Charged, Conducting, Rotating Sphere

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1 Problem

A well-known example of a tiny relativistic correction to an everyday phenomenon is the small bulk charge density inside a conductor that carries a steady current.[1] Discuss the bulk charge distribution in a charged, conducting, rotating sphere of radius a, where the angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ obeys $a\omega \ll c$, and c is the speed of light in vacuum. You may suppose that the bulk of the sphere remains rigid (and spherical), although the distribution of conduction electrons is perturbed slightly.

2 Solution

In the first approximation, the total charge Q consists of a uniform surface charge density,

$$\sigma_0 = \frac{Q}{4\pi a^2}\,, \tag{1}$$

while the interior of the sphere contains equal and opposite bulk charge densities $\pm\rho_0$. These charge densities are at rest with respect to the rotating sphere (as an external energy source would be required to maintain any motion of charges with respect to the bulk conductor). There is no (macroscopic) electric field inside the sphere, but the rotating surface charge, with surface current density

$$\mathbf{K}_0 = a\sigma_0\omega\sin\theta\,\hat{\boldsymbol{\phi}} = \frac{Q\omega\sin\theta}{4\pi a}\,\hat{\boldsymbol{\phi}}, \tag{2}$$

in a spherical coordinate system (r, θ, ϕ) centered on the sphere. This current distribution is the same as that of a sphere with uniform magnetization density $\mathbf{M}_0 = M_0\,\hat{\mathbf{z}}$, where[2]

$$M_0 = \frac{K_0}{c\sin\theta} = \frac{Q\omega}{4\pi ac}\,, \tag{3}$$

in Gaussian units, and so the interior of the sphere has a uniform magnetic field[3]

$$\mathbf{B}_0 = \frac{8\pi\mathbf{M}_0}{3} = \frac{2Q\omega}{3ac}\,\hat{\mathbf{z}}. \tag{4}$$

[1] See, for example, prob. 3 of http://puhep1.princeton.edu/~mcdonald/examples/ph501set4.pdf

[2] See, for example, prob. 12(a) of http://puhep1.princeton.edu/~mcdonald/examples/ph501set4.pdf

[3] See, for example, p. 98 of http://puhep1.princeton.edu/~mcdonald/examples/ph501/ph501lecture8.pdf

As a result, the conduction electrons (with bulk charge density $\rho^-(r,\theta) \neq -\rho_0$ in the second approximation) experience a Lorentz force density

$$\mathbf{f}_B = \rho^- \frac{\mathbf{v}}{c} \times \mathbf{B}_0 = \rho^- \frac{\boldsymbol{\omega} \times \mathbf{r}}{c} \times \mathbf{B}_0 = \frac{\omega \rho^- B_0}{c} \mathbf{r}_\perp = \frac{2\rho^- Q \omega^2}{3ac^2} \mathbf{r}_\perp \tag{5}$$

To maintain their circular motion, the conduction electrons of charge $-e$, mass m and mass density $\rho_m = -m\rho^-/e$ must be subject to a total inward force density

$$\mathbf{f}_{\rm total} = -\rho_m \omega^2 \mathbf{r}_\perp = \frac{m\rho^- \omega^2}{e} \mathbf{r}_\perp \tag{6}$$

In general, the force densities (5) and (6) are not equal, so there must be an electric field $\mathbf{E}$ present such that

$$\mathbf{f}_{\rm total} = \frac{m\rho^- \omega^2}{e} \mathbf{r}_\perp = \rho^- \mathbf{E} + \mathbf{f}_B = \rho^- \mathbf{E} + \frac{2\rho^- Q \omega^2}{3ac^2} \mathbf{r}_\perp. \tag{7}$$

Hence, the electric field in the interior of the sphere must be

$$\mathbf{E} = \left(\frac{mc^2}{e} - \frac{2Q}{3a} \right) \frac{\omega^2}{c^2} \mathbf{r}_\perp. \tag{8}$$

If we write the total charge as

$$Q = Ne, \tag{9}$$

and introduce the classical electron radius $r_0 = e^2/mc^2 = 2.8 \times 10^{-13}$ cm, then we require that

$$\mathbf{E} = e \left(\frac{1}{a^2 r_0} - \frac{2N}{3a^3} \right) \frac{\omega^2 a^2}{c^2} \mathbf{r}_\perp \equiv A \mathbf{r}_\perp. \tag{10}$$

For a sphere of radius $a = 1$ cm, the required electric field will be positive or negative depending on whether N is smaller or larger than 10^{13}.

The total charge density $\rho = \rho_0 + \rho^-$ is related to the electric field according to[4]

$$\rho = \frac{\boldsymbol{\nabla} \cdot \mathbf{E}}{4\pi} = \frac{1}{4\pi r_\perp} \frac{\partial (r_\perp E_{r_\perp})}{\partial r_\perp} = \frac{A}{2\pi} \, . \tag{11}$$

Hence, the charge density of conduction electrons is

$$\rho^- = -\rho_0 + \rho = -\rho_0 + \frac{e}{2\pi} \left(\frac{1}{a^2 r_0} - \frac{2N}{3a^3} \right) \frac{\omega^2 a^2}{c^2} \, . \tag{12}$$

The bulk charge density ρ_0 is of order 10^{24} electrons/cm^3, so for, say, $N \approx 1$ Coulomb $\approx 10^{19}$ electrons, and $\omega a = 1$ cm/s the difference between ρ^- and $-\rho_0$ is $\approx 10^{-26} \rho_0$.

The total charge in the interior of the sphere is now

$$Q_{\rm in} = \frac{4\pi a^3 \rho}{3} = \frac{2a^3 A}{3} = \frac{2e}{3} \left(\frac{a}{r_0} - \frac{2N}{3} \right) \frac{\omega^2 a^2}{c^2} \, . \tag{13}$$

For the numerical example above, this is less than one electron.

This problem was suggested by Dragan Redzic.

[4]That a uniform bulk charge density ρ leads to an electric field in the $\mathbf{r}_\perp$ direction, rather than along $\mathbf{r}$, is possible as there is a small change to the surface charge density $\sigma(\theta)$ which breaks the spherical symmetry of the problem.