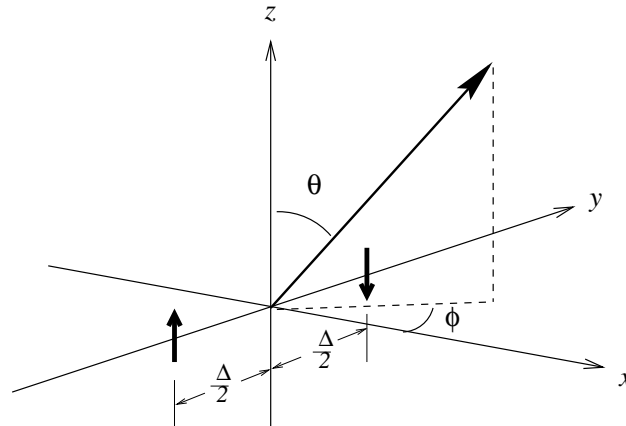


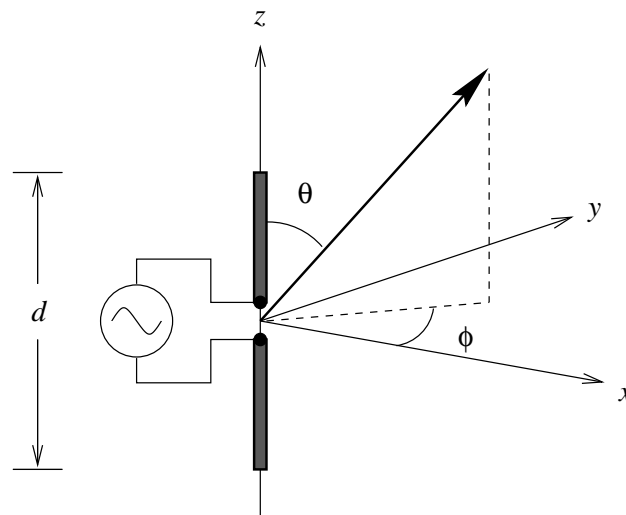
A Phased Antenna Array

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Two “short” dipole antennas form a small “phased array” as shown in the figure. The second dipole is placed a distance $\Delta = \lambda/2$ away from the first along the y axis. The two dipoles are parallel to one another and are driven 180° out of phase of one another.



Each antenna is a center-fed dipole radiator formed from two wires, each of length $d/2 \ll \lambda$ and driven by a current source as shown in the figure below. The wires are aligned parallel to the z axis ($\theta = (0, \pi)$). The current source produces a time-dependent current given by $I(t) = I_0 e^{-i\omega t}$. You may assume that the charge that enters the wires is uniformly distributed along their lengths.



Calculate the time averaged angular distribution of the radiated power for this arrangement in the radiation zone as a function of θ and ϕ , *i.e.*, calculate $\langle dP(\theta, \phi)/d\Omega \rangle$.

Solution

The amplitude A_0 of the radiation from a single antenna varies with angle as $\sin \theta$. The radiated power has angular distribution

$$P(\theta, \phi) \propto A_0^2 \sin^2 \theta. \quad (1)$$

We first discuss the angular distribution of the two-antenna system, then return to the issue of the normalization of eq. (1).

Consider 2 antennas separated in space by vector distance \mathbf{L} , and operated with phase difference $\delta\varphi_0$ between them. When viewed by a distant observer along direction $\hat{\mathbf{n}}$, the path length difference of the radiation of the two antennas to the observer is $\hat{\mathbf{n}} \cdot \mathbf{L}$. The total phase difference between the radiation from the two antennas is therefore

$$\delta\varphi = 2\pi \frac{\hat{\mathbf{n}} \cdot \mathbf{L}}{\lambda} + \delta\varphi_0. \quad (2)$$

The total amplitude of the radiation from the two antennas is

$$A = A_0 \sin \theta (1 + e^{i\delta\varphi}) = 2A_0 e^{i\delta\varphi/2} \sin \theta \cos \frac{\delta\varphi}{2}. \quad (3)$$

The radiated power is then

$$P \propto |A^2| = 4A_0^2 \sin^2 \theta \cos^2 \frac{\delta\varphi}{2} = 4A_0^2 \sin^2 \theta \cos^2 \left(\pi \frac{\hat{\mathbf{n}} \cdot \mathbf{L}}{\lambda} + \frac{\delta\varphi_0}{2} \right). \quad (4)$$

In the present example, $\mathbf{L} = \lambda \hat{\mathbf{y}}/2$, $\delta\varphi_0 = \pi$, and, of course, $\hat{\mathbf{n}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$, so that

$$P(\theta, \phi) \propto 4A_0^2 \sin^2 \theta \cos^2 \left(\frac{\pi}{2} + \frac{\pi}{2} \sin \theta \sin \phi \right) = 4A_0^2 \sin^2 \theta \sin^2 \left(\frac{\pi}{2} \sin \theta \sin \phi \right). \quad (5)$$

In the horizontal plane ($\theta = \pi/2$), the radiation pattern is

$$P(\phi) \propto 4A_0^2 \sin^2 \left(\frac{\pi}{2} \sin \phi \right). \quad (6)$$

This vanishes for $\phi = 0^\circ, 180^\circ$, and is maximal when $\phi = \pm 90^\circ$. That is, the radiation is preferential emitted along the y axis, the axis of the antenna array.

For completeness, we relate A_0^2 to the current I_0 by considerations of a single antenna. The total radiated power is given by an appropriate version of Larmor's formula (in Gaussian units):

$$P = \frac{2\ddot{p}^2}{3c^3} = \frac{2\omega^4 p^2}{3c^3}, \quad (7)$$

where p is the dipole moment and ω is the frequency. The time-averaged power is then

$$\langle P \rangle = \frac{\omega^4 p_0^2}{3c^3}, \quad (8)$$

The angular distribution varies as $\sin^2 \theta$, and so must be

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 p_0^2 \sin^2 \theta}{8\pi c^3}. \quad (9)$$

The hint is that the charge distribution $\rho(z, t) = \rho_0(z)e^{-i\omega t}$ is actually uniform in each wire of the antenna: $\rho_0(0 < z < d/2) = \rho_0 = \text{constant}$. Of course, $\rho_0(-z) = -\rho_0(z)$. The dipole moment is

$$p_0 = \int_{-d/2}^{d/2} \rho z dz = \frac{\rho_0 d^2}{4}. \quad (10)$$

To get ρ_0 , we must relate the charge to the current, which is usefully done via the continuity equation, $\nabla \cdot \mathbf{j} = -\dot{\rho}$. For our one-dimensional problem, this can be re-expressed as

$$\frac{\partial I}{\partial z} = -\dot{\rho} = i\omega \rho_0 e^{-i\omega t} \quad (0 < z < d/2), \quad (11)$$

which integrates to

$$I(z, t) = C(t) + i\rho_0 \omega z e^{-i\omega t} \quad (0 < z < d/2). \quad (12)$$

Now, $I(d/2) = 0$, so $C(t) = -i\rho_0 \omega (d/2) e^{-i\omega t}$, and

$$I(z, t) = -\frac{i\rho_0 \omega d}{2} \left(1 - \frac{2z}{d}\right) e^{-i\omega t} \quad (0 < z < d/2). \quad (13)$$

That is,

$$I_0 = -\frac{i\rho_0 \omega d}{2}, \quad \text{and} \quad \rho_0 = \frac{2I_0}{\omega d}. \quad (14)$$

The full expression for the current distribution is

$$I(z, t) = I_0 \left(1 - \frac{2|z|}{d}\right) e^{-i\omega t} \quad (-d/2 < z < d/2). \quad (15)$$

Combining eqs. (1), (9), (10), and (14), and noting that $\omega/c = k = 2\pi/\lambda$, we have

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 d^2 I_0^2 \sin^2 \theta}{32\pi c^3} = \frac{\pi I_0^2 \sin^2 \theta}{8c} \left(\frac{d}{\lambda}\right)^2 \equiv A_0^2 \sin^2 \theta. \quad (16)$$

Hence,

$$A_0^2 = \frac{\pi I_0^2}{8c} \left(\frac{d}{\lambda}\right)^2. \quad (17)$$

Inserting this in eq. (5), the time-averaged power radiated by the two antennas is

$$\left\langle \frac{dP(\theta, \phi)}{d\Omega} \right\rangle = \frac{\pi I_0^2}{2c} \left(\frac{d}{\lambda}\right)^2 \sin^2 \theta \sin^2 \left(\frac{\pi}{2} \sin \theta \sin \phi\right). \quad (18)$$