Free Precession

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1 Problem

Calculate the angular frequency Ω of free precession of a planet or star whose angular frequency of rotation about its axis is ω.

For this you may use the following slightly contradictory model. First suppose the shape of the object, whose density ρ is uniform, can be determined by the condition of hydrostatic equilibrium to relate the equatorial radius to the polar radius in the form $r_E = r_P(1+\epsilon)$. Deduce an expression for ϵ in terms of ω, M and r_P, where $M \approx 4\pi\rho r_P^3/3$ is the mass of the object. Then, suppose the object can be treated as a rigid body whose principal moments of inertia obey $(I_P - I_E)/I_P = \epsilon$ to deduce Ω.

This model works fairly well for the Earth, whose observed free precession period of 430 days (Chandler, 1891) is about 1.6 times that as estimated above. The Chandler wobble is thought to be driven by surface wind and water; see Science **289**, 710 (4 Aug. 2000). First evidence for free precession of a pulsar, PSR B1828-11, has recently been reported by Princeton Ph.D. I.H. Stairs, Nature **406**, 484 (2000), with a period about 1/150 that of the above model. This discrepancy is ascribed to little understood aspects of the superfluid interior of the pulsar.

2 Solution

2.1 Parameter ϵ

We calculate in the rest frame of the rotating object, and suppose that the surface follows an equipotential of the combined gravitational potential ϕ_G and centrifugal potential ϕ_C. The latter corresponds to the centrifugal force

$$\mathbf{F}_C = \omega^2 r_\perp \hat{\mathbf{r}}_\perp = -\nabla\phi_C, \tag{1}$$

where $r_\perp = r\sin\theta$ is the distance between the axis of rotation and a point on the surface, in the obvious spherical coordinate system. Thus, the centrifugal potential has the well-known form

$$\phi_C = -\frac{\omega^2 r_\perp^2}{2} = -\frac{\omega^2 r^2 \sin^2\theta}{2}. \tag{2}$$

Because the object is oblate, with radius $r \approx r_P(1+\epsilon\sin\theta)$, its gravitational potential is not simply GM/r. We include the effect of the quadrupole moment M_2 in a multipole expansion of the potential:

$$\phi_G \approx -\frac{GM}{r} - \frac{GM_2 P_2(\cos\theta)}{r^3}, \tag{3}$$

where

$$
\begin{aligned}
M_2 &= \int \rho r^2 P_2 d\text{Vol} \\
&= 2\pi\rho \int_0^\pi \sin\theta \; d\theta \frac{3\cos^2\theta - 1}{2} \int_0^{r_P(1+\epsilon\sin\theta)} r^4 dr \\
&= \pi\rho r_P^5 \int_0^\pi \sin\theta \; d\theta (3\cos^2\theta - 1) \frac{(1+\epsilon\sin\theta)^5}{5} \\
&\approx \pi\rho r_P^5 \int_0^\pi \sin\theta \; d\theta (3\cos^2\theta - 1) \left(\frac{1}{5} + \epsilon\sin\theta\right) \\
&= \pi\epsilon\rho r_P^5 \int_0^\pi \sin^2\theta \; d\theta (3\cos^2\theta - 1) \\
&= \pi\epsilon\rho r_P^5 \int_0^\pi d\theta \left(\frac{3\sin^2 2\theta}{4} - \sin^2\theta\right) \\
&= -\frac{\pi^2\epsilon\rho r_P^5}{8}. \\
&\approx -\frac{3\pi\epsilon M r_P^2}{32}.
\end{aligned} \tag{4}
$$

In the above, we approximated the total mass M by $4\pi\rho r_P^3/3$, but in detail the assumption of a shape $r = r_P(1+\epsilon\sin\theta)$ leads to $M = (4\pi\rho r_P^3/3)(1+3\epsilon/4)$. The resulting correction to eq. (4) is of order ϵ^2, and is neglected.

In this approximation, the potential ϕ is

$$
\phi(r,\theta) = -\frac{GM}{r} + \frac{3\pi\epsilon GM r_P^2 P_2(\cos\theta)}{32 r^3} - \frac{\omega^2 r^2 \sin^2\theta}{2}. \tag{5}
$$

Taking the surface to be an equipotential, we can write

$$
\begin{aligned}
\phi(r_P, 0) = -\frac{GM}{r_P} + \frac{3\pi\epsilon GM}{32 r_P} = \phi(r_E, \pi/2) &= -\frac{GM}{r_P(1+\epsilon)} - \frac{3\pi\epsilon GM r_P^2}{64 r_E^3} - \frac{\omega^2 r_E^2}{2}. \\
&\approx -\frac{GM}{r_P}(1-\epsilon) - \frac{3\pi\epsilon GM}{64 r_P} - \frac{\omega^2 r_P 2}{2},
\end{aligned} \tag{6}
$$

where we note that $\omega^2 r_P^2 \ll GM/r_P$. Thus,

$$
\epsilon \approx \frac{\omega^2 r_P^3}{2GM(1 - 9\pi/64)} = \frac{\omega^2 r_P}{1.12g} = \frac{3.6\pi^2 r_P}{gT^2}, \tag{7}
$$

in terms of the surface gravity $g = GM/r_P^2$ and the period of rotation $T = 2\pi/\omega$.

For example, the polar radius of the Earth is $r = 6,356,752$ m [1], the equatorial radius is 6,378,137 m, the surface gravity is $g = 9.8$ m/s^2 and $T = 8.64 \times 10^4$ s, so that prediction is $\epsilon = 0.0037$, compared to the observed result of 0.0033. The pulsar PSR 1828-11 has $T = 0.4$ s, and we estimate that $M = 2.8 \times 10^{30}$ kg (the Chandrasekhar mass) and radius $r = 10^4$ m, for which our model predicts that $\epsilon = 7 \times 10^{-7}$.

Remark: If we ignore the effect of the quadrupole deformation on the gravitational potential, we find from eq. (6) that $\epsilon \approx \omega^2 r_p/2g$, which is still not too bad an approximation.

2.2 The Free Precession Rate

Following Euler, we write the torque-free equation of motion as

$$\mathbf{N} = 0 = \frac{d\mathbf{L}}{dt}, \tag{8}$$

where $\mathbf{L} = \vec{\vec{I}} \cdot \vec{\omega}$ is the angular momentum and $\vec{\vec{I}}$ is the inertia tensor. To avoid the complication of a time-dependent inertia tensor, we introduce the (orthogonal) body axes $\hat{\mathbf{1}}$ = the axis of rotation, and $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$. The body axes rotate with angular velocity $\vec{\omega}$. In the body frame the inertia tensor is constant in time and diagonal with $I_{11} = I_P$ and $I_{22} = I_{33} = I_E$, so that the constant angular momentum can be written

$$\mathbf{L} = I_P\omega_1\hat{\mathbf{1}} + I_E\omega_2\hat{\mathbf{2}} + I_E\omega_3\hat{\mathbf{3}} = (I_P - I_E)\omega_1\hat{\mathbf{1}} + I_E\vec{\omega}. \tag{9}$$

If we write the time rate of change of a vector $\mathbf{a}$ in the body frame as $\delta\mathbf{a}/\delta t$, then the lab-frame time derivative $d\mathbf{a}/dt$ is

$$\frac{d\mathbf{a}}{dt} = \frac{\delta\mathbf{a}}{\delta t} + \vec{\omega} \times \mathbf{a}. \tag{10}$$

The equation of motion (8) now becomes

$$\begin{aligned} 0 &= (I_P - I_E)\dot{\omega}_1\hat{\mathbf{1}} + I_E\dot{\vec{\omega}} + \vec{\omega} \times [(I_P - I_E)\omega_1\hat{\mathbf{1}} + I_E\vec{\omega}] \\ &= (I_P - I_E)\dot{\omega}_1\hat{\mathbf{1}} + I_E\dot{\vec{\omega}} - (I_P - I_E)\omega_1\hat{\mathbf{1}} \times \vec{\omega}, \end{aligned} \tag{11}$$

where the dot indicates time differentiation in the lab frame. [For a scalar quantity such as ω_1, $d\omega_1/dt = \delta\omega_1/\delta t$, and the vector $\vec{\omega}$ obeys $d\vec{\omega}/dt = \delta\vec{\omega}/\delta t$ according to eq. (10).] The $\hat{\mathbf{1}}$ component of this equation is simply $0 = I_P\dot{\omega}_1$, so that $\dot{\omega}_1 = 0$. We can therefore rewrite eq. (11) as

$$\dot{\vec{\omega}} = \frac{I_P - I_E}{I_E}\omega_1\hat{\mathbf{1}} \times \vec{\omega}. \tag{12}$$

Thus, in the body frame the angular velocity precesses about the polar axis with angular velocity

$$\Omega = \frac{I_P - I_E}{I_E}\omega_1, \tag{13}$$

which is called the angular velocity of free precession.

For an oblate spheroid with $r_E = r_P(1 + \epsilon)$, we have that

$$\Omega = \epsilon\omega_1, \tag{14}$$

using $\epsilon = (I_P - I_E)/I_E$, as verified in the Appendix.

The period of free precession is then,

$$T_{\rm precess} = \frac{2\pi}{\epsilon\omega_1} \approx \frac{T}{\epsilon}, \tag{15}$$

as the model for ϵ makes sense only for $\vec{\omega} \approx \omega_1\hat{\mathbf{1}}$.

This model predicts that $T_{\rm precess} \approx 1/0.0037 = 270$ days, compared to the observed period of 430 days (Chandler, 1891). The predicted period of free precession for the pulsar PSR 1828-11 is 7 days, compared to the observed period of about 1000 days.

3 Appendix: The Moment of Inertia of a Uniform Ellipsoid about a Principal Axis

Given an ellipsoid described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{16}$$

with uniform mass density ρ, the moment of inertia about the x axis is

$$\begin{aligned}
I_x &= \rho \int_{-a}^{a} dx \int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} dy \int_{-(c/ab)\sqrt{b^2(a^2-x^2)-a^2y^2}}^{(c/ab)\sqrt{b^2(a^2-x^2)-a^2y^2}} dz\ (y^2+z^2) \\
&= \frac{8\rho c}{a^3b^3} \int_0^a dx \int_0^{(b/a)\sqrt{a^2-x^2}} dy \left(a^2b^2y^2\sqrt{b^2(a^2-x^2)-a^2y^2} + \frac{c^2}{3}(b^2(a^2-x^2)-a^2y^2)^{3/2} \right) \\
&= \frac{8\rho c}{a^4b^3} \int_0^a dx \int_0^{b\sqrt{a^2-x^2}} du \left(b^2u^2\sqrt{b^2(a^2-x^2)-u^2} + \frac{c^2}{3}(b^2(a^2-x^2)-u^2)^{3/2} \right) \\
&= \frac{\pi\rho bc(b^2+c^2)}{2a^4} \int_0^a dx\ (a^2-x^2)^2 = \frac{4}{15}\pi\rho abc(b^2+c^2) = \frac{M}{5}(b^2+c^2),
\end{aligned} \tag{17}$$

where $M = 4\pi\rho abc/3$ is the mass of the ellipsoid.

For an oblate spheriod with $a = r$ and $b = c = r(1+\epsilon)$, we have that

$$I_P = I_x = \frac{2}{5}Mr^2(1+\epsilon)^2 \approx \frac{2}{5}Mr^2(1+2\epsilon), \tag{18}$$

and

$$I_E = I_y = \frac{1}{5}M(a^2+c^2) = \frac{1}{5}Mr^2(1+(1+\epsilon)^2) \approx \frac{2}{5}Mr^2(1+\epsilon) \tag{19}$$

Then,

$$\frac{I_P - I_E}{I_E} \approx \epsilon, \tag{20}$$

as claimed.

4 References

[1] P. Mohazzabi and M.C. James, *Plumb line and the shape of the earth*, Am. J. Phys. **68**, 1038 (2000).