8], a classical wave corresponds to a photon state with minimum uncertainty products $\Delta E \Delta B$, where E and B are the electric and magnetic field amplitudes of the wave, respectively. In case of a pulse, we expect classically that both its energy U and phase ϕ are well defined, but the closest quantum equivalent is a coherent state with minimal uncertainty to the product $\Delta U \Delta \phi$. This state is a superposition of states of various photons numbers n whose expectation value for n follows a Poisson distribution with $\langle n \rangle = U/\hbar\omega$. For large n, the variance in photon number is $\sqrt{n}$.

Hence, in an experiment in which large numbers N_1 and N_2 of photons are observed at the two output ports of the beam splitter, we can say that the numbers n_1 and n_2 of photons at the input ports obeyed $n_1 + n_2 = N_1 + N_2$, but we cannot know n_1 and n_2 separately (if the inputs beams are "classical"). All we can know are the mean values $\langle n_1 \rangle$ and $\langle n_2 \rangle$. Therefore, we should rewrite the probability distribution (17) as

$$P(N_1, N_2| \langle n_1 \rangle, \langle n_2 \rangle) = \left| \sum_{n_1, n_2} a_{n_1} a_{n_2} A(N_1, N_2|n_1, n_2) \right|^2, \tag{21}$$

where a_{n_i} is the amplitude that input beam i contained n_i photons when the mean number of photons in this beam is $\langle n_i \rangle$. I conjecture that a detailed calculation of eq. (21) would agree with the classical prediction (14), but I have not confirmed this.

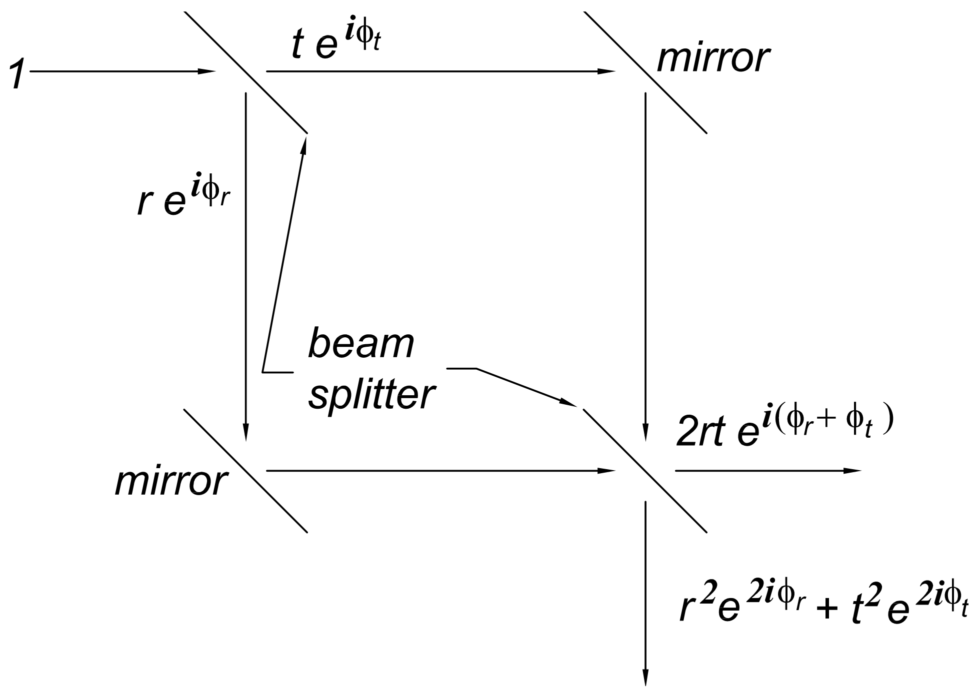
3 Appendix: Phase Shift in a Lossless Beam Splitter

We give a classical argument based on a Mach-Zehnder interferometer, shown in the figure below, that there is a 90° phase shift between the reflected and transmitted beams in a lossless, symmetric beam splitter. Then, following Dirac's dictum, we suppose that this result applies to a single photon.

A beam of light of unit amplitude is incident on the interferometer from the upper left. The reflected and transmitted amplitudes are $re^{i\phi_r}$ and $te^{i\phi_t}$, where magnitudes r and t are real numbers. The condition of a lossless beam splitter is that

$$r^2 + t^2 = 1. \tag{22}$$

The reflected and transmitted beams are reflected off mirrors and recombined in a second lossless beam splitter, identical to the first.



Then, the amplitude for transmission at the first beam splitter, followed by reflection at the second, is $tre^{i(\phi_t+\phi_r)}$, *etc.* Hence, the recombined beam that moves to the right has amplitude

$$A_1 = 2rte^{i(\phi_r+\phi_t)}, \tag{23}$$

while the recombined beam that moves downwards has amplitude

$$A_2 = r^2e^{2i\phi_r} + t^2e^{2i\phi_t}. \tag{24}$$

The intensity of the first output beam is

$$I_1 = |A_1|^2 = 4r^2t^2, \tag{25}$$

and that of the second output beam is

$$I_2 = |A_2|^2 = r^4 + t^4 + 2r^2t^2\cos 2(\phi_t - \phi_r). \tag{26}$$

For lossless splitters, the total output intensity must be unity,

$$I_1 + I_2 = 1 = (r^2 + t^2)^2 + 2r^2t^2[1 + \cos 2(\phi_t - \phi_r)]. \tag{27}$$

Recalling eq. (22), we must have

$$\phi_t - \phi_r = \pm 90^\circ, \tag{28}$$

for any value of the splitting ratio $r^2 : t^2$.

The preceding argument does not clarify where that phase difference (28) is 90° or −90°, but more detailed arguments [2] show the phase difference to be −90°. That is,

$$\phi_r = \phi_t + 90^\circ. \tag{29}$$

Furthermore, if the beam splitter is thin compared to a wavelength, then $\phi_t \approx 0$ and $\phi_r \approx 90^\circ$.

4 References

[1] P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon Press, London, 1958), p. 9.

[2] See, for example, Problem 4 of the Princeton Ph501 problem set 6 at
http://puhep1.princeton.edu/~mcdonald/examples/ph501set6.pdf
An argument very similar to that in the Appendix is given by V. Degiorgio, *Phase shift between the reflected and transmitted optical fields of a semireflecting lossless mirror is* $\pi/2$, Am. J. Phys. **48**, 81 (1980),
http://puhep1.princeton.edu/~mcdonald/examples/optics/degiorgio_ajp_48_81_80.pdf
and also by Z.Y. Ou and L. Mandel, *Derivation of reciprocity relations for a beam splitter from energy balance*, Am. J. Phys. **57**, 66 (1989),
http://puhep1.princeton.edu/~mcdonald/examples/optics/ou_ajp_57_66_89.pdf
Early discussions of phase shifts in beam splitters include G.B. Airy, *On the Phenomenon of Newton's Rings when formed between two transparent Substances of different refactive Powers*, Phil. Mag. **2**, 20 (1833),
http://puhep1.princeton.edu/~mcdonald/examples/optics/airy_pm_2_30_33.pdf
G.G. Stokes, *On the Perfect Blackness of the Central Spot in Newton's Rings, and on the Verification of Fresnel's Formulæ for the Intensities of Reflected and Refracted Rays*, Cambridge and Dublin Math. J. **4**, 1 (1849), reprinted in *Mathematical and Physical Papers of G.G. Stokes*, V. 2 (Cambridge U. Press, Cambridge, 1883), pp. 89-103,
http://puhep1.princeton.edu/~mcdonald/examples/optics/stokes_cdmj_4_1_49.pdf
See also sec. 1.6.4 of M. Born and E.Wolf, *Principles of Optics*, 7th ed. (Cambridge U. Press, Cambridge, 1999).

[3] R.A. Campos, B.E.A. Saleh and M.C. Teich, *Quantum-mechanical lossless beam splitter: SU(2) symmetry and photon statistics*, Phys. Rev. A **40**, 1371 (1989), http://puhep1.princeton.edu/~mcdonald/examples/QM/campos_pra_40_1371_89.pdf

[4] M.G.A. Paris, *Homodyne Photocurrent, Symmetries in Photon Mixing and Number State Synthesis*, Int. J. Mod. Phys. B **11**, 1913 (1997), http://puhep1.princeton.edu/~mcdonald/examples/optics/paris_ijmp_b11_1913_97.pdf

[5] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison Wesley, Reading, MA, 1965), Vol. III, secs. 4.2-3.

[6] C.K. Hong, Z.Y. Ou and L. Mandel, *Measurement of Subpicosecond Time Intervals between Two Photons by Interference*, Phys. Rev. Lett. **59**, 2044 (1987), http://puhep1.princeton.edu/~mcdonald/examples/optics/hong_prl_59_2044_87.pdf

[7] Z.Y. Ou, J.-K. Rhee and L.J. Wang, *Observation of Four-Photon Interference with a Beam Splitter by Pulsed Parametric Down-Conversion*, Phys. Rev. Lett. **83**, 959 (1999), http://puhep1.princeton.edu/~mcdonald/examples/optics/ou_prl_83_959_99.pdf
Photon bunching and multiphoton interference in parametric down-conversion, Phys. Rev. A **60**, 593 (1999), http://puhep1.princeton.edu/~mcdonald/examples/optics/ou_pra_60_593_99.pdf

[8] R.J. Glauber, *Photon Correlations*, http://puhep1.princeton.edu/~mcdonald/examples/QM/glauber_prl_10_84_63.pdf
The Quantum Theory of Optical Coherence, Phys. Rev. **130**, 2529 (1963), http://puhep1.princeton.edu/~mcdonald/examples/QM/glauber_pr_130_2529_63.pdf
Coherent and Incoherent States of the Radiation Field, http://puhep1.princeton.edu/~mcdonald/examples/QM/glauber_pr_131_2766_63.pdf