

Force-Free Magnetic Fields aka Eigenfunctions of the Curl Operator

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(August 10, 2011)

1 Problem

Deduce forms of a static magnetic field $\mathbf{B}(\mathbf{x})$ such that the Lorentz force density $\mathbf{J} \times \mathbf{B}$ on the associated current density \mathbf{J} is everywhere zero.^{1,2}

Assuming that the medium has permeability μ_0 (and that any electric field is also static), the current density is proportional to $\nabla \times \mathbf{B}$, so the Lorentz force vanishes if $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$, which obtains when

$$\nabla \times \mathbf{B} = f(\mathbf{x})\mathbf{B} \quad (1)$$

for any scalar function $f(\mathbf{x})$, noting that $\nabla \cdot \mathbf{B} = 0$. In particular, the function f can be a constant k , such that any (vector) eigenfunction of the curl operator is a possible force-free magnetic field.³

2 Solution

2.1 Cowling's Theorem

Force-free magnetic fields are a possible model of the magnetic fields of planets, stars and other astrophysical regions, which fields are observed to be quasistatic. The question of static, force-free magnetic fields seems to have been first considered by Cowling [5, 6], who concluded that they cannot exist if they are to be axially symmetric. This result is sometimes called Cowling's Theorem. A corollary is that the Earth's magnetic field is dynamic and/or nonaxisymmetric.

However, it appears that this theorem holds only with the additional assumption that the magnetic field has no azimuthal component B_ϕ [7], contrary to the claim of Cowling.

A static, force-free magnetic field has $\mathbf{J} \propto \nabla \times \mathbf{B} \propto \mathbf{B}$, so the magnetic field exists only where the current density \mathbf{J} is nonzero. Thus, there is no force-free magnetic field external to the current distribution, and such a field cannot apply to astrophysical objects such as the

¹There is no such thing as a force-free electric field, since force density $\rho\mathbf{E}$ on charge density ρ can be zero only if $\mathbf{E} = 0$ wherever $\rho \neq 0$, but the first Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ implies that \mathbf{E} is nonzero wherever the volume charge density ρ is nonzero.

²The conducting medium is subject to internal stresses described by the Maxwell stress tensor, $(1/\mu_0)(B_i B_j - \delta_{ij} B^2/2)$, which are always nonzero for nonzero \mathbf{B} and can lead to deformations of the medium even if the Lorentz force is small/zero [1].

³If the vector \mathbf{B} represents the velocity \mathbf{v} of an incompressible fluid, then condition (1) corresponds to so-called Beltrami flow (1889). Vectors that obey eq. (1) are sometimes called Trkalian (1919). See, for example, [2, 3, 4].

Earth and Sun that have external magnetic fields. Thus, the corollary of Cowling's theorem that the Earth's magnetic field is dynamic and/or nonaxisymmetric appears to be basically correct.⁴ However, the concept of a static, force-free magnetic field remains interesting in principle.

2.2 Lundquist's Solution

The first demonstration of a static, force-free magnetic field is due to Lundquist [9, 10],⁵ who considered eq. (1) with $f = k$ in cylindrical coordinates (ρ, ϕ, z) for fields with dependence only ρ ,

$$\frac{\partial B_z}{\partial \rho} = -k B_\phi, \quad (2)$$

$$\frac{1}{\rho} \frac{\partial(\rho B_\phi)}{\partial \rho} = k B_z. \quad (3)$$

A particular solution to eqs. (2)-(3) is

$$B_\rho = 0, \quad B_\phi = J_1(k\rho), \quad B_z = J_0(k\rho), \quad (4)$$

where J_0 and J_1 are Bessel functions. The field lines are helices [9], and since the Bessel functions are oscillatory in ρ there are both left- and righthanded helices, and ones with both positive and negative B_z . Such a complex field pattern seems somewhat unlikely to occur in Nature, but it is suggestive that other force-free forms exist as well.

2.3 Other Simple Force-Free Magnetic Fields

A force-free field that depends only on z in cylindrical coordinates must obey

$$\frac{\partial B_\phi}{\partial z} = -k B_\rho, \quad (5)$$

$$\frac{\partial B_\rho}{\partial z} = k B_\phi. \quad (6)$$

A particular solution to eqs. (5)-(6) is

$$B_\rho = \cos kz, \quad B_\phi = -\sin kz, \quad B_z = 0. \quad (7)$$

However, $\nabla \cdot \mathbf{B} = (1/\rho) \cos kz$, so eq. (7) cannot represent a magnetic field (contrary to a claim in sec. II(a) of [12]).

In rectangular coordinates a force-free field that depends only on z obeys

$$\frac{\partial B_y}{\partial z} = -k B_x, \quad (8)$$

$$\frac{\partial B_x}{\partial z} = k B_y. \quad (9)$$

⁴For a simplified discussion, see pp. 6-7 of [8].

⁵Equation (4) with \mathbf{B} interpreted as fluid velocity \mathbf{v} dates back to [11].

A particular solution to eqs. (8)-(9) is

$$B_x = \cos kz, \quad B_y = -\sin kz, \quad B_z = 0, \quad (10)$$

for which $\nabla \cdot \mathbf{B} = 0$. The lines of \mathbf{B} are straight in any plane of constant z , making angle $\phi = kz$ to the x -axis. As with the example in sec. 2.2, this is not a physically plausible field configuration.

In spherical coordinates (r, θ, ϕ) a force free field that depends only on r obeys

$$\frac{\partial(rB_\phi)}{\partial r} = -krB_\theta, \quad (11)$$

$$\frac{\partial(rB_\theta)}{\partial r} = krB_\phi. \quad (12)$$

A particular solution to eqs. (11)-(12) is

$$B_r = 0, \quad B_\theta = \frac{\cos kr}{r}, \quad B_\phi = -\frac{\sin kr}{r}, \quad (13)$$

but $\nabla \cdot \mathbf{B} = (1/r^2) \cot \theta \cos kr$, so that eq. (13) does not represent a magnetic field (contrary to a claim in sec. III(a) of [12]).

It appears that a more general method is needed to deduce the forms of additional force-free magnetic fields.

2.4 A General Solution

Considerations [13] subsequent to Lundquist's soon led to a general solution for force-free magnetic fields [14, 15, 16, 17].⁶ Taking the curl of eq. (1) with $f = k$, we have that

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = k^2 \mathbf{B}, \quad (14)$$

and hence, force-free magnetic fields are a subset of solutions to the vector Helmholtz equation,

$$(\nabla^2 + k^2)\mathbf{B} = 0. \quad (15)$$

A useful decomposition of solutions to the vector Helmholtz equation is due to Hansen [18] (see also sec. 7.1 of [19]), in which we write the field \mathbf{B} as a linear sum of three fields,

$$\mathbf{S} = \nabla\psi, \quad \mathbf{T} = \nabla \times \psi \mathbf{a} = \nabla\psi \times \mathbf{a}, \quad \text{and} \quad \mathbf{P} = \frac{1}{k} \nabla \times \mathbf{T}, \quad (16)$$

for any function ψ that obeys the scalar Helmholtz equation,

$$(\nabla^2 + k^2)\psi = 0, \quad (17)$$

where \mathbf{a} is either a constant vector or the position vector \mathbf{x} ($= r\hat{\mathbf{r}}$ in spherical coordinates (r, θ, ϕ)). The three fields \mathbf{S} , \mathbf{T} and \mathbf{P} have been named *scaloidal*, *toroidal* and *poloidal*,

⁶Independently, general solutions to eq. (1) with \mathbf{B} interpreted as fluid velocity \mathbf{v} has been developed by several authors, as summarized in [2, 3].

respectively, by Elasser [20].⁷ The scaloidal/irrotational term \mathbf{S} does not contribute to magnetic fields, which obey $\nabla \cdot \mathbf{B} = 0$, and we have that

$$\mathbf{B} = \mathbf{P} + \mathbf{T}. \quad (18)$$

Since \mathbf{T} obeys eq. (15), and $\nabla \cdot \mathbf{T} = 0$, it follows from eq. (16) that

$$\nabla \times \mathbf{P} = \frac{1}{k} \nabla \times (\nabla \times \mathbf{T}) = -\frac{1}{k} \nabla^2 \mathbf{T} = k\mathbf{T}, \quad \text{and} \quad \mathbf{T} = \frac{1}{k} \nabla \times \mathbf{P}, \quad (19)$$

and hence,

$$\nabla \times \mathbf{B} = \nabla \times \mathbf{P} + \nabla \times \mathbf{T} = k\mathbf{T} + k\mathbf{P} = k\mathbf{B}. \quad (20)$$

Thus, the form (18) is an eigenfunction of the curl operator, and is a force-free magnetic field.⁸

It remains to consider a general set of solutions ψ to the scalar Helmholtz wave equation (17), which has separable solutions in 11 coordinate systems [23]. Here, we consider the basic three.^{9,10}

2.4.1 Solution in Rectangular Coordinates

Solutions to the scalar Helmholtz wave equation (17) in rectangular coordinates have the form of plane waves

$$\psi = e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (22)$$

where the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ can have complex components, so long as $k^2 = k_x^2 + k_y^2 + k_z^2$. Then,

$$\nabla \psi = i\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (23)$$

and the toroidal component of the force-free magnetic field can be taken as

$$\mathbf{T} = \nabla \psi \times \mathbf{x} = i\mathbf{k} \times \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (24)$$

from which we obtain the poloidal component as

$$\begin{aligned} \mathbf{P} &= \frac{1}{k} \nabla \times \mathbf{T} = \nabla \times (i\hat{\mathbf{k}} \times \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}}) = i\hat{\mathbf{k}} \nabla \cdot (\mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}}) - i(\hat{\mathbf{k}} \cdot \nabla) \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \\ &= 3i\hat{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} - \hat{\mathbf{k}}(\mathbf{k} \cdot \mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} - i\hat{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + k\mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}}. \end{aligned} \quad (25)$$

Thus, a force-free magnetic field can be written as

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = [2i\hat{\mathbf{k}} - (\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{k}} + k\mathbf{x} + i\mathbf{k} \times \mathbf{x}] e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (26)$$

⁷Equation (16) is a variant on the Helmholtz decomposition of any vector field (see, for example, [21]), in which \mathbf{S} corresponds to the irrotational part, and $\mathbf{P} + \mathbf{T}$ to the rotational part, of \mathbf{B} .

⁸A variant on the above is that for any magnetic field \mathbf{B}' that satisfies the vector Helmholtz equation (15), the field

$$\mathbf{B} = \mathbf{B}' + \frac{1}{k} \nabla \times \mathbf{B}' \quad (21)$$

is force free [22], which can be used to deduce time-dependent forms.

⁹For a solution in toroidal coordinates, see [24].

¹⁰For a different characterization of eigenfunctions of the curl operator, see [25].

For example, if $\mathbf{k} = (0, 0, k)$, then

$$\mathbf{B} = [2i\hat{\mathbf{z}} - kz\hat{\mathbf{z}} + k\mathbf{x} - ik_y\hat{\mathbf{x}} + ik_x\hat{\mathbf{y}}] e^{ikz} = [k(x - iy)\hat{\mathbf{x}} + k(y + ix)\hat{\mathbf{y}} + 2i\hat{\mathbf{z}}] e^{ikz}. \quad (27)$$

Alternatively, the toroidal component of the force-free magnetic field can be taken as

$$\mathbf{T} = \nabla\psi \times \mathbf{a} = i\mathbf{k} \times \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (28)$$

for any constant vector \mathbf{a} . In this case the poloidal component is

$$\mathbf{P} = \frac{1}{k}\nabla \times \mathbf{T} = \nabla \times (i\hat{\mathbf{k}} \times \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{x}}) = -\mathbf{k} \times (\hat{\mathbf{k}} \times \mathbf{a}) e^{i\mathbf{k}\cdot\mathbf{x}} = [k\mathbf{a} - (\mathbf{k} \cdot \mathbf{a})\hat{\mathbf{k}}] e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (29)$$

Thus, a force-free magnetic field can also be written as

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = [k\mathbf{a} - (\mathbf{k} \cdot \mathbf{a})\hat{\mathbf{k}} + i\mathbf{k} \times \mathbf{a}] e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (30)$$

For example, if $\mathbf{k} = (0, 0, k)$, then

$$\mathbf{B} = [k\mathbf{a} - ka_z\hat{\mathbf{z}} + ik\hat{\mathbf{z}} \times \mathbf{a}] e^{ikz}. \quad (31)$$

With $\mathbf{a} = \hat{\mathbf{x}}/k$ we obtain

$$\mathbf{B} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{ikz}, \quad (32)$$

whose real part is the form (10).

2.4.2 Solution in Cylindrical Coordinates

In cylindrical coordinates (ρ, ϕ, z) , solutions to the Helmholtz equation (17) that are finite on the z -axis can be written (see, for example, sec. 7.1 of [19])

$$\psi_n = J_n(k_\rho\rho) e^{i(k_z z + n\phi)}, \quad (33)$$

where n is a non-negative integer, J_n is a Bessel function and $k_\rho^2 + k_z^2 = k^2$. Then,

$$\nabla\psi_n = \frac{dJ_n(k_\rho\rho)}{d\rho} e^{i(k_z z + n\phi)} \hat{\boldsymbol{\rho}} + \frac{in}{\rho} J_n(k_\rho\rho) e^{i(k_z z + n\phi)} \hat{\boldsymbol{\phi}} + ik_z J_n(k_\rho\rho) e^{i(k_z z + n\phi)} \hat{\mathbf{z}}. \quad (34)$$

We consider only the choice of $\mathbf{a} = \hat{\mathbf{z}}/k$ in eq. (16), such that

$$\mathbf{T}_n = -\frac{in}{k\rho} J_n(k_\rho\rho) e^{i(k_z z + n\phi)} \hat{\boldsymbol{\rho}} + \frac{dJ_n(k_\rho\rho)}{kd\rho} e^{i(k_z z + n\phi)} \hat{\boldsymbol{\phi}}, \quad (35)$$

and

$$\mathbf{P}_n = -\frac{ik_z}{k^2} \frac{dJ_n(k_\rho\rho)}{d\rho} e^{i(k_z z + n\phi)} \hat{\boldsymbol{\rho}} + \frac{k_z n}{k^2 \rho} J_n(k_\rho\rho) e^{i(k_z z + n\phi)} \hat{\boldsymbol{\phi}} - \frac{k_\rho^2}{k^2} J_n(k_\rho\rho) e^{i(k_z z + n\phi)} \hat{\mathbf{z}}, \quad (36)$$

noting that Bessel's equation has the form

$$\frac{d}{d\rho} \left[\rho \frac{dJ_n(k_\rho\rho)}{d\rho} \right] = \left(\frac{n^2}{\rho} - k_\rho^2 \rho \right) J_n(k_\rho\rho). \quad (37)$$

Of, course, the force-free magnetic field has the form¹¹

$$\mathbf{B}_n = \mathbf{P}_n + \mathbf{T}_n. \quad (38)$$

For example,

$$\psi_0 = J_0(k_\rho \rho) e^{ik_z z}, \quad (39)$$

$$\mathbf{B}_0 = \frac{ik_\rho k_z}{k^2} J_1(k_\rho \rho) e^{ik_z z} \hat{\boldsymbol{\rho}} - \frac{k_\rho}{k} J_1(k_\rho \rho) e^{ik_z z} \hat{\boldsymbol{\phi}} - \frac{k_\rho^2}{k^2} J_0(k_\rho \rho) e^{ik_z z} \hat{\mathbf{z}}. \quad (40)$$

In particular, if $k_z = 0$ then $k_\rho = k$ and we obtain (to within a minus sign) the form (4),

$$\mathbf{B}_0(k_z = 0) = J_1(k\rho) \hat{\boldsymbol{\phi}} + J_0(k\rho) \hat{\mathbf{z}}, \quad (41)$$

as found by Lundquist [9].

2.4.3 Solution in Spherical Coordinates

In spherical coordinates (r, θ, ϕ) , solutions to the scalar Helmholtz equation (17) can be written in various ways, as discussed in sec. 7.3 of [19], sec. 9.6 of [27], *etc.* A form that is finite at the origin and on the z -axis is

$$\psi_n^m = j_n(kr) P_n^m(\cos \theta) e^{im\phi}, \quad (42)$$

m and n are integers, $n \geq 0$, $|m| \leq n$, j_n is a so-called spherical Bessel function,

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2(x) = \left(\frac{3}{x^2} - \frac{1}{x} \right) \sin x - \frac{3 \cos x}{x^2}, \quad \dots, \quad (43)$$

and $P_n^m(y)$ is an associated Legendre function,

$$P_0^0(y) = 1, \quad P_1^0(y) = y, \quad P_1^{\pm 1}(y) = \pm \sqrt{1 - y^2}, \quad P_2^0 = \frac{3y^2 - 1}{2}, \quad \dots \quad (44)$$

Then,

$$\begin{aligned} \nabla \psi_n^m &= \frac{\partial \psi_n^m}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi_n^m}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial \psi_n^m}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \frac{dj_n(kr)}{dr} P_n^m(\cos \theta) e^{im\phi} \hat{\mathbf{r}} + \frac{j_n(kr)}{r} \frac{dP_n^m(\cos \theta)}{d\theta} e^{im\phi} \hat{\boldsymbol{\theta}} + \frac{im}{r \sin \theta} j_n(kr) P_n^m(\cos \theta) e^{im\phi} \hat{\boldsymbol{\phi}}. \end{aligned} \quad (45)$$

$\mathbf{a} = r \hat{\mathbf{r}}$

We consider first the choice of $\mathbf{a} = \mathbf{x} = r \hat{\mathbf{r}}$ in eq. (16), such that [16, 28, 29]

$$\mathbf{T}_n^m = \frac{im}{\sin \theta} j_n(kr) P_n^m(\cos \theta) e^{im\phi} \hat{\boldsymbol{\theta}} - j_n(kr) \frac{dP_n^m(\cos \theta)}{d\theta} e^{im\phi} \hat{\boldsymbol{\phi}}, \quad (46)$$

¹¹The forms (35)-(36) and (38) are often called the Chandrasekhar-Kendall eigenfunctions, although they were not explicitly displayed in [16]. They form a complete set of eigenfunctions of the curl operator [26].

and

$$\begin{aligned} \mathbf{P}_n^m &= \frac{n(n+1)}{kr} j_n(kr) P_n^m(\cos \theta) e^{im\phi} \hat{\mathbf{r}} + \frac{1}{kr} \frac{d[rj_n(kr)]}{dr} \frac{dP_n^m(\cos \theta)}{d\theta} e^{im\phi} \hat{\boldsymbol{\theta}} \\ &+ \frac{im}{kr \sin \theta} \frac{d[rj_n(kr)]}{dr} P_n^m(\cos \theta) e^{im\phi} \hat{\boldsymbol{\phi}}, \end{aligned} \quad (47)$$

noting that the associated Legendre functions obey the differential equation

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP_n^m(\cos \theta)}{d\theta} \right) = \left(\frac{m^2}{\sin^2 \theta} - n(n+1) \right) P_n^m(\cos \theta). \quad (48)$$

Of course, the force-free magnetic fields are

$$\mathbf{B}_n^m = \mathbf{P}_n^m + \mathbf{T}_n^m. \quad (49)$$

For example,

$$\psi_0^0 = \frac{\sin kr}{kr}, \quad \mathbf{B}_0^0 = 0, \quad (50)$$

$$\psi_1^0 = \left(\frac{\sin kr}{k^2 r^2} - \frac{\cos kr}{kr} \right) \cos \theta, \quad (51)$$

$$\begin{aligned} \mathbf{B}_1^0 &= 2 \left(\frac{\sin kr}{k^3 r^3} - \frac{\cos kr}{k^2 r^2} \right) \cos \theta \hat{\mathbf{r}} - \left[\frac{\sin kr}{kr} \left(1 - \frac{1}{k^2 r^2} \right) + \frac{\cos kr}{k^2 r^2} \right] \sin \theta \hat{\boldsymbol{\theta}} \\ &+ \left(\frac{\sin kr}{k^2 r^2} - \frac{\cos kr}{kr} \right) \sin \theta \hat{\boldsymbol{\phi}}. \end{aligned} \quad (52)$$

For small r , such that $kr \ll 1$,

$$\mathbf{B}_1^0(kr \ll 1) \approx \frac{2}{3} (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}) - \frac{kr \sin \theta}{3} \hat{\boldsymbol{\phi}} = \frac{2}{3} \hat{\mathbf{z}} - \frac{kr \sin \theta}{3} \hat{\boldsymbol{\phi}}. \quad (53)$$

$\mathbf{a} = \hat{\mathbf{z}}$

We can also consider that $\mathbf{a} = \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$ in eq. (16) [3], for which

$$\mathbf{T} = \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\mathbf{r}} + \frac{\cot \theta}{r} \frac{\partial \psi}{\partial \phi} \hat{\boldsymbol{\theta}} - \left(\sin \theta \frac{\partial \psi}{\partial r} + \frac{\cos \theta}{r} \frac{\partial \psi}{\partial \theta} \right) \hat{\boldsymbol{\phi}}, \quad (54)$$

and

$$\begin{aligned} \mathbf{P} &= -\frac{1}{kr \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin 2\theta \frac{\partial \psi}{\partial r} + \frac{\sin \theta \cos \theta}{r} \frac{\partial \psi}{\partial \theta} \right) + \frac{\cot \theta}{r} \frac{\partial^2 \psi}{\partial \phi^2} \right] \hat{\mathbf{r}} \\ &+ \frac{1}{kr} \left[\frac{1}{r \sin \theta} \frac{\partial^2 \psi}{\partial \phi^2} + \sin \theta \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \cos \theta \frac{\partial^2 \psi}{\partial r \partial \theta} \right] \hat{\boldsymbol{\theta}} \\ &+ \frac{1}{kr} \left[\cot \theta \frac{\partial^2 \psi}{\partial r \partial \phi} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial \phi} \right] \hat{\boldsymbol{\phi}}. \end{aligned} \quad (55)$$

For the case of no azimuthal dependence, $\partial\psi/\partial\phi = 0$, the force-free magnetic field has the form

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = \frac{1}{kr^2 \sin\theta} \frac{\partial\Psi}{\partial\theta} \hat{\mathbf{r}} - \frac{1}{kr \sin\theta} \frac{\partial\Psi}{\partial r} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \Psi \hat{\boldsymbol{\phi}}, \quad (56)$$

where¹²

$$\Psi = - \left(r \sin^2\theta \frac{\partial\psi}{\partial r} + \sin\theta \cos\theta \frac{\partial\psi}{\partial\theta} \right) = -\rho \frac{\partial\psi}{\partial\rho}, \quad (58)$$

with $\rho = r \sin\theta$. Then, since $(\nabla \times \mathbf{B})_\phi = kB_\phi$, the auxiliary function Ψ obeys the differential equation

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial\Psi}{\partial\theta} \right) + k^2\Psi = 0. \quad (59)$$

For example,

$$\Psi_0 = \frac{\sin kr}{k}, \quad \mathbf{B}_0 = -\frac{\cos kr}{kr \sin\theta} \hat{\boldsymbol{\theta}} + \frac{\sin kr}{kr \sin\theta} \hat{\boldsymbol{\phi}}, \quad (60)$$

$$\Psi_1 = \frac{\sin kr}{k} \cos\theta, \quad \mathbf{B}_1 = -\frac{\sin kr}{k^2 r^2} \hat{\mathbf{r}} - \frac{\cos kr}{kr} \cot\theta \hat{\boldsymbol{\theta}} + \frac{\sin kr}{kr} \cot\theta \hat{\boldsymbol{\phi}}, \quad (61)$$

$$\Psi_2 = \left(\frac{\sin kr}{k^2 r^2} - \frac{\cos kr}{kr} \right) \sin^2\theta, \quad (62)$$

$$\begin{aligned} \mathbf{B}_2 = & 2 \left(\frac{\sin kr}{k^3 r^3} - \frac{\cos kr}{k^2 r^2} \right) \cos\theta \hat{\mathbf{r}} - \left[\frac{\sin kr}{kr} \left(1 - \frac{1}{k^2 r^2} \right) + \frac{\cos kr}{k^2 r^2} \right] \sin\theta \hat{\boldsymbol{\theta}} \\ & + \left(\frac{\sin kr}{k^2 r^2} - \frac{\cos kr}{kr} \right) \sin\theta \hat{\boldsymbol{\phi}} = \mathbf{B}_1^0. \end{aligned} \quad (63)$$

Note that \mathbf{B}_0 and \mathbf{B}_1 are infinite on the z -axis, which reminds us that the P_n^m in eq. (42) could also be the associated Legendre functions of the second kind, Q_n^m .¹³

The fields obtained using $\mathbf{a} = r \hat{\mathbf{r}}$ are not independent of those found using $\mathbf{a} = \hat{\mathbf{z}}$. It is shown in [29] that the former set of fields is complete.

2.5 Exponential Decay of a Force-Free Magnetic Field

The fourth Maxwell equation relates the curl of the magnetic field to the conduction current \mathbf{J} and the so-called displacement current $\epsilon_0 \partial\mathbf{E}/\partial t$,

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial\mathbf{E}}{\partial t} \right). \quad (64)$$

¹²The function Ψ is akin to a stream function in fluid dynamics, as discussed in secs. 4.5 and 5.1 of [2]. Of course, $\Psi = -\rho \partial\psi/\partial\rho$ can also be introduced in cylindrical coordinates (sec. 2.4.2) in case of azimuthal symmetry, for which

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = \frac{1}{k\rho} \frac{\partial\Psi}{\partial z} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \Psi \hat{\boldsymbol{\phi}} + \frac{1}{k\rho} \frac{\partial\Psi}{\partial\rho} \hat{\mathbf{z}}. \quad (57)$$

¹³The form \mathbf{B}_0 is probably what was meant to have been found in sec. III(a) of [12].

In astrophysical situations the time dependence of the currents and fields may be sufficiently slow that the displacement-current term in eq. (64) can be neglected. In this case we can write

$$\mathbf{J}(t) \approx \frac{1}{\mu_0} \nabla \times \mathbf{B}(t). \quad (65)$$

If the currents flow in a medium of electrical conductivity σ , they are related to the electric field by $\mathbf{J} = \sigma \mathbf{E}$, and eq. (65) tells us that

$$\mathbf{E}(t) \approx \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B}(t). \quad (66)$$

Faraday's law then gives

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \approx -\frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \mathbf{B}) = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}. \quad (67)$$

If the quasistatic magnetic field is force-free, then from eq. (15) we have

$$\frac{\partial \mathbf{B}}{\partial t} \approx -\frac{k^2}{\mu_0 \sigma} \mathbf{B}, \quad (68)$$

such that [9]

$$\mathbf{B}(\mathbf{x}, t) \approx \mathbf{B}_0(\mathbf{x}) e^{-k^2 t / \mu_0 \sigma}, \quad (69)$$

where $\mathbf{B}_0(\mathbf{x})$ is a static, force-free magnetic field. Hence, if a force-free magnetic field could be established in a (poorly) conducting medium, it would decay away slowly without change to its spatial configuration [9].

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