

The Fractal Dimension of a Ball of Aluminum Foil

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1 Problem

If a sheet of aluminum foil is crumpled into a ball, the mass at radius less than r can be taken as kr^D where D will lie between 2 and 3. We may call D the fractal (Hausdorff) dimension of the crumpled aluminum ball. (In practice the above relation could only hold for r larger than the thickness of the foil.)

Explain how the fractal dimension D could be determined from knowledge of the velocity v attained by the ball upon rolling without slipping down an incline of height h . Ignore air resistance, rolling friction, *etc.*

(A standard model of crumpling predicts $D = 2.5$.)

2 Solution

We need some other property of the ball that depends on the fractal dimension D . The moment of inertia about a diameter suggests itself.

$$I = 2\pi \int_0^R \int_0^\pi \rho(r)r^2 \sin\theta dr d\theta r^2 \sin^2\theta,$$

where the density $\rho(r)$ is related by

$$\rho = \frac{1}{4\pi r^2} \frac{dm}{dr} = \frac{Dkr^{D-3}}{4\pi}.$$

Hence the moment of inertia is

$$I = \frac{2}{3} \frac{DMR^2}{D+2}.$$

For the rolling experiment, conservation of energy tells us

$$Mgh + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2,$$

where the condition of rolling without slipping is $\omega = v/r$. Combining things, we find

$$\frac{D}{D+2} = \frac{3}{2} \left(\frac{2gh}{v^2} - 1 \right).$$