Electromagnetic Momentum of a Capacitor in a Uniform Magnetic Field

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1 Problem

Analytic calculations of the electric field of a parallel-plate capacitor are notoriously difficult [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The usual goal of such efforts is a calculation of the correction to the hard-edge approximation, $C = A/4\pi d$ (in Gaussian units), of the capacitance for plates of area A separated by distance d.

Calculate instead the electromagnetic momentum of the parallel-plate capacitor if it resides in a uniform magnetic field that is parallel to the capacitor plates.

Consider also the case of a capacitor whose electrodes are caps of polar angle $\theta_0 < \pi/2$ on a sphere of radius a.

In both cases, the remaining space is vacuum.

2 Solution

The principal result of this exercise is that the electromagnetic momentum of a parallel-plate capacitor of central field $\mathbf{E}_0$ in a uniform magnetic field $\mathbf{B}_0$ is only 1/2 of the naive estimate of $\mathbf{E}_0 \times \mathbf{B}_0$ times (Volume/$4\pi c$).[1] Furthermore, if the electromagnetic momentum is interpreted as being stored in the electromagnetic fields, then only 2/3 of the electromagnetic momentum is localized near the capacitor, while the remaining 1/3 is localized near the coils of the magnet, if there is no shielding of the fringe field of the capacitor.

2.1 Electromagnetic Momentum

For systems in which effects of radiation and of retardation can be ignored, the electromagnetic momentum can be calculated in various equivalent ways [15],[2]

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\Phi \mathbf{J}}{c^2}\, d{\rm Vol}, \tag{1}$$

where ϱ is the electric charge density, $\mathbf{A}$ is the magnetic vector potential (in the Coulomb gauge where $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$), $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, Φ is the electric (scalar) potential, $\mathbf{J}$ is the electric current density, and c is the speed of light. The first

[1] A previous discussion [14] of the electromagnetic momentum of a capacitor in an electric field missed the factor of 1/2.

[2] The third form of eq. (1) indicates that if the magnetic field is created by steady currents in a good/super conductor, over which the scalar potential Φ is constant, then $\mathbf{P}_{\rm EM} = (\Phi/c^2)\int \mathbf{J}\, d{\rm Vol} = 0$. Hence, we restrict our attention to magnetic fields create by structures that need not be equipotentials, such as permanent magnets made from ferromagnetic grains embedded in a nonconducting matrix.

form is due to Faraday [16] and Maxwell [17], the second form is due to Poynting [19] and Abraham [18], and the third form was introduced by Furry [20].

The three forms of eq. (1) lead to ambiguities of interpretation as to where the electromagnetic momentum is located. The first form, which we recognize as the electromagnetic part of the canonical momentum of charges in an electromagnetic field, suggests that the electromagnetic momentum is a property of the charges in the system, even if they are at rest. The second form suggests that the electromagnetic momentum is stored in the electromagnetic fields, while the third form suggests that the electromagnetic momentum is associated with the charges, but only if they are moving.

This ambiguity suggests that the electromagnetic momentum does not have a clear meaning when associated with charges and currents, in contrast to the case of free fields where it is the classical precursor of the momentum of quantum photons [21]. Nonetheless, consistency between mechanics and electromagnetism is only achieved if an electromagnetic momentum can be associated with charges, currents and fields. Other examples by the author the illustrate this point include include [22, 23, 24, 25, 26, 27, 28].

The existence of three equivalent methods of calculation of the electromagnetic momentum permits us to choose whichever form is most convenient in a particular situation. However, it also provides some general guidance as to the sensitivity of these calculations to details at large distances. For example, if the electric charge distribution ϱ is spatially localized then the first form of eq. (1) permits a calculation using only knowledge of the vector potential in that localized region. This might suggest that details of the electric and magnetic fields at large distances will be unimportant if we use the second form of eq. (1) to calculate the electromagnetic momentum. However, the third form eq. (1) requires detailed knowledge of the electric potential at the location of the currents that generate the magnetic field, which may be far from the electric charges. This warns us that detailed knowledge of the electric field far from the charges is in general needed when using the second form of eq. (1).

Another perspective is that the electromagnetic momentum (1) is a (bi)linear function of the electric and magnetic fields, so its value is more sensitive to the behavior of the fields at large distances than, say, a calculation of field energy which is quadratic in the electric and magnetic field strengths.

This point is illustrated in sec. 2.3 by the possibly surprising result for the electromagnetic momentum of a capacitor in a uniform magnetic field.

2.2 Electric Dipole $\mathbf{p} = q\mathbf{d}$

We first consider the case of an electric dipole $\mathbf{p} = q\mathbf{d}$ that consists of point charges $\pm q$ at positions $\mathbf{d}^+$ and $\mathbf{d}^-$ where $\mathbf{d} = \mathbf{d}^+ - \mathbf{d}^-$.

The uniform magnetic field of strength $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$ is created by a (long) solenoid whose axis of symmetry is the z axis. In the vicinity of the capacitor the vector potential of the magnetic field is azimuthal, with value $A_\phi = \rho B_0/2$, so that we can write

$$\mathbf{A} = \frac{\rho}{2} B_0\,\hat{\boldsymbol{\phi}} = \frac{B_0}{2}(-\rho\sin\phi\,\hat{\mathbf{x}} + \rho\cos\phi\,\hat{\mathbf{y}}) = \frac{B_0}{2}(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) = -\frac{B_0}{2}\,\mathbf{r}\times\hat{\mathbf{z}} = -\frac{\mathbf{r}\times\mathbf{B}_0}{2}\,, \quad (2)$$

at a point $\mathbf{r} = (x, y, z) = (\rho, \phi, z)$ in a cylindrical coordinate system.

It is most straightforward in the present case to evaluate the electromagnetic momentum using the first form of eq. (1). Then,

$$\mathbf{P}_{\rm EM,1} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = -(q\mathbf{d}^+ - q\mathbf{d}^-) \times \frac{\mathbf{B}_0}{2c} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,. \qquad (3)$$

Since the electromagnetic momentum (1) is a linear function of the charge distribution ϱ (as well as a linear function of the electric field $\mathbf{E}$ and the scalar potential Φ), the electromagnetic momentum of an extended charge distribution in a uniform magnetic field can be calculated by linear superposition. In particular, the electromagnetic momentum for any charge distribution that is a superposition of electric dipoles obeys eq. (3), where $\mathbf{p}$ is the total electric dipole moment of the charge distribution.

Additional calculations of the electromagnetic momentum of an electric dipole are given in the Appendix.

2.3 Parallel-Plate Capacitor

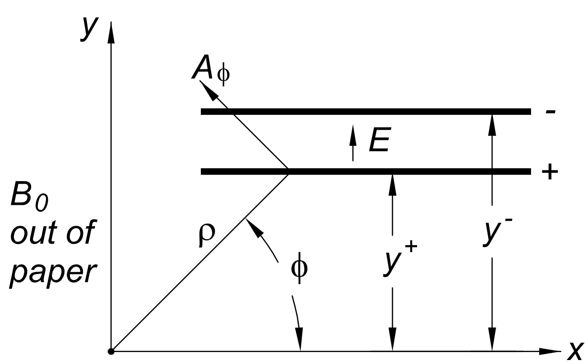
We next consider a parallel-plate capacitor that has plates of area A and separation d that are parallel to the x-z plane and at distances y^+ and y^- from it, where $d = y^- - y^+$, as shown in the figure on the next page.

The uniform magnetic field of strength B_0 is again created by a (long) solenoid whose axis of symmetry is the z axis, so its vector potential is again given by eq. (2).

If the two parallel plates of the capacitor have the same shape, then the surface charge distributions on the two plates have the same form, $\sigma^+ = -\sigma^- \equiv \sigma$, and the total charge distribution is a superposition of electric dipoles parallel to the y axis. In this case we can evaluate the electromagnetic momentum using eq. (3),

$$\mathbf{P}_{\rm EM,1} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c} = \frac{B_0\,\hat{\mathbf{z}} \times -Qd\,\hat{\mathbf{y}}}{2c} = \frac{QdB_0}{2c}\,\hat{\mathbf{x}} = \frac{E_0 B_0}{8\pi c}\,{\rm Vol}\,\hat{\mathbf{x}}\,, \qquad (4)$$

where $\mathbf{p} = -Qd\,\hat{\mathbf{y}}$ is the total electric dipole moment, $Q = \int \sigma\; d{\rm Area}$ is the charge on each plate of the capacitor, and $E_0 = 4\pi Q/A$ is the electric field in the capacitor neglecting edge effects.



We could also use the first form of eq. (1), together with eq. (2), to find

$$\mathbf{P}_{\rm EM,1} \quad = \quad \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = \frac{1}{c}\left[\int_+ \sigma^+ \mathbf{A}^+\; d{\rm Area}^+ + \int_- \sigma^- \mathbf{A}^-\; d{\rm Area}^-\right]$$

$$
\begin{aligned}
&= \frac{B_0}{2c}\left[\int \sigma(-y^+\,\hat{\mathbf{x}} + x^+\,\hat{\mathbf{y}})\; d\text{Area} - \int \sigma(-y^-\,\hat{\mathbf{x}} + x^-\,\hat{\mathbf{y}})\; d\text{Area}\right] \\
&= \frac{B_0}{2c}(y^- - y^+)\int \sigma\; d\text{Area}\,\hat{\mathbf{x}} = \frac{B_0 Q d}{2c}\,\hat{\mathbf{x}} = \frac{B_0 p}{2c}\,\hat{\mathbf{x}} \qquad (5)
\end{aligned}
$$

The result (4) holds no matter what the shape of the capacitor plates, and they are valid for any separation d. This result is remarkable in that it is exactly 1/2 of the naive expectation based on form 2 of eq. (1),

$$
\mathbf{P}_{\text{EM},2} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol} \overset{?}{\approx} \frac{E_0 B_0}{4\pi c}\,\text{Vol}\,\hat{\mathbf{x}}. \qquad (6)
$$

In the interpretation that the electromagnetic momentum is stored in the field, we infer that outside the capacitor, in its fringe field, the stored momentum is approximately $-1/2$ that stored within the nominal volume.[3] This result is surprising in that we can typically neglect the region outside the capacitor in considerations of capacitance and stored energy. However, energy is quadratic while momentum is linear in the electric field strength, so that field momentum is much more sensitive to fringe-field effects than is field energy.

2.4 Spherical Capacitor inside a Spherical Magnet

In this section we consider a capacitor whose electrodes are conducting spherical caps on a sphere of radius a. The caps extend over polar angle $0 \le \theta \le \theta_0 < \pi/2$ in a spherical coordinate system whose z axis is the symmetry axis of the capacitor.[4]

We suppose that the uniform external magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{y}}$ is along the y-axis. Then, the electric potential Φ has the form

$$
\Phi(r,\theta) = \begin{cases} \sum_{n\ \text{odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) & (r < a), \\ \sum_{n\ \text{odd}} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) & (r > a). \end{cases} \qquad (7)
$$

It is not easy to determine the Fourier coefficients A_n for the general case of spherical caps of angle θ_0, but it turns out that we need to know only the coefficient A_1, which is related to the dipole moment of the capacitor. Of course, the Fourier coefficients are nonzero only for odd n, since the charge distributions on the two plates are equal and opposite.

We are interested in the x-component of the electromagnetic momentum, which we calculate using the second form of eq. (1),

$$
P_{\text{EM},2,x} = -\int \frac{E_z B_0}{4\pi c}\, d\text{Vol}. \qquad (8)
$$

[3]In the following section we deduce that 1/3 of the total electromagnetic momentum is located far from the capacitor, so that the momentum in the fringe field near the capacitor is actually $-2/3$ that stored in the nominal volume.

[4]The case of a sphere with a fixed surface charge distribution that varies as $\cos\theta$ was considered by Romer [21], which first interested the author in this type of problem.

The z-component of the electric field is given by

$$
\begin{aligned}
E_z &= \cos\theta E_r - \sin\theta E_\theta = -P_1(\cos\theta)\frac{\partial\Phi}{\partial r} + \frac{\sin\theta}{r}\frac{\partial\Phi}{\partial\theta} \\
&= \begin{cases} -\sum_{n \text{ odd}} A_n \frac{r^{n-1}}{a^n}\left(nP_1P_n + \sin^2\theta P_n'\right) & (r < a), \\ \sum_{n \text{ odd}} A_n \frac{a^{n+1}}{r^{n+2}}\left((n+1)P_1P_n - \sin^2\theta P_n'\right) & (r > a), \end{cases}
\end{aligned} \tag{9}
$$

noting that $dP_n(\cos\theta)/d\theta = -\sin\theta\, dP_n(\cos\theta)/d\cos\theta = -\sin\theta P_n'$.

We first consider the volume integral for the region $r > a$. In particular, if the magnetic field is uniform for all $r > a$, then this integral includes the factor

$$
\begin{aligned}
&\int_{-1}^{1} d\cos\theta\,\left((n+1)P_1P_n - \sin^2\theta P_n'\right) = \frac{4}{3}\delta_{1n} - \int_{-1}^{1} d\mu\,(1-\mu^2)P_n'(\mu) \\
&= \frac{4}{3}\delta_{1n} - P_n|_{-1}^{1} + \mu^2 P_n|_{-1}^{1} - \int_{-1}^{1} d\mu\, 2\mu P_n(\mu) = \frac{4}{3}\delta_{1n} - \frac{4}{3}\delta_{1n} = 0.
\end{aligned} \tag{10}
$$

Thus, the contribution to the electromagnetic momentum for $r > a$ vanishes at every order n, and so it appears that the total electromagnetic momentum stored in this region is zero.

For the region $r < a$ the θ integral is

$$
\begin{aligned}
&\int_{-1}^{1} d\cos\theta\,\left(nP_1P_n + \sin^2\theta P_n'\right) = \frac{2}{3}\delta_{1n} + \int_{-1}^{1} d\mu\,(1-\mu^2)P_n'(\mu) \\
&= \frac{2}{3}\delta_{1n} + P_n|_{-1}^{1} - \mu^2 P_n|_{-1}^{1} + \int_{-1}^{1} d\mu\, 2\mu P_n(\mu) = \frac{2}{3}\delta_{1n} + \frac{4}{3}\delta_{1n} = 2\delta_{1n}.
\end{aligned} \tag{11}
$$

Thus, only the $n = 1$ term of the potential for $r < a$ contributes to the electromagnetic momentum. We can identify the coefficient A_1 as p/a^2, where p is the dipole moment of the charged capacitor (according to an observer at $r > a$). The electric field $E_{z,1} = -p/a^3$ due to the $n = 1$ term is constant within the sphere of radius a.

The electromagnetic momentum according to the second form of eq. (1) is then

$$
P_{\text{EM},2,x} = -\int \frac{E_z B_0}{4\pi c}\, d\text{Vol} = -\frac{E_{z,1}B_0}{4\pi c}\ \text{Vol} = \frac{B_0 p}{3c}\,, \tag{12}
$$

for any angle θ_0 of the spherical capacitor.

This simplicity of this result is gratifying, but it disagrees with the previous result (3) that the electromagnetic momentum of an electric dipole at right angles to a uniform magnetic field is $B_0p/2c$.

The defect of the result (12) is that it is based on the assumption that the magnetic field is uniform at large distances from the capacitor. But any real magnetic field, whose source currents lie within a bounded volume, falls to zero at very large distances. And as we noted at the end of sec. 2.1, the details of the fields at large distance are important when using form 2 of eq. (1) to calculate the electromagnetic momentum.

A model for a magnetic field that is uniform near the origin and well defined at large distances is a sphere of (large) radius $b > a$ on which there exists a surface current density that varies as $\mathbf{K} = 3c\mathbf{B}_0 \times \hat{\mathbf{r}}/8\pi$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the sphere. The magnetic field is uniform within the sphere, while outside the sphere the field is that of the magnetic dipole $\mathbf{m} = b^3\mathbf{B}_0/2 = b^3 B_0\,\hat{\mathbf{y}}/2$ located at the center of the sphere. See also sec. 3.3.

The calculation of the electromagnetic momentum via form 2 of eq. (1) for the spherical capacitor plus spherical magnet is the same as that given above for $r < b$, where the magnetic field is uniform.

Now, we must calculate the electromagnetic momentum in the region $r > b$, where the magnetic field has the dipole form

$$\begin{aligned}\mathbf{B}(r > b) &= \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \\ &= \frac{m}{r^3}[3\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{x}} + (3\sin^2\theta\sin^2\phi - 1)\,\hat{\mathbf{y}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{z}}], \qquad (13)\end{aligned}$$

since

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}. \qquad (14)$$

The electric field of the spherical capacitor has rectangular components

$$\begin{aligned}\mathbf{E} &= E_r\,\hat{\mathbf{r}} + E_\theta\,\hat{\boldsymbol{\theta}} \qquad (15)\\ &= (E_r\sin\theta + E_\theta\cos\theta)\cos\phi\,\hat{\mathbf{x}} + (E_r\sin\theta + E_\theta\cos\theta)\sin\phi\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}},\end{aligned}$$

noting that

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}. \qquad (16)$$

The cross product is

$$\begin{aligned}\mathbf{E} \times \mathbf{B}(r > b) \qquad (17)\\ = \frac{m}{r^3}[3(E_r\sin\theta + E_\theta\cos\theta)\sin\theta\cos\theta\sin^2\phi - E_z(3\sin^2\theta\sin^2\phi - 1)]\,\hat{\mathbf{x}} \\ -\frac{3m}{r^3}E_\theta\sin\theta\sin\phi\cos\phi\,\hat{\mathbf{y}} - \frac{m}{r^3}(E_r\sin\theta + E_\theta\cos\theta)\cos\phi\,\hat{\mathbf{z}}.\end{aligned}$$

On performing the azimuthal part of the volume integral, the only surviving terms are

$$\begin{aligned}\int_0^{2\pi}\mathbf{E}\times\mathbf{B}(r > b)\,d\phi &= \frac{\pi m}{r^3}[3(E_r\sin\theta + E_\theta\cos\theta)\sin\theta\cos\theta - E_z(3\sin^2\theta - 2)]\,\hat{\mathbf{x}} \\ &= \frac{\pi m}{r^3}\sum_{n\ \mathrm{odd}} A_n\frac{a^{n+1}}{r^{n+2}}\left(2(n+1)\mu P_n + (1-\mu^2)P_n'\right)\hat{\mathbf{x}} \qquad (18)\end{aligned}$$

recalling eq. (9), and where $\mu = \cos\theta = P_1$. The polar part of the volume integral includes the factor

$$\int_{-1}^{1} d\mu\,\left(2(n+1)\mu P_n + (1-\mu^2)P_n'\right) = \int_{-1}^{1} d\mu\,(2(n+1)\mu P_n + 2\mu P_n) = 4\delta_{1n}. \qquad (19)$$

The contribution to the electromagnetic momentum at $r > b$ is thus,

$$\mathbf{P}_{\mathrm{EM},2}(r > b) = \int_{r>b}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} = \frac{a^2 A_1 m}{c}\int_b^\infty \frac{r^2\,dr}{r^6}\,\hat{\mathbf{x}} = \frac{a^2 A_1 m}{3b^3 c}\,\hat{\mathbf{x}} = \frac{B_0 p}{6c}\,\hat{\mathbf{x}}. \qquad (20)$$

Combining this with eq. (12), which gives the electromagnetic momentum for $r < b$, we find the total electromagnetic momentum of a spherical capacitor inside a $\sin\theta$ spherical magneto to be

$$\mathbf{P}_{\rm EM,2} = \frac{B_0 p}{2c}\,\hat{\mathbf{x}} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,, \tag{21}$$

which now agrees with eq. (3).

It is noteworthy that 1/3 of the electromagnetic momentum is located outside the coil of the spherical magnet, according to form 2 of eq. (1), no matter how large is the magnet compared to the capacitor inside it. This result holds for any two-electrode capacitor with midplane symmetry, since its potential obeys eq. (7) for $r > a$ where a is the radius of a sphere that completely enclosed the capacitor.

Thus we find the result (5) of sec. 2.3 to be even more impressive as we now understand that the electromagnetic momentum located close to the capacitor sums to only 1/3 of the naive estimate $\mathbf{E}_0 \times \mathbf{B}_0\,{\rm Vol}/4\pi c$.

For completeness, we also calculate the electromagnetic momentum using the third form of eq. (1):

$$\begin{aligned}
\mathbf{P}_{\rm EM,3} &= \int_{r=b} \frac{\Phi(b)\mathbf{K}}{c^2}\, d{\rm Area} = \frac{3\mathbf{B}_0}{8\pi c} \times \sum_{n\ \rm odd} A_n \left(\frac{a}{b}\right)^{n+1} \int_{-1}^{1} 2\pi b^2\, d\cos\theta\ P_n(\cos\theta)\,\hat{\mathbf{r}} \\
&= \frac{3b^2\mathbf{B}_0}{4c} \times \sum_{n\ \rm odd} A_n \left(\frac{a}{b}\right)^{n+1} \int_{-1}^{1} d\cos\theta\ P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\
&= \frac{\mathbf{B}_0 \times a^2 A_1\,\hat{\mathbf{z}}}{2c} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}\,,
\end{aligned} \tag{22}$$

in agreement with eq. (3).

Thus, we have an explicit example of a capacitor in a magnetic field for which the electromagnetic momentum is calculated to have the same value using all three forms of eq. (1).

2.5 Hidden Mechanical Momentum

The electromagnetic momentum (1) can be nonzero for configurations of static charge distributions combined with steady electric currents. From a mechanical point of view, such systems are at rest, so it is counterintuitive that they can contain a nonzero momentum.

In footnote 2 we saw that if the currents flow in perfect conductors, which are equipotentials, the total electromagnetic momentum of the system of currents plus fixed charges would be zero. Hence, in the examples above we have tacitly assumed that the currents do not flow in superconductors.

An idealized model for the surface currents $\mathbf{K}$ of the preceding sections is that they are due to a nonconducting tubes at rest that contain a circulating incompressible liquid of charged molecules, and that adjacent tubes have oppositely charged molecules whose flow has opposite senses of rotation. In this model the only matter in motion is the charge carriers, and the structure is electrically neutral. The electric potential on the incompressible charged fluid is that due to the fixed charges elsewhere in the system.

In this case, as first noted by Shockley [29], and more explicity by Coleman and Van Vleck [30] and by Furry [20], as the charges move through a spatially varying electric potential Φ their relativistic mass changes according to $-e\Phi/c^2$ so that the system possesses a "hidden" mechanical momentum

$$\mathbf{P}_{\rm mech} = -\int \frac{\Phi \mathbf{J}}{c^2}\, d{\rm Vol} = -\mathbf{P}_{\rm EM}. \tag{23}$$

A system of currents and charges "at rest" therefore contains zero total momentum, in agreement with one's expectations.

For further discussion of "hidden" mechanical momentum see, for example, [31, 32, 33, 34, 35, 36].

2.6 Spherical Magnet inside a Spherical Capacitor

A variant of the example of sec. 2.4 is a spherical magnet with a $\sin\theta$ winding of radius b inside a spherical capacitor of radius $a > b$.

The electromagnetic momentum according to the third form of eq. (1) is, recalling eq. (7) for the scalar potential of the capacitor,

$$\begin{aligned}
\mathbf{P}_{\rm EM,3} &= \int_{r=b} \frac{\Phi(b)\mathbf{K}}{c^2}\, d{\rm Area} = \frac{3\mathbf{B}_0}{8\pi c} \times \sum_{n\ {\rm odd}} A_n \left(\frac{b}{a}\right)^n \int_{-1}^{1} 2\pi b^2\, d\cos\theta\ P_n(\cos\theta)\,\hat{\mathbf{r}} \\
&= \frac{3b^2\mathbf{B}_0}{4c} \times \sum_{n\ {\rm odd}} A_n \left(\frac{b}{a}\right)^n \int_{-1}^{1} d\cos\theta\ P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\
&= \frac{b^3\mathbf{B}_0 \times A_1/a\,\hat{\mathbf{z}}}{2c} = \frac{\mathbf{E}_0 \times \mathbf{m}}{c},
\end{aligned} \tag{24}$$

where $\mathbf{m} = b^3\mathbf{B}_0/2$ is the magnetic dipole moment, and $\mathbf{E}_0 = -(A_1/a)\,\hat{\mathbf{z}}$ is the electric field at the origin as given by eq. (9).

To calculate the electromagnetic momentum using the first form of eq. (1), we recall that the vector potential associated with a magnetic moment $\mathbf{m}$ at the origin is

$$\mathbf{A} = \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \tag{25}$$

and that the combined surface charge σ density on the inner and outer surfaces of the capacitor at radius a is given by

$$\sigma = \frac{E_r(a^+) - E_r(a^-)}{4\pi} = \sum_{n\ {\rm odd}} (2n+1)\frac{A_n}{4\pi a} P_n(\cos\theta). \tag{26}$$

Then,

$$\begin{aligned}
\mathbf{P}_{\rm EM,1} &= \int_{r=a} \frac{\sigma \mathbf{A}(a)}{c}\, d{\rm Area} = \frac{\mathbf{m}}{4\pi a^2 c} \times \sum_{n\ {\rm odd}} (2n+1)\frac{A_n}{a} \int_{-1}^{1} 2\pi a^2\, d\cos\theta\ P_n(\cos\theta)\,\hat{\mathbf{r}} \\
&= \frac{\mathbf{m}}{2c} \times \sum_{n\ {\rm odd}} (2n+1)\frac{A_n}{a} \int_{-1}^{1} d\cos\theta\ P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\
&= \frac{\mathbf{m} \times A_1/a\,\hat{\mathbf{z}}}{c} = \frac{\mathbf{E}_0 \times \mathbf{m}}{c}.
\end{aligned} \tag{27}$$

We partition the calculation of the electromagnetic momentum via form 2 of eq. (1) into the three regions, $r < b$, $b < r < a$ and $r > a$. The electric and magnetic fields for $r < b$ in the present case are the same as those for $r < a$ in sec. 2.4, where we suppose that $\mathbf{B}_0 = B_0\,\hat{\mathbf{y}}$. Hence, from eq. (12) we see that

$$\mathbf{P}_{\rm EM,2}(r < b) = -\int_{r<b} \frac{E_z B_0}{4\pi c}\, d{\rm Vol}\,\hat{\mathbf{x}} = -\frac{E_{z,1} B_0}{4\pi c}\,{\rm Vol}\,\hat{\mathbf{x}} = \frac{E_0 b^3 B_0}{3c}\,\hat{\mathbf{x}} = \frac{2\mathbf{E}_0 \times \mathbf{m}}{3c}\,. \tag{28}$$

Similarly, the electric and magnetic fields for $r > a$ in the present case are the same as those for $r > b$ in sec. 2.4, so from eq. (20) we see that

$$\mathbf{P}_{\rm EM,2}(r > a) = \int_{r>a} \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol} = \frac{a^2 A_1 m}{c}\int_a^\infty \frac{r^2\, dr}{r^6}\,\hat{\mathbf{x}} = \frac{A_1 m}{3ac}\,\hat{\mathbf{x}} = \frac{\mathbf{E}_0 \times \mathbf{m}}{3c}\,. \tag{29}$$

For the region $b < r < a$ we again use eqs. (13)-(17), but we note that eq. (18) becomes

$$\begin{aligned}\int_0^{2\pi} \mathbf{E}(r < a) \times \mathbf{B}(r > b)\, d\phi &= \frac{\pi m}{r^3}[3(E_r \sin\theta + E_\theta \cos\theta)\sin\theta\cos\theta - E_z(3\sin^2\theta - 2)]\,\hat{\mathbf{x}} \\ &= \frac{\pi m}{r^3}\sum_{n\ {\rm odd}} A_n \frac{r^{n-1}}{a^n}\left(-2n\mu P_n + (1-\mu^2)P_n'\right)\hat{\mathbf{x}},\end{aligned} \tag{30}$$

again using eq. (9). The polar part of the volume integral includes the factor

$$\int_{-1}^1 d\mu\,\left(-2n\mu P_n + (1-\mu^2)P_n'\right) = \int_{-1}^1 d\mu\,(-2n\mu P_n + 2\mu P_n) = 0. \tag{31}$$

Thus, the total electromagnetic momentum of a $\sin\theta$ spherical magnet inside a spherical capacitor is the sum of eqs. (28) and (29),

$$\mathbf{P}_{\rm EM,2} = \frac{\mathbf{E}_0 \times \mathbf{m}}{c}\,, \tag{32}$$

in agreement with the calculations (24) and (27) using forms 1 and 3 of eq. (1).

Again, according to the interpretation of momentum being stored in the electromagnetic field, 2/3 of the momentum is within the inner sphere, and 1/3 is outside the outer sphere.

A Appendix: Additional Calculations for an Electric Dipole

If we evaluate the electromagnetic momentum for the electric dipole $\mathbf{p} = q\mathbf{d}$ in a long solenoid using either the second or third forms of eq. (1) we obtain divergent contributions of opposite sign from the two charges $\pm q$. It is delicate to obtain the finite resultant of these canceling divergences.

Section 3.1 presents a calculation in which a judicious cancelation of divergent terms using the third form of eq. (1) yields the same result as with use of the first form. Section 3.2 avoids the problem of divergences in form 3 by consideration of a point dipole at the origin. In sec. 3.3 the long solenoid magnet is replaced by a spherical magnet with a $\sin\theta$ winding.

A.1 Calculation Using Form 3 of Eq. (1) for a Long Solenoid

We first consider the third form of eq. (1). The current in the solenoid of radius b whose symmetry axis is the z-axis can be described by the surface current density vector $\mathbf{K} = cB_0\,\boldsymbol{\phi}/4\pi$. Then for a charge q at position $\mathbf{d} = (d, 0, z)$, where $d < b$, only the y-component of $\int \Phi\mathbf{K}\,d\text{Area}$ is nonzero, and we find using Dwight 200.01,[5]

$$
\begin{aligned}
\mathbf{P}_{\text{EM},3} &= \hat{\mathbf{y}} \int \frac{\Phi K_y}{c^2}\, d\text{Area} = \hat{\mathbf{y}} \int_0^{2\pi} b\, d\phi \int_{-\infty}^{\infty} dz\, \frac{qK\cos\phi}{c^2\sqrt{b^2 - 2bd\cos\phi + d^2 + z^2}} \\
&= \frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\, \lim_{z\to\infty} \Big[\ln(z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2}) \\
&\qquad\qquad - \ln(-z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2})\Big] \\
&= -\frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\, \Big[\ln(b^2 - 2bd\cos\phi + d^2) \\
&\qquad\qquad -2 \lim_{z\to\infty} \ln(z + \sqrt{b^2 - 2bd\cos\phi + d^2 + z^2})\Big]. \qquad (33)
\end{aligned}
$$

The divergent term in the last line of eq. (33) is independent of d, so when we add the contribution of charge $-q$ at some other position d', we argue that the divergences cancel. If so, we continue the evaluation of the finite part of eq. (33), integrating by parts and then using Dwight 859.131,

$$
\begin{aligned}
\mathbf{P}_{\text{EM},3,\text{finite part}} &= -\frac{dqB_0\,\hat{\mathbf{y}}}{4\pi c} \int_0^{2\pi} \cos\phi\, d\phi\; \ln(b^2 - 2bd\cos\phi + d^2) \\
&= \frac{b^2 qdB_0\,\hat{\mathbf{y}}}{2\pi c} \int_0^{2\pi} d\phi\, \frac{\sin^2\phi}{b^2 - 2bd\cos\phi + d^2} = \frac{qdB_0\,\hat{\mathbf{y}}}{2c}. \qquad (34)
\end{aligned}
$$

Then, for an electric dipole $\mathbf{p}$ with charges $\pm q$ at $(d^+, 0, z^+)$ and $(d^-, 0, z^-)$, eq. (34) gives

$$
\mathbf{P}_{\text{EM},3} = \frac{q(d^+ - d^-)B_0\,\hat{\mathbf{y}}}{2c} = \frac{\mathbf{B}_0 \times \mathbf{p}}{2c}, \qquad (35)
$$

in agreement with eq. (3).

A.2 Calculation for a Point Dipole via Form 3 of Eq. (1)

If we use form 3 of eq. (1) to evaluate the electromagnetic momentum of a point dipole in a uniform magnetic field, we again obtain eq. (3), without having to cancel divergent terms. For simplicity, we suppose that the dipole is at the origin, with moment $\mathbf{p} = p\,\hat{\mathbf{x}}$. Then the electric potential of this dipole at position $\mathbf{r}$ is $\Phi = p\cos\theta/r^2$, where θ is the angle between vector $\mathbf{r}$ and dipole moment $\mathbf{p}$. Point $\mathbf{r} = (b, \phi, z)$ on the solenoid winding of radius a has rectangular coordinates $(b\cos\phi, b\sin\phi, z)$, so that $r = \sqrt{b^2 + z^2}$ and $\cos\theta = b\cos\phi/r$. The

[5] We neglect the effect of the variation of electric potential along the wire of the solenoid winding in case this wire has finite conductivity. We have discussed the electromagnetic momentum of circuits of finite conductivity in [26].

current in the solenoid is again described by the surface current density vector $\mathbf{K} = cB_0\boldsymbol{\phi}/4\pi$, and again only the y-component of $\int \Phi\mathbf{K}\, d$Area is nonzero. Hence,

$$\begin{aligned}\mathbf{P}_{\rm EM,3} &= \hat{\mathbf{y}}\int \frac{\Phi K_y}{c^2}\, d\text{Area} = \hat{\mathbf{y}}\int_0^{2\pi} a\, d\phi \int_{-\infty}^{\infty} dz\, \frac{p\cos\theta}{c^2 r^2} K\cos\phi \\ &= \frac{b^2 B_0 p}{4\pi c}\hat{\mathbf{y}}\int_0^{2\pi} d\phi\, \cos^2\phi \int_{-\infty}^{\infty} \frac{dz}{(b^2+z^2)^{3/2}} = \frac{B_0 p}{2c}\hat{\mathbf{y}} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}, \qquad (36)\end{aligned}$$

in further agreement with eq. (3).

A.3 Calculation for a Spherical Magnet Using Form 3

The use of a long solenoid as a model of a uniform magnetic field involves unrealistic conditions at "infinity", which may lead to subtle inaccuracies in some calculations of the electromagnetic momentum. A better model for a uniform magnetic field $\mathbf{B}_0$ may be a sphere of (large) radius b on which there exists a surface current density that varies as $\mathbf{K} = 3c\mathbf{B}_0\times\hat{\mathbf{r}}/8\pi$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the sphere. The magnetic field is uniform within the sphere, while outside the sphere the field is that of the magnetic dipole $\mathbf{m} = b^3\mathbf{B}_0/2$ located at the center of the sphere.

We first consider a charge q at position $d/2\,\hat{\mathbf{z}}$ where $d/2 < a$. Then the electric scalar potential of a charge q at position $d/2\,\hat{\mathbf{z}}$ where $d/2 < b$, can be written in a spherical coordinate system (r, θ, ϕ) for $r > d/2$ as

$$\Phi_q(r > d/2) = \frac{2q}{d}\sum_n \left(\frac{d}{2r}\right)^{n+1} P_n(\cos\theta), \qquad (37)$$

where P_n is a Legendre polynomial.

Then, the electric dipole $\mathbf{p} = qd\,\hat{\mathbf{z}}$ consisting of charges $\pm q$ at positions $\pm d/2\,\hat{\mathbf{z}}$ has potential

$$\Phi_p(r > d/2) = \frac{4q}{d}\sum_{n\ \rm odd} \left(\frac{d}{2r}\right)^{n+1} P_n(\cos\theta), \qquad (38)$$

recalling that $P_n(-\mu) = (-1)^n P_n(\mu)$.

The electromagnetic momentum associated with the electric dipole $\mathbf{p}$ and the spherical magnet is, according to form 3 of eq. (1),

$$\begin{aligned}\mathbf{P}_{\rm EM,3} &= \int \frac{\Phi_p(b)\mathbf{K}}{c^2}\, d\text{Area} = \frac{4q}{d}\frac{3\mathbf{B}_0}{8\pi c}\times\sum_{n\ \rm odd}\left(\frac{d}{2b}\right)^{n+1}\int_{-1}^{1} 2\pi b^2\, d\cos\theta\ P_n(\cos\theta)\,\hat{\mathbf{r}} \\ &= \frac{3b^2 q\mathbf{B}_0}{cd}\times\sum_{n\ \rm odd}\left(\frac{d}{2b}\right)^{n+1}\int_{-1}^{1} d\cos\theta\ P_n(\cos\theta)(\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}) \\ &= \frac{\mathbf{B}_0\times qd\,\hat{\mathbf{z}}}{2c} = \frac{\mathbf{B}_0\times\mathbf{p}}{2c}, \qquad (39)\end{aligned}$$

in agreement with eq. (3).

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