Reactance of Small Antennas

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1 Problem

Estimate the capacitance and inductance of a short, center-fed, linear dipole antenna whose arms each have length h and radius a. Also estimate the inductance of a small loop antenna of major radius b and minor radius a.

For completeness, consider also the real part, its so-called radiation resistance, of the antenna impedance in the approximation of perfect conductors.

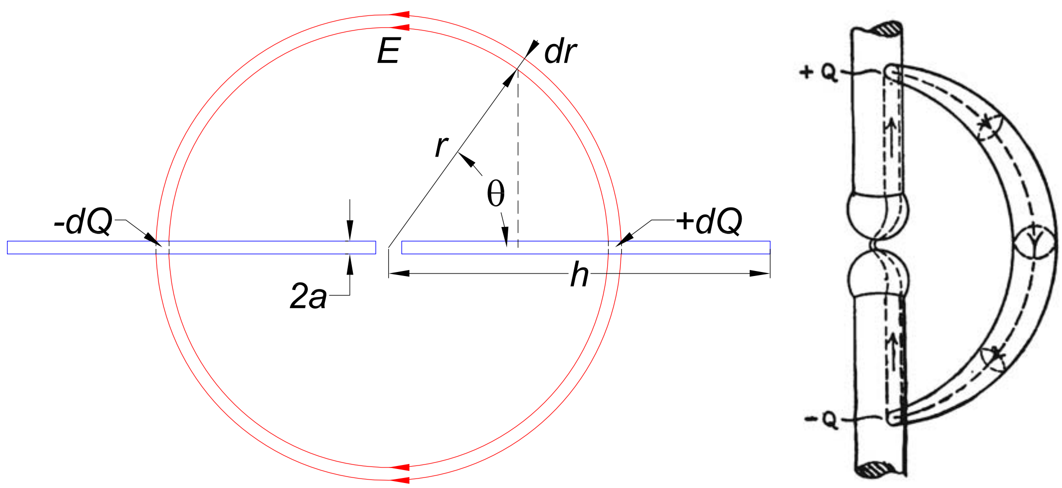
2 Solution

2.1 Short, Center-Fed, Linear Dipole Antenna

This solution follows sec. 10.3 of [1].

2.1.1 Capacitance

The key assumption is that the electric field lines from one arm of the dipole antenna to the other follow semicircular paths (the principal mode), as shown in the figure below.[1]



If so, all the field lines emanating from charge dQ in interval dr at distance r from the center of the antenna cross a surface of area $2\pi r\, dr\, \sin\theta$ that lies on a cone of half angle θ, so the electric field strength at (r, θ) is

$$E = \frac{dQ/dr}{2\pi\epsilon_0 r \sin\theta}\,. \tag{1}$$

[1] On the right is Fig. 86 from [2].

The voltage difference between the two arms of the antenna is[2]

$$\Delta V = 2\int_{\theta_{\min}}^{\pi/2} Er\, d\theta = \frac{dQ/dr}{\pi\epsilon_0}\int_{a/r}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{dQ/dr}{\pi\epsilon_0}\ln[\tan(\theta/2)]_{a/r}^{\pi/2} = \frac{dQ/dr}{\pi\epsilon_0}\ln(2r/a). \quad (2)$$

This voltage difference should be independent of position along the antenna.[3] The charge distribution dQ/dr is indeed constant to a good approximation for short dipole antennas, but the factor $\ln(2r/a) = -\ln(\theta_{\min}/2)$ is constant only for a biconical dipole antenna (as much favored theoretically by Schelkunoff). A reasonable approximation for a linear dipole antenna is to use $r = h/2$ as a representative length in eq. (2), which leads to the estimate

$$\Delta V \approx \frac{dQ/dr}{\pi\epsilon_0}\ln(h/a). \quad (3)$$

The corresponding capacitance per unit length along the antenna is

$$\frac{dC}{dr} \approx \frac{\pi\epsilon_0}{\ln(h/a)}\,, \quad (4)$$

and the total capacitance is

$$C \approx \frac{\pi\epsilon_0 h}{\ln(h/a)}\,. \quad (5)$$

This estimate ignores the contribution to the capacitance of roughly $\pi\epsilon_0 a^2/d$ associated with the electric field in the gap d between the terminals of the antenna, as is reasonable when $d \approx a$ since then $\ln(h/a) \ll h/a \approx dh/a^2$.

2.1.2 Inductance

For a quick estimate of the inductance of the antenna we note when the arms carry current I the magnetic field near the conductors varies with distance as

$$B \approx \frac{\mu_0 I}{2\pi r}\,. \quad (6)$$

The magnetic flux associated with the linear antenna is

$$\Phi = LI \approx h\int_a^h B\, dr \approx \frac{\mu_0 h I}{2\pi}\ln\frac{h}{a}\,, \quad (7)$$

where we note that the current drops from I to 0 over length h of each arm. Then, our rough estimate of the inductance L is

$$L \approx \frac{\mu_0 h}{2\pi}\ln\frac{h}{a}\,. \quad (8)$$

[2]In general the electric field is related to the scalar and vector potentials by $\mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t = -\nabla V - i\omega\mathbf{A}$, assuming a time dependence of the form $e^{i\omega t}$. Then, $\int_1^2 \mathbf{E}\cdot d\mathbf{l} = V_1 - V_2 - i\omega\int_1^2 \mathbf{A}\cdot d\mathbf{l}$. However, close to a small linear dipole antenna the electric field is much larger than the magnetic field (see, for example, [3]), and the contribution of the vector potential to the electric field in negligible in this region.

[3]The vanishing of the tangential component of the electric field along the (ideal) conductor implies that this conductor is an equipotential only if the vector potential can be neglected. For examples where this does not hold, see [4, 5].

2.1.3 Reactance

The reactance of a short linear antenna ($h \ll \lambda$) is largely due to its capacitance,

$$X_{\rm small\ linear} = \omega L - \frac{1}{\omega C} \approx -\frac{1}{ckC} \approx -\frac{\ln(h/a)}{\pi\epsilon_0 ckh} = -\frac{Z_0}{\pi}\frac{\ln(h/a)}{kh} = -\frac{Z_0}{\pi^2}\frac{\lambda}{2h}\ln(h/a), \qquad (9)$$

where ($c = 1/\sqrt{\epsilon_0\,\mu_0}$ being the speed of light in vacuum)

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0\, c = \frac{1}{\epsilon_0\, c} = 377\ \Omega. \qquad (10)$$

The reactance (9) falls with increasing length h of the arms of the antenna, and vanishes when

$$\omega = \frac{1}{\sqrt{LC}} \approx \sqrt{\frac{2\pi}{\mu_0 h}\frac{1}{\pi\epsilon_0 h}} = \sqrt{2}\frac{c}{h} = kc = \frac{2\pi c}{\lambda}, \qquad i.e., \qquad h \approx \frac{\sqrt{2}\lambda}{2\pi} = \frac{\lambda}{4.44}. \qquad (11)$$

Thus, the rough estimates (5) and (8) of C and L for a linear antenna give a fairly good prediction of "resonance" when half-length $h \approx \lambda/4$.

2.1.4 Relation between Reactance and "Free Oscillation"

As an aside, we note that frequencies at which the terminal reactance vanishes correspond to those of "free oscillation of the antenna (with its terminals shorted).

In a "free oscillation"[4] radiation is ignored and the (near) fields are standing waves that obey the Helmholtz equation, $(\nabla^2 + k^2)\psi = 0$, where ψ is any scalar component of the electric and magnetic fields. Electromagnetic energy is stored in the (near) fields, which oscillates between "electric" and "magnetic" terms, there being no exchange of energy with the perfect conductor.

For driven oscillations of a conductor, a nonzero terminal reactance implies an exchange of energy between the energy/voltage source and the (near) electromagnetic fields.

Thus, if the reactance in nonzero at some frequency, that frequency cannot correspond to a "free oscillation" (for which there is no exchange of energy between fields and conductors).

2.2 Small Loop Antenna

2.2.1 Inductance

One definition of a small loop antenna is that the spatial variation of the current around the loop can be neglected. In this case the inductance is essentially that of a circular loop/torus of, say, major radius b and minor radius a, supposing that all the current in on the surface because of the skin effect.

For a quick estimate we note when the loop carries current I the magnetic field near the conductor varies with distance as

$$B \approx \frac{\mu_0 I}{2\pi r}, \qquad (12)$$

[4] "Free oscillations" of (perfect) conductors were perhaps first discussed in [6]. See also, [7].

so the magnetic flux linked by the loop is

$$\Phi = LI \approx 2\pi b \int_a^b B\, dr \approx \mu_0 b I \ln \frac{b}{a}\,, \tag{13}$$

and the inductance L is

$$L \approx \mu_0 b \ln \frac{b}{a} = \mu_0 b \left(\ln \frac{8b}{a} - 2.08 \right). \tag{14}$$

A more exact calculation using toroidal coordinates [8] shows that the number $2.08 = \ln 8$ in eq. (14) is actually 2 when $b \gg a$.

2.2.2 Capacitance

A loop antenna has a small capacitance C associated with the gap between its terminal. However, the capacitive reactance $1/i\omega C$ is negligible in practice, so we skip estimating the capacitance C.[5]

2.2.3 Reactance

The reactance of a small loop antenna is essentially that due to its inductance,

$$X_{\text{small loop}} \approx \omega L \approx \mu_0 \omega b \ln \frac{b}{a} = \mu_0 c k b \ln \frac{b}{a} = Z_0 \frac{2\pi b}{\lambda} \ln \frac{b}{a}\,. \tag{15}$$

A Appendix: Radiation Resistance of Small Antennas

For completeness, we include the well-known calculations of the radiation resistance R_{rad} of small antennas, noting that the time-average radiated power P is related to the peak current I_0 at the antenna terminals by

$$P = \frac{I_0^2 R_{\text{rad}}}{2} = \frac{\mu_0 \left|\ddot{p}\right|^2}{12\pi c} = \frac{\mu_0\, \omega^4 \left|p_0\right|^2}{12\pi c}\,, \qquad i.e., \qquad R_{\text{rad}} = \frac{\mu_0\, \omega^4 \left|p_0\right|^2}{6\pi c I_0^2}\,, \tag{16}$$

where p_0 is the peak electric dipole moment of the antenna (or $p_0 = m_0/c$ in case the antenna has magnetic dipole moment m).

A.1 Short Linear Antenna

A linear antenna (of half length h, along the z-axis) has electric dipole moment p related to its linear charge density ρ by

$$p = \int_{-h}^{h} \rho z\, dz. \tag{17}$$

The charge density is related to the current distribution

$$I(z,t) \approx I_0 (1 - |z|/h)\, e^{i\omega t}, \tag{18}$$

[5]An estimate of the terminal capacitance is given in [9].

by the continuity equation,

$$\dot{\rho} = -\frac{dI}{dz} \approx \pm \frac{I_0}{h}\, e^{i\omega t}, \tag{19}$$

so that

$$\rho = \mp \frac{iI_0}{\omega h}\, e^{i\omega t}, \tag{20}$$

and

$$p_0 = -\frac{iI_0 h}{\omega} \tag{21}$$

from eq. (17). Then, according to eq. (16) the radiation resistance is, recalling eq. (10),

$$R_{\rm rad} = \frac{\mu_0\, \omega^4 \left|p_0\right|^2}{6\pi c I_0^2} = \frac{\mu_0\, \omega^2 h^2}{6\pi c} = \frac{\pi \mu_0\, c\, (2h)^2}{6} = \frac{\pi Z_0}{6} \frac{(2h)^2}{\lambda^2}\,. \tag{22}$$

The approximation (19) is not very accurate for "resonance" with $h \approx \lambda/4$, for which eq. (22) gives $R_{\rm rad,resonance} \approx 377\pi/24 = 49\ \Omega$ rather than 71 Ω.

For an ("unmatched") small linear antenna with terminal impedance $Z \approx iX$ and reactance X given by eq. (9), the time-average radiated power when driven by a voltage source V_0 is, noting that $I_0 = |V_0/Z|$,

$$P_{\rm linear,unmatched} = \frac{V_0^2 R_{\rm rad}}{2\left|Z\right|^2} \approx \frac{V_0^2 R_{\rm rad}}{2X^2} \approx \frac{\pi^5 V_0^2}{12 Z_0 \ln^2(h/a)} \frac{(2h)^4}{\lambda^4}\,. \tag{23}$$

If the small linear antenna is "matched" to a line of (real) impedance $Z_{\rm line}$ ($\gg R_{\rm rad}$) then

$$P_{\rm linear,matched} = \frac{V_0^2 R_{\rm rad}}{2Z_{\rm line}^2} \approx \frac{\pi V_0^2 Z_0}{12 Z_{\rm line}^2} \frac{(2h)^2}{\lambda^2}\,. \tag{24}$$

A.2 Small Loop Antenna

A small loop antenna (of radius b) has azimuthally symmetric current $I(\phi, t) = I_0\, e^{-i\omega t}$, such that the peak magnetic dipole moment is

$$m_0 = \pi b^2 I_0, \tag{25}$$

and radiation resistance

$$R_{\rm rad} = \frac{\mu_0 \omega^4 \left|m_0\right|^2}{6\pi c^3 I_0^2} = \frac{\pi \mu_0 \omega^4 b^4}{6c^3} = \frac{\pi \mu_0 c}{6} \frac{(2\pi b)^4}{\lambda^4} = \frac{\pi Z_0}{6} \frac{(2\pi b)^4}{\lambda^4}\,. \tag{26}$$

For an ("unmatched") small loop antenna with reactance X given by eq. (15) the time-average radiated power when driven by a voltage source V_0 is

$$P_{\rm loop,unmatched} \approx \frac{V_0^2 R_{\rm rad}}{2X^2} \approx \frac{\pi V_0^2}{12 Z_0 \ln^2(b/a)} \frac{(2\pi b)^2}{\lambda^2}\,. \tag{27}$$

If the small loop antenna is "matched" to a line of impedance $Z_{\rm line}$ *then*

$$P_{\rm loop,matched} = \frac{V_0^2 R_{\rm rad}}{2Z_{\rm line}^2} \approx \frac{\pi V_0^2 Z_0}{12 Z_{\rm line}^2} \frac{(2\pi b)^4}{\lambda^4}\,. \qquad (28)$$

Thus, a "matched," small loop antenna of circumference $2\pi b$ *radiates much less power than a "matched," small linear antenna of total length* $2h = 2\pi b$.[6]

References

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[9] K.T. McDonald, *Radiation by an AC Voltage Source* (Jan. 9, 2005),
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[6] An "unmatched," small loop antenna of circumference $2\pi b$ radiates more power than an "unmatched," small linear antenna of total length $2h = 2\pi b$ provided $2\pi b \lesssim \lambda/10$, but the radiated power is quite small.