

Stress and Momentum in a Capacitor That Moves with Constant Velocity

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1 Problem

Consider a parallel-plate capacitor whose plates are held apart by a nonconducting slab of unit (relative) dielectric constant and unit (relative) magnetic permeability.¹ Discuss the energy, momentum and stress in this (isolated) system when at rest and when moving with constant velocity parallel or perpendicular to the electric field.

Does the system contain hidden momentum, $\mathbf{P}_{\text{hidden}}$, defined for a subsystem by

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}}) (\mathbf{p} - \rho\mathbf{v}_b) \cdot d\text{Area}, \quad (1)$$

where \mathbf{P} is the total momentum of the subsystem, $M = U/c^2$ is its total “mass,” U is its total energy, c is the speed of light in vacuum, \mathbf{x}_{cm} is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, \mathbf{p} is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, and \mathbf{v}_b is the velocity (field) of its boundary?²

Fringe-field effects can be ignored. The velocity can be large or small compared to the speed of light.

2 Solution

This problem is concerned with the relativistic transformation of properties of the capacitor. It represents a macroscopic application of the concepts of “Poincaré stresses” [3] that were introduced into classical models of the electron. Versions of this problem have also appeared in [4, 5, 6, 7].

We suppose that the capacitor supports a uniform electric field $\mathbf{E}^* = E^* \hat{\mathbf{z}}$ between its plates, in its rest frame. *Taking the (relative) dielectric constant ϵ of the material between the plates to be unity, the electric displacement is given by $\mathbf{D}^* = \mathbf{E}^*$ (in Gaussian units). The macroscopic magnetic fields vanish in the capacitor’s rest frame, $\mathbf{B}^* = \mathbf{H}^* = 0$, noting that the relative permeability is $\mu = 1$.*

Associated with these electromagnetic fields is the 4-dimensional, macroscopic (symmetric) electromagnetic energy-momentum-stress tensor (secs. 32-33 of [8], sec. 12.10B of [9]),

$$T_{\text{EM}}^{\mu\nu} = \left(\begin{array}{c|c} u_{\text{EM}} & c\mathbf{P}_{\text{EM}} \\ \hline c\mathbf{P}_{\text{EM}} & -T_{\text{EM}}^{ij} \end{array} \right), \quad (2)$$

¹The use of unit dielectric constant and unit permeability avoids entering into the interesting controversy as to the so-called Abraham and Minkowski forms of the energy-momentum-stress tensor [1].

²The definition (8) was suggested by Daniel Vanzella. See also [2].

where indices μ and ν take on values 0, 1, 2, 3, spatial indices i and j take on values 1, 2, 3, u_{EM} is the electromagnetic field energy density,

$$u_{\text{EM}} = \frac{E^2 + B^2}{4\pi}, \quad (3)$$

\mathbf{p}_{EM} is the electromagnetic momentum density,

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \quad (4)$$

and T_{EM}^{ij} is the 3-dimensional electromagnetic stress tensor

$$T_{\text{EM}}^{ij} = \frac{E_i E_j + B_i B_j}{4\pi} - \delta_{ij} \frac{E^2 + B^2}{8\pi}. \quad (5)$$

In the rest frame of the capacitor the electromagnetic energy-momentum-stress tensor has components

$$\mathbb{T}_{\text{EM}}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \frac{E^{\star 2}}{8\pi} & & & \mathbf{0} \\ \hline & \frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & \frac{E^{\star 2}}{8\pi} & \\ & & & -\frac{E^{\star 2}}{8\pi} \end{array} \right), \quad (6)$$

in the region between the capacitor plates. The nonzero diagonal elements $\mathbb{T}^{\star 11}$, $\mathbb{T}^{\star 22}$ and $\mathbb{T}^{\star 33}$, indicate that there are internal electric forces on the material between the capacitor plates. If the isolated capacitor is to be at rest, there must be internal mechanical stresses that are equal and opposite to the electromagnetic ones.³ Thus, we are led to consider also the mechanical energy-momentum stress tensor,

$$\mathbb{T}_{\text{mech}}^{\mu\nu} = \left(\begin{array}{c|ccc} u_{\text{mech}} & & & c \mathbf{p}_{\text{mech}} \\ \hline & & & \\ c \mathbf{p}_{\text{mech}} & & & -T_{\text{mech}}^{ij} \end{array} \right), \quad (7)$$

where the mechanical energy density is $u_{\text{mech}} = \rho_m c^2$, the mass density when there is no electric field in the capacitor is ρ_m , the density of mechanical momentum is \mathbf{p}_{mech} , and T_{mech}^{ij} is the 3-dimensional mechanical stress tensor.

We infer that in the rest frame of the isolated capacitor, the mechanical energy-momentum-stress tensor has components

$$\mathbb{T}_{\text{mech}}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho_m^{\star} c^2 & & & \mathbf{0} \\ \hline & -\frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & -\frac{E^{\star 2}}{8\pi} & \\ & & & \frac{E^{\star 2}}{8\pi} \end{array} \right), \quad (8)$$

³In a related analysis [10], it is suppose that the capacitor plates are held apart by the pressure of a gas in an external box. The stresses in the walls of the box are ignored, which seems to provide a less complete analysis than that of the present model.

The total energy-momentum-stress tensor is the sum of the electromagnetic and mechanical tensors (2) and (7). In the rest frame of the capacitor the total energy-momentum-stress tensor has components

$$\mathbb{T}_{\text{total}}^{*\mu\nu} = \mathbb{T}_{\text{EM}}^{*\mu\nu} + \mathbb{T}_{\text{mech}}^{*\mu\nu} = \left(\begin{array}{c|c} \rho_m^* c^2 + \frac{E^{*2}}{8\pi} & \mathbf{0} \\ \hline \mathbf{0} & 0 \end{array} \right), \quad (9)$$

We interpret the component \mathbb{T}^{*00} as implying the total mass density in the rest frame to be

$$\rho_{\text{total}}^* = \rho_m^* + \frac{E^2}{8\pi c^2}. \quad (10)$$

2.1 The Capacitor Has Velocity $\mathbf{v} \parallel \mathbf{E}^*$

In a frame where the capacitor has constant velocity $\mathbf{v} = v \hat{\mathbf{z}}$, the electric and magnetic fields between its plates are given by the transformation

$$\mathbf{E}_{\parallel} = \mathbf{E}_{\parallel}^* = E^* \hat{\mathbf{z}}, \quad (11)$$

$$\mathbf{E}_{\perp} = \gamma(\mathbf{E}_{\perp}^* - \frac{\mathbf{v}}{c} \times \mathbf{B}^*) = 0, \quad (12)$$

$$\mathbf{B}_{\parallel} = \mathbf{B}_{\parallel}^* = 0, \quad (13)$$

$$\mathbf{B}_{\perp} = \gamma(\mathbf{B}_{\perp}^* + \frac{\mathbf{v}}{c} \times \mathbf{E}^*) = 0, \quad (14)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. That is, the electromagnetic fields have the same values inside the capacitor in its rest frame and in frames where the capacitor moves with velocity \mathbf{v} parallel to \mathbf{E} . Hence, the electromagnetic energy-momentum-stress tensor has the same component values in all such frames,

$$\mathbb{T}_{\text{EM}}^{\mu\nu} = \mathbb{T}_{\text{EM}}^{*\mu\nu}. \quad (15)$$

It is useful to confirm this result via a Lorentz transformation of the stress tensor. The transformation \mathbb{L}_z from the rest frame to a frame in which the capacitor has velocity $v \hat{\mathbf{z}}$ can be expressed in tensor form as

$$\mathbb{L}_z^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma\beta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array} \right), \quad (16)$$

where $\beta = v/c$. Then the transform of a tensor,

$$\mathbb{T}^{*\mu\nu} = \left(\begin{array}{c|ccc} \mathbb{T}^{*00} & 0 & 0 & 0 \\ \hline 0 & \mathbb{T}^{*11} & 0 & 0 \\ 0 & 0 & \mathbb{T}^{*22} & 0 \\ 0 & 0 & 0 & \mathbb{T}^{*33} \end{array} \right), \quad (17)$$

that is diagonal in the rest frame is given by

$$\mathbb{T}^{\mu\nu} = (\mathbb{L}_z \mathbb{T}^* \mathbb{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathbb{T}^{*00} + \gamma^2 \beta^2 \mathbb{T}^{*33} & 0 & 0 & \gamma^2 \beta (\mathbb{T}^{*00} + \mathbb{T}^{*33}) \\ \hline 0 & \mathbb{T}^{*11} & 0 & 0 \\ 0 & 0 & \mathbb{T}^{*22} & 0 \\ \gamma^2 \beta (\mathbb{T}^{*00} + \mathbb{T}^{*33}) & 0 & 0 & \gamma^2 \beta^2 \mathbb{T}^{*00} + \gamma^2 \mathbb{T}^{*33} \end{array} \right). \quad (18)$$

In particular, the transformation of $\mathbb{T}_{\text{EM}}^{*\mu\nu}$, eq. (6), is

$$\mathbb{T}_{\text{EM}}^{\mu\nu} = \frac{E^{*2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1 - \beta^2) & 0 & 0 & \gamma^2 \beta(1 - 1) \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma^2 \beta(1 - 1) & 0 & 0 & -\gamma^2(1 - \beta^2) \end{array} \right) = \frac{E^{*2}}{8\pi} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) = \mathbb{T}_{\text{EM}}^{*\mu\nu}, \quad (19)$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathbb{T}_{\text{mech}}^{*\mu\nu}$, eq. (8), is

$$\mathbb{T}_{\text{mech}}^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \rho_m^* c^2 + \gamma^2 \beta^2 \frac{E^{*2}}{8\pi} & 0 & 0 & \gamma^2 \beta \left(\rho_m^* c^2 + \frac{E^{*2}}{8\pi} \right) \\ \hline 0 & -\frac{E^{*2}}{8\pi} & 0 & 0 \\ 0 & 0 & -\frac{E^{*2}}{8\pi} & 0 \\ \gamma^2 \beta \left(\rho_m^* c^2 + \frac{E^{*2}}{8\pi} \right) & 0 & 0 & \gamma^2 \beta^2 \rho_m^* c^2 + \gamma^2 \frac{E^{*2}}{8\pi} \end{array} \right), \quad (20)$$

and the transformation of the total stress tensor $\mathbb{T}_{\text{total}}^{*\mu\nu}$, eq. (9), is

$$\mathbb{T}_{\text{total}}^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \rho_{\text{total}}^* c^2 & 0 & 0 & \gamma^2 \beta \rho_{\text{total}}^* c^2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma^2 \beta \rho_{\text{total}}^* c^2 & 0 & 0 & \gamma^2 \beta^2 \rho_{\text{total}}^* c^2 \end{array} \right) = \mathbb{T}_{\text{EM}}^{\mu\nu} + \mathbb{T}_{\text{mech}}^{\mu\nu}, \quad (21)$$

with the total mass density ρ_{total}^* given by eq. (10).

One noteworthy feature of eq (21) is the nonzero value of $\mathbb{T}_{\text{total}}^{33}$. We recall that the purely spatial components of a stress-energy-momentum tensor have the dual interpretation as the momentum-flux tensor. In the present case, the flux of momentum is in the z direction, with magnitude equal to the momentum density times v , namely $(\gamma \rho_{\text{total}}^* v) \cdot v = \gamma^2 \beta^2 \rho_{\text{total}}^* c^2 = \mathbb{T}_{\text{total}}^{33}$.

Turning to the component $\mathbb{T}_{\text{total}}^{00}$, we note that the mass of the material between the capacitor plates, when moving with velocity \mathbf{v} , is larger than its rest mass by the factor γ . However, a moving volume element is smaller by the factor $1/\gamma$ than when that element is

at rest. Hence, the mass density ρ_m is larger by a factor of γ^2 for the moving capacitor than when at rest,

$$\rho_m = \gamma^2 \rho_m^*. \quad (22)$$

Thus, the component $\mathbb{T}_{\text{total}}^{00}$ transforms as expected for a mass density.

Furthermore, the four components

$$(\mathbb{T}_{\text{total}}^{00}, \mathbb{T}_{\text{total}}^{01}, \mathbb{T}_{\text{total}}^{02}, \mathbb{T}_{\text{total}}^{03}) \quad (23)$$

of the total energy-momentum-stress tensor transform as an energy-momentum-density 4-vector, although this is not the case for the sets of components

$$(\mathbb{T}_{\text{EM}}^{00}, \mathbb{T}_{\text{EM}}^{01}, \mathbb{T}_{\text{EM}}^{02}, \mathbb{T}_{\text{EM}}^{03}) \quad \text{or} \quad (\mathbb{T}_{\text{mech}}^{00}, \mathbb{T}_{\text{mech}}^{01}, \mathbb{T}_{\text{mech}}^{02}, \mathbb{T}_{\text{mech}}^{03}) \quad (24)$$

separately. This illustrates a general result that within volumes that contain both electromagnetic fields and matter, the concepts of electromagnetic momentum density, $\mathbb{T}_{\text{EM}}^{0i}/c = \mathbf{E} \times \mathbf{B}/4\pi c$, and mechanical momentum density, $\mathbb{T}_{\text{mech}}^{0i}/c$, are not consistent with being components of a energy-momentum 4-vector; only the total momentum density, $\mathbb{T}_{\text{total}}^{0i}/c$, is satisfactory in this respect. The great utility of the concept of “electromagnetic” momentum in matter-free regions leads us to attach similar significance to it in systems containing matter. However, this often results in difficulties in the interpretation of the “mechanical” part of the momentum.

As noted in [5], $\mathbb{T}_{\text{EM}}^{*0i} = 0$, so there is no flow of electromagnetic energy along with the moving capacitor when $\mathbf{v} \parallel \mathbf{E}$. However, the flow of mechanical energy density associated with the moving capacitor is $c\mathbb{T}_{\text{mech}}^{*0i}$, and we see in eq.(20) that $c\mathbb{T}_{\text{mech}}^{*03}$ has a term $\gamma^2 E^* \mathbf{v}/8\pi = \mathbb{T}_{\text{EM}}^{*00} \mathbf{v}$, which is the value perhaps naïvely expected for $c\mathbb{T}_{\text{EM}}^{*0i}$. That is, the flow of energy in the moving capacitor is “mechanical,” not “electromagnetic.” The electromagnetic field energy inside the moving capacitor is at rest, while the “bottom” plate of the capacitor “sweeps up” this energy, converts it to mechanical energy that flows up to the “top” plate inside the stressed dielectric, where it is converted back into electromagnetic energy. This is an example of the relativity of steady energy flow [11, 12, 13].

For a capacitor moving with $\mathbf{v} \parallel \mathbf{E}$ the electromagnetic-field momentum density is zero,

$$\mathbf{P}_{\text{EM}} = 0, \quad (25)$$

in that $\mathbb{T}_{\text{EM}}^{0i} = 0$, while the electromagnetic field energy is $U_{\text{EM}} = \mathbb{T}_{\text{mech}}^{00} V = E^{*2} V/8\pi$, where V is the volume of the capacitor. The effective mass of this energy is $M_{\text{EM}} = E^{*2} V/8\pi c^2$, and it moves with velocity \mathbf{v} , so that

$$M_{\text{EM}} \mathbf{v}_{\text{cm,EM}} = \frac{E^{*2} V}{8\pi c^2} \mathbf{v}. \quad (26)$$

Thus, according to the definition (1) the electromagnetic field of the moving capacitor possesses “hidden” momentum,

$$\mathbf{P}_{\text{hidden,EM}} = \mathbf{P}_{\text{EM}} - M_{\text{EM}} \mathbf{v}_{\text{cm,EM}} = -\frac{E^{*2} V}{8\pi c^2} \mathbf{v}. \quad (27)$$

This momentum is “hidden” in the sense that the electromagnetic field has no momentum, but its center of mass/energy is in motion.

The mechanical momentum density is given from eq. (20) as

$$\mathbf{p}_{\text{mech}} = \gamma^2 \left(\rho_m^* + \frac{E^{*2}}{8\pi c^2} \right) \mathbf{v} = \rho_{\text{total}} \mathbf{v}. \quad (28)$$

The mechanical momentum is the momentum density times the volume V ,

$$\mathbf{P}_{\text{mech}} = \mathbf{p}_{\text{mech}} V = \gamma^2 V \left(\rho_m^* + \frac{E^{*2}}{8\pi c^2} \right) \mathbf{v}. \quad (29)$$

The mechanical mass density is

$$\rho_{\text{mech}} = \mathsf{T}_{\text{mech}}^{00} = \gamma^2 \left(\rho_m^* + \beta^2 \frac{E^{*2}}{8\pi c^2} \right), \quad (30)$$

and this moves with velocity \mathbf{v} such that

$$M_{\text{mech}} \mathbf{v}_{\text{cm,mech}} = \rho_{\text{mech}} V \mathbf{v} = \gamma^2 V \left(\rho_m^* + \beta^2 \frac{E^{*2}}{8\pi c^2} \right) \mathbf{v}. \quad (31)$$

According to definition (1) the matter of the moving capacitor possesses “hidden” momentum,

$$\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{mech}} - M_{\text{mech}} \mathbf{v}_{\text{cm,mech}} = \frac{E^{*2} V}{8\pi c^2} \mathbf{v} = -\mathbf{P}_{\text{hidden,EM}}. \quad (32)$$

This result reflects that the energy and momentum of stress in a moving subsystem do not transform like a 4-vector if that subsystem interacts with another subsystem (here, the electromagnetic field).

The total “hidden” momentum of the system is zero.⁴

2.2 The Capacitor Has Velocity $\mathbf{v} \perp \mathbf{E}^*$

In a frame where the capacitor has constant velocity $\mathbf{v} = v \hat{\mathbf{x}}$, the electric and magnetic fields between its plates are given by the transformation

$$\mathbf{E}_{\parallel} = \mathbf{E}_{\parallel}^* = 0, \quad (33)$$

$$\mathbf{E}_{\perp} = \gamma \left(\mathbf{E}_{\perp}^* - \frac{\mathbf{v}}{c} \times \mathbf{B}^* \right) = \gamma E^* \hat{\mathbf{z}}, \quad (34)$$

$$\mathbf{B}_{\parallel} = \mathbf{B}_{\parallel}^* = 0, \quad (35)$$

$$\mathbf{B}_{\perp} = \gamma \left(\mathbf{B}_{\perp}^* + \frac{\mathbf{v}}{c} \times \mathbf{E}^* \right) = -\gamma \frac{v}{c} E^* \hat{\mathbf{y}}. \quad (36)$$

⁴If we consider the system to consist of two subsystems, A = capacitor plates and charge thereon, B = dielectric + electromagnetic fields, then subsystem B has the same properties as the entire system considered above (where we neglected the mass/energy of the capacitor plates). Hence, subsystem B has zero “hidden” momentum.

Using eqs. (2)-(5) together with eqs. (33)-(36), the electromagnetic energy-momentum-stress tensor of the moving capacitor is

$$\mathbb{T}_{\text{EM}}^{\mu\nu} = \frac{E^{*2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right). \quad (37)$$

We confirm this result using the Lorentz transformation \mathbb{L}_x from the rest frame to a frame in which the capacitor has velocity $v \hat{\mathbf{x}}$,

$$\mathbb{L}_x^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & \gamma\beta & 0 & 0 \\ \hline \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \quad (38)$$

Then, the transform of a tensor (17) that is diagonal in the rest frame is given by

$$\mathbb{T}^{\mu\nu} = (\mathbb{L}_x \mathbb{T}^* \mathbb{L}_x)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathbb{T}^{*00} + \gamma^2 \beta^2 \mathbb{T}^{*11} & \gamma^2 \beta (\mathbb{T}^{*00} + \mathbb{T}^{*11}) & 0 & 0 \\ \hline \gamma^2 \beta (\mathbb{T}^{*00} + \mathbb{T}^{*11}) & \gamma^2 \beta^2 \mathbb{T}^{*00} + \gamma^2 \mathbb{T}^{*11} & 0 & 0 \\ 0 & 0 & \mathbb{T}^{*22} & 0 \\ 0 & 0 & 0 & \mathbb{T}^{*33} \end{array} \right), \quad (39)$$

In particular, the transformation of $\mathbb{T}_{\text{EM}}^{*\mu\nu}$, eq. (6), is

$$\mathbb{T}_{\text{EM}}^{\mu\nu} = \frac{E^{*2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right), \quad (40)$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathbb{T}_{\text{mech}}^{*\mu\nu}$, eq. (8), is

$$\mathbb{T}_{\text{mech}}^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \rho_m^* c^2 - \gamma^2 \beta^2 \frac{E^{*2}}{8\pi} & \gamma^2 \beta \left(\rho_m^* c^2 - \frac{E^{*2}}{8\pi} \right) & 0 & 0 \\ \hline \gamma^2 \beta \left(\rho_m^* c^2 - \frac{E^{*2}}{8\pi} \right) & \gamma^2 \beta^2 \rho_m^* c^2 - \gamma^2 \frac{E^{*2}}{8\pi} & 0 & 0 \\ 0 & 0 & -\frac{E^{*2}}{8\pi} & 0 \\ 0 & 0 & 0 & \frac{E^{*2}}{8\pi} \end{array} \right), \quad (41)$$

the transformation of the total stress tensor $\mathbb{T}_{\text{total}}^{\star\mu\nu}$, eq. (9), is

$$\mathbb{T}_{\text{total}}^{\mu\nu} = \left(\begin{array}{c|cc} \gamma^2 \rho_{\text{total}}^{\star} c^2 & \gamma^2 \beta \rho_{\text{total}}^{\star} c^2 & 0 & 0 \\ \hline \gamma^2 \beta \rho_{\text{total}}^{\star} c^2 & \gamma^2 \beta^2 \rho_{\text{total}}^{\star} c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = \mathbb{T}_{\text{EM}}^{\mu\nu} + \mathbb{T}_{\text{mech}}^{\mu\nu}, \quad (42)$$

with the total mass density $\rho_{\text{total}}^{\star}$ again given by eq. (10).⁵

For a capacitor moving with $\mathbf{v} \perp \mathbf{E}$ the electromagnetic-field momentum is found from from eq. (40) to be

$$\mathbf{P}_{\text{EM}} = \frac{\mathbb{T}_{\text{EM}}^{01} V}{c} \hat{\mathbf{x}} = \frac{2\gamma^2 E^{\star 2} V}{8\pi c^2} \mathbf{v}, \quad (43)$$

and

$$M_{\text{EM}} \mathbf{v}_{\text{cm,EM}} = \frac{\mathbb{T}_{\text{EM}}^{00} V}{c^2} \mathbf{v} = \frac{\gamma^2 (1 + \beta^2) E^{\star 2} V}{8\pi c^2} \mathbf{v}. \quad (44)$$

According to definition (1) the electromagnetic field of the moving capacitor possesses “hidden” momentum,

$$\mathbf{P}_{\text{hidden,EM}} = \mathbf{P}_{\text{EM}} - M_{\text{EM}} \mathbf{v}_{\text{cm,EM}} = \frac{E^{\star 2} V}{8\pi c^2} \mathbf{v}. \quad (45)$$

The mechanical momentum is given from eq. (41) as

$$\mathbf{P}_{\text{mech}} = \frac{\mathbb{T}_{\text{mech}}^{01} V}{c} \hat{\mathbf{x}} = \gamma^2 V \left(\rho_m^{\star} - \frac{E^{\star 2}}{8\pi c^2} \right) \mathbf{v}, \quad (46)$$

and

$$M_{\text{mech}} \mathbf{v}_{\text{cm,mech}} = \frac{\mathbb{T}_{\text{mech}}^{00} V}{c^2} \mathbf{v} = \gamma^2 V \left(\rho_m^{\star} - \beta^2 \frac{E^{\star 2}}{8\pi c^2} \right). \quad (47)$$

According to definition (1) the matter of the moving capacitor possesses “hidden” momentum,

$$\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{mech}} - M_{\text{mech}} \mathbf{v}_{\text{cm,mech}} = -\frac{E^{\star 2} V}{8\pi c^2} \mathbf{v} = -\mathbf{P}_{\text{hidden,EM}}. \quad (48)$$

The total “hidden” momentum of the system is zero.

⁵The top row of eq. (42) shows that the flow of total energy between the capacitor plates equals the energy density there times the velocity of the capacitor. However, the top rows of eqs. (40)-(41) indicate that this is not true separately for the electromagnetic and mechanical energies. If the dielectric between the capacitor plates does not cover the entire area of the plates, such that there are regions in which the only form of energy density is electromagnetic, then the flow of energy here is not simply the energy density times the bulk velocity of the capacitor. Since this is a steady-state example, there must be a closed circulation of energy flow, which therefore includes some flow opposite to the direction of motion of the capacitor. To account for this circulation, we must consider both the fringe field of the capacitor, and the mechanical energy density inside the (stressed) plates of the capacitor [5]. This more complicated variant has counterintuitive aspects also encountered in examples such as the belt drive considered by Taylor and Wheeler [11, 12]; see also [13].

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