Distortionless Transmission Line

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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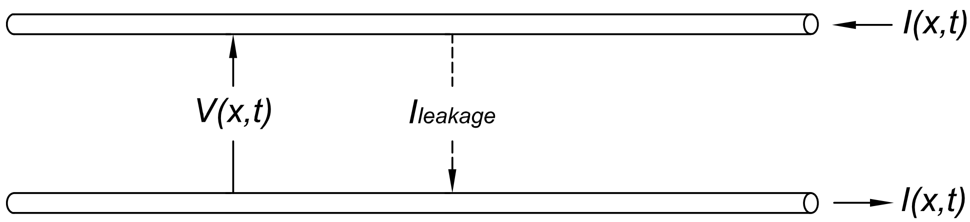
1 Problem

Deduce the differential equation for current (or voltage) in a two-conductor transmission line that is characterized by resistance R (summed over both conductors), inductance L, capacitance C and leakage conductivity K, all defined per unit length. The leakage conductivity describes the undesirable current that flows directly from one conductor to the other across the dielectric that separate them according to

$$I_{\rm leakage} = KV, \tag{1}$$

where $V(x,t)$ is the voltage between the two conductors, taken to be along the x axis.



Deduce a relation among R, L, C and K that permits 'distortionless' waves of the form

$$e^{-\gamma x} f(x - vt) \tag{2}$$

to propagate along the transmission line. Give expressions for v and γ in terms of R, L and C; relate γ to the transmission line impedance defined by $Z = V/I$ in the limit that R and K vanish.

2 Historical Note

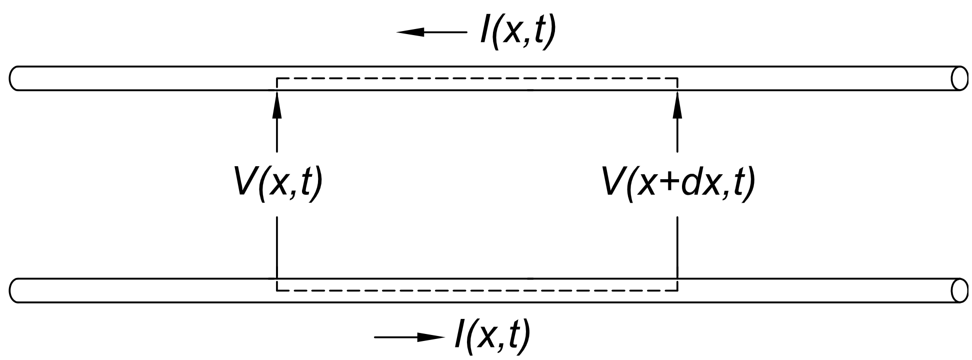
A wave equation for electromagnetic effects propagating along wires was first given by G. Kirchhoff [1] in 1857 in the context of Weber's electrodynamics, with the consequence that the speed of propagation in vacuum was $c/\sqrt{2}$, where c is the speed of light in vacuum. This speed was corrected to be c in a Maxwellian context by O. Heaviside in 1876 [2], but his argument is hard to follow.

The kind of derivation of the "telegrapher's equation" found in textbooks today was first given by Heaviside in 1887 [3], who argued that long-distance telegraph lines (including trans-Atlantic cables) should be designed to be "distortionless." Previous cables were fairly far from this ideal, being based on a theory of W Thomson (Lord Kelvin) [4] that ignored the effect of inductance (and so considered telegraphy a diffusion phenomenon rather than a wave phenomenon). However, long cables are expensive so there was considerable hesitation to abandon the large existing capital investment and implement the proposed improvements. Indeed, the editor of the journal that published Heaviside's papers was fired for being too sympathetic to Heaviside's views that were initially quite unpopular with industry. Heaviside, who was unemployed for most of his life, could not be fired! Large-scale implementation of "distortionless" telegraphy occured only after 1900 following vigorous advocacy by M. Pupin of the U.S.A., for whom the physics building of Columbia U. is named.

3 Solution

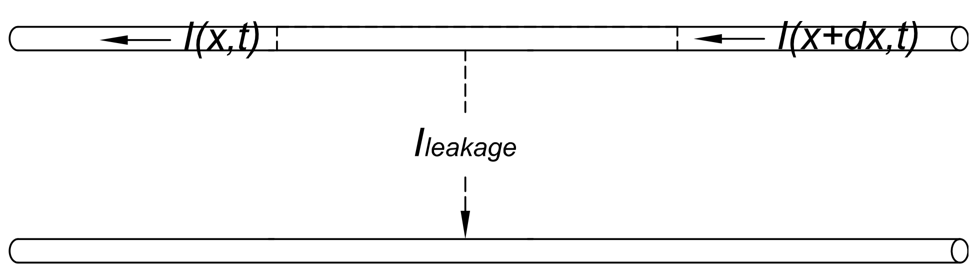
Referring to the sketch, Kirchoff's rule for the circuit of length dz shown by dashed lines tells us

$$V(x) - I(Rdx) - V(x+dx) - (Ldx)\frac{\partial I}{\partial t} = 0, \qquad \text{or} \qquad -\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t} + IR. \tag{3}$$



Next, the charge dQ that accumulates on length dx of the upper wire during time dt is $(Cdx)dV$ in terms of the change of voltage dV between the wires, which also can be written in terms of currents as

$$Q = (Cdx)dV = (I(x) - I(x+dx) - I_{\rm leakage})dt, \qquad \text{so} \qquad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t} + KV. \tag{4}$$



Together these imply the desired wave equation

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + KL)\frac{\partial I}{\partial t} + KRI. \tag{5}$$

We seek solutions of the form

$$I = e^{-\gamma x} f(x - vt), \tag{6}$$

for which

$$\frac{\partial I}{\partial t} = -ve^{-\gamma x} f', \qquad \text{and} \qquad \frac{\partial^2 I}{\partial t^2} = v^2 e^{-\gamma x} f'', \tag{7}$$

while

$$\frac{\partial I}{\partial x} = -\gamma e^{-\gamma x} f + e^{-\gamma x} f', \qquad \text{so} \qquad \frac{\partial^2 I}{\partial x^2} = \gamma^2 e^{-\gamma x} f - 2\gamma e^{-\gamma x} f' + e^{-\gamma x} f''. \tag{8}$$

Inserting these into the wave equation we find

$$\gamma^2 f - 2\gamma f' + f'' = v^2 LC f'' - v(RC + KL) f' + KRf. \tag{9}$$

This should be true for an arbitrary function f, so the coefficients of each derivative of f must separately be equal:

$$v = \sqrt{\frac{1}{LC}}, \qquad \gamma = \sqrt{KR}, \qquad \text{and} \qquad 2\frac{\gamma}{v} = RC + KL = 2\sqrt{RCKL}, \tag{10}$$

where we have used the first two relations in obtaining the second form of the third. In general, $a + b \neq 2\sqrt{ab}$; this only holds when $a = b$. So we deduce the desired condition

$$RC = KL, \tag{11}$$

for distortionless telegraphy.

With this condition, we can re-express γ as

$$\gamma = R\sqrt{\frac{C}{L}}. \tag{12}$$

Finally, we relate this to the impedance $Z = V/I$ when $R = 0 = K$. For this, we suppose that $V = V_0 f(x - vt)$ and $I = I_0 f$, where V_0 and I_0 can be related by either of the first-order differential equations above. We quickly find that $V_0 = vLI_0$, so $Z = \sqrt{L/C}$. Then,

$$\gamma = \frac{R}{Z}, \tag{13}$$

once we have arranged that $RC = KL$.

Remark: A typical cable has $RC \gg KL$. It costs a lot to reduce RC, although this was the direction of industry prior to Heaviside. He noted that one shouldn't even try to reduce leakage K, so long as the signal is not attenuated until it is undetectable – and the distortion-free condition makes it much easier to detect small signals. Rather one should increase the inductance, or leakage, or both! This counterintuitive result did not sit well with industry leaders, who, needless to say, were little guided by partial differential equations.

References

[1] G. Kirchhoff, *Ueber die Bewegung der Elektricität in Drähten*, Ann. Phys. Chim. **102**, 529 (1857),
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[2] O. Heaviside, *Electrical Papers*, Vol. 2, p. 82)),
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[4] W. Thomson, *On the Theory of the Electric Telegraph*, Proc. Roy. Soc. London **7**, 382 (1855), http://puhep1.princeton.edu/~mcdonald/examples/EM/thomson_prsl_7_382_55.pdf