

# Second-Order Paraxial Gaussian Beam

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Many discussions of Gaussian beams emphasize a single electric field component, such as  $E_y = f(r, z)e^{i(kz - \omega t)}$ , of a cylindrically symmetric beam of angular frequency  $\omega$  and wave number  $k = n\omega/c$  propagating along the  $z$  axis in a medium with index of refraction  $n$ . Here, we generalize to the case of a beam with an elliptical cross section. Of course, the electric field must satisfy the free-space Maxwell equation  $\nabla \cdot \mathbf{E} = 0$ . If  $f(r, z)$  is not constant and  $E_x = 0$ , then we must have nonzero  $E_z$ . That is, the desired electric field has more than one vector component.

To deduce all components of the electric and magnetic fields of a Gaussian beam from a single scalar wave function, we follow the suggestion of Davis [2] and seek solutions for a vector potential  $\mathbf{A}$  that has only a single Cartesian component (such that  $(\nabla^2 \mathbf{A})_j = \nabla^2 A_j$  [4]). We work in the Lorenz gauge (and SI units), so that the electric scalar potential  $\Phi$  is related to the vector potential  $\mathbf{A}$  by

$$\nabla \cdot \mathbf{A} = -\frac{n^2}{c^2} \frac{\partial \Phi}{\partial t} = i \frac{n^2 \omega}{c^2} \Phi = i \frac{k^2}{\omega} \Phi. \quad (1)$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component.

Of course, the electric and magnetic fields can be deduced from the potentials via

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = i \frac{\omega}{k^2} \nabla (\nabla \cdot \mathbf{A}) + i\omega \mathbf{A}, \quad (2)$$

using the Lorenz condition (1), and

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3)$$

The vector potential satisfies the free-space (Helmholtz) wave equation,

$$\nabla^2 \mathbf{A} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = (\nabla^2 + k^2) \mathbf{A} = 0. \quad (4)$$

We seek a solution in which the vector potential is described by a single Cartesian component  $A_j$  that propagates in the  $+z$  direction with the form

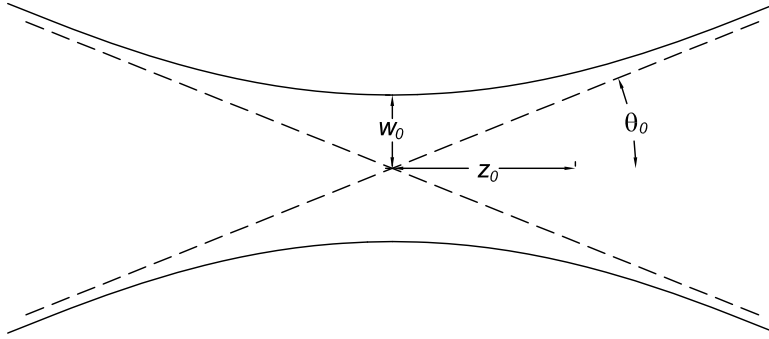
$$A_j(\mathbf{r}) = \psi(\mathbf{r})e^{i(kz - \omega t)}. \quad (5)$$

Inserting trial solution (5) into the wave equation (4) we find that

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z} = 0. \quad (6)$$

In the usual analysis, one now assumes that the beam is cylindrically symmetric about the  $z$  axis and can be described in terms of three geometric parameters the diffraction angle  $\theta_0$ , the waist  $w_0$ , and the depth of focus (Rayleigh range)  $z_0$ , which are related by

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \quad \text{and} \quad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}. \quad (7)$$



Changing variables and noting relations (7), eq. (6) takes the form

$$\nabla_{\perp}^2 \psi + 4i \frac{\partial \psi}{\partial \zeta} + \theta_0^2 \frac{\partial^2 \psi}{\partial \zeta^2} = 0, \quad (8)$$

where

$$\nabla_{\perp}^2 \psi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial v^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right), \quad (9)$$

since  $\psi$  is assumed to be independent of the azimuth  $\phi$ .

The form of eq. (8) suggests the series expansion

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + \dots \quad (10)$$

in terms of the small parameter  $\theta_0^2$ . Inserting this into eq. (8) and collecting terms of order  $\theta_0^0$  and  $\theta_0^2$ , we find

$$\nabla_{\perp}^2 \psi_0 + 4i \frac{\partial \psi_0}{\partial \zeta} = 0, \quad (11)$$

and

$$\nabla_{\perp}^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \zeta} = -\frac{\partial^2 \psi_0}{\partial \zeta^2}, \quad (12)$$

*etc.*

## 1 Zeroth-Order Gaussian Beam

Equation (11) is called the the **paraxial** wave equation, whose solution we obtain by an “educated guess”. Namely, we expect the transverse behavior of the wave function  $\psi_0$  to be Gaussian, but with a width that varies with  $z$ . Also, the amplitude of the wave should vary with  $z$ , asymptotically falling as  $1/z$ . We work in the scaled coordinates  $\rho$  and  $\zeta$ , and write a trial solution as

$$\psi_0 = h(\zeta) e^{-f(\zeta) \rho^2}, \quad (13)$$

where the possibly complex functions  $f$  and  $h$  are defined to obey  $f(0) = 1 = h(0)$ . Since the transverse coordinate  $\rho$  is scaled by the waist  $w_0$ , we see that  $Re(f) = w_0^2/w^2(\zeta)$  where  $w(\zeta)$  is the beam width at position  $\zeta$ . From the geometric parameters (6) we see  $w(\zeta) \approx \theta_0 z = w_0 \zeta$  for large  $\zeta$ . Hence, we expect that  $Re(f) \approx 1/\zeta^2$  for large  $\zeta$ . Also, we expect the amplitude  $h$  to obey  $|h| \approx 1/\zeta$  for large  $\zeta$ .

Plugging the trial solution (13) into the paraxial wave equation (11) we find that

$$-fh + ih' + \rho^2 h(f^2 - if') = 0. \quad (14)$$

For this to be true at all values of  $\rho$  we must have

$$\frac{f'}{f^2} = -i, \quad \text{and} \quad \frac{h'}{fh} = -i. \quad (15)$$

We see that  $f = h$  is a solution – despite the different physical origin of these two functions as the transverse width and amplitude of the wave. We integrate the first of eq. (15) to obtain

$$\frac{1}{f} = C + i\zeta. \quad (16)$$

Our definition  $f(0) = 1$  determines that  $C = 1$ . That is,

$$f = \frac{1}{1 + i\zeta} = \frac{1 - i\zeta}{1 + \zeta^2} = \frac{e^{-i \tan^{-1} \zeta}}{\sqrt{1 + \zeta^2}}. \quad (17)$$

Note that  $Re(f) = 1/(1 + \zeta^2) = w_0^2/w^2(\zeta)$ , while  $|f| = 1/\sqrt{1 + \zeta^2}$ , so that  $f = h$  is in fact consistent with the asymptotic expectations discussed above. The longitudinal dependence of the width of the Gaussian beam is now seen to be

$$w(\zeta) = w_0 \sqrt{1 + \zeta^2}. \quad (18)$$

The lowest-order wave function is

$$\psi_0 = f e^{-f\rho^2} = \frac{e^{-i \tan^{-1} \zeta}}{\sqrt{1 + \zeta^2}} e^{-\rho^2/(1+\zeta^2)} e^{i\zeta\rho^2/(1+\zeta^2)}. \quad (19)$$

The factor  $e^{-i \tan^{-1} \zeta}$  in  $\psi_0$  is the so-called Gouy phase shift, which changes from 0 to  $\pi/2$  as  $z$  varies from 0 to  $\infty$ , with the most rapid change near the  $z_0$ . For large  $z$  the phase factor  $e^{i\zeta\rho^2/(1+\zeta^2)}$  can be written  $e^{ikr_\perp^2/(2z)}$ , recalling eq. (6). When this is combined with the traveling wave factor  $e^{i(kz - \omega t)}$  we have

$$e^{i[kz(1+r_\perp^2/2z^2) - \omega t]} \approx e^{i(kr - \omega t)}, \quad (20)$$

where  $r = \sqrt{z^2 + r_\perp^2}$ . Thus, the wave function  $\psi_0$  is a modulated spherical wave for large  $z$ , but is a modulated plane wave near the focus.

## 2 Second-Order Gaussian Beam

The solution to eq. (12) for  $\psi_2$  has been given in [2], and that for  $\psi_4$  has been discussed in [3].

In particular,

$$\psi_2 = \left( \frac{f}{2} - \frac{f^3 \rho^4}{4} \right) \psi_0 = \left( \frac{f^2}{2} - \frac{f^4 \rho^4}{4} \right) e^{-f\rho^2}. \quad (21)$$

We now verify that the form (22) satisfies eq. (12). First,

$$\begin{aligned}
-\frac{\partial^2 \psi_0}{\partial \zeta^2} &= -\frac{\partial^2}{\partial \zeta^2} f e^{-f\rho^2} = -\frac{\partial}{\partial \zeta} f'(1 - f\rho^2) e^{-f\rho^2} = i \frac{\partial}{\partial \zeta} (f^2 - f^3 \rho^2) e^{-f\rho^2} \\
&= i f' [2f - 3f^2 \rho^2 - \rho^2 (f^2 - f^3 \rho^2)] e^{-f\rho^2} \\
&= (2f^3 - 4f^4 \rho^2 + f^5 \rho^4) e^{-f\rho^2},
\end{aligned} \tag{22}$$

recalling eq. (15). Next,

$$\begin{aligned}
\frac{\partial^2 \psi_2}{\partial \xi^2} &= \frac{\partial^2}{\partial \xi^2} \left( \frac{f^2}{2} - \frac{f^4 \rho^4}{4} \right) e^{-f\rho^2} = \frac{\partial}{\partial \xi} \xi \left( -f^3 + \frac{f^5 \rho^4}{2} - f^4 \rho^2 \right) e^{-f\rho^2} \\
&= \left[ -f^3 + \frac{f^5 \rho^4}{2} - f^4 \rho^2 + \xi^2 (4f^5 \rho^2 - f^6 \rho^4) \right] e^{-f\rho^2}
\end{aligned} \tag{23}$$

Hence,

$$\nabla_{\perp}^2 \psi_2 = (-2f^3 + 5f^5 \rho^4 - 2f^4 \rho^2 - f^6 \rho^4) e^{-f\rho^2} \tag{24}$$

Finally,

$$\begin{aligned}
4i \frac{\partial \psi_2}{\partial \zeta} &= 4i \frac{\partial}{\partial \zeta} \left( \frac{f^2}{2} - \frac{f^4 \rho^4}{4} \right) e^{-f\rho^2} = 4i f' \left( f - f^3 \rho^4 - \frac{f^2 \rho^2}{2} + \frac{f^4 \rho^4}{4} \right) e^{-f\rho^2} \\
&= (4f^3 - 4f^5 \rho^4 - 2f^4 \rho^2 + f^6 \rho^4) e^{-f\rho^2}
\end{aligned} \tag{25}$$

Thus,

$$\nabla_{\perp}^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \zeta} = (2f^3 + f^5 \rho^4 - 4f^4 \rho^2) e^{-f\rho^2} = -\frac{\partial^2 \psi_0}{\partial \zeta^2}. \tag{26}$$

### 3 Third-Order Electric and Magnetic Fields

To obtain the electric and magnetic fields of a second-order Gaussian beam that is polarized in the  $y$  direction we take the vector potential to be

$$A_x = 0, \quad A_y = \frac{E_0}{i\omega} (\psi_0 + \theta_0^2 \psi_2) e^{i(kz - \omega t)} = \frac{E_0}{i\omega} \left[ f + \theta_0^2 \left( \frac{f^2}{2} - \frac{f^4 \rho^4}{4} \right) \right] e^{-f\rho^2} e^{i(kz - \omega t)}, \quad A_z = 0. \tag{27}$$

Then,

$$i \frac{\omega}{k^2} \nabla \cdot \mathbf{A} = -\frac{E_0 \theta_0^2 y}{4} \left[ 2f^2 + \theta_0^2 \left( f^3 - \frac{f^5 \rho^4}{4} - f^4 \rho^2 \right) \right] e^{-f\rho^2} e^{i(kz - \omega t)} \approx -\frac{E_0 \theta_0^2 y}{2} f^2 e^{-f\rho^2} e^{i(kz - \omega t)}. \tag{28}$$

and the electric field follows from eq. (2) as

$$\begin{aligned}
E_x &\approx \theta_0^2 \frac{xy}{w_0^2} f^2 E_{0y}, \\
E_y &\approx \left[ 1 + \theta_0^2 \left( \frac{f^2 y^2}{w_0^2} - \frac{f^3 \rho^4}{4} \right) \right] E_{0y} \quad \text{where} \quad E_{0y} = E_0 f e^{-f\rho^2} e^{i(kz - \omega t)}, \\
E_z &\approx -i\theta_0 \frac{y}{w_0} \left[ f + \theta_0^2 \left( 1 - \frac{f\rho^2}{2} \right) \right] E_{0y},
\end{aligned} \tag{29}$$

where we neglect terms of order  $\theta_0^4$ , and note that  $f' = -if^2/z_0 = -if^2k\theta_0^2/2 = -if^2\theta_0/w_0$ . Similarly, the magnetic field follows from eq. (3) as

$$\begin{aligned}
 B_x &= -\left(1 - \theta_0^2 \frac{f^3 \rho^4}{4}\right) \frac{n}{c} E_{0y}, \\
 B_y &= 0, \\
 B_z &= i\theta_0 \frac{x}{w_0} f \left[1 + \theta_0^2 \left(\frac{f}{2} - \frac{f^3 \rho^4}{4} + \frac{f^2 \rho^2}{2}\right)\right] \frac{n}{c} E_{0y}.
 \end{aligned}
 \tag{30}$$

## References

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