

The Helmholtz Decomposition and the Coulomb Gauge

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1 Problem

Helmholtz showed in 1858 [1] (in a hydrodynamic context) that any vector field, say \mathbf{E} , that vanishes suitably quickly at infinity can be decomposed as¹

$$\mathbf{E} = \mathbf{E}_{\text{irr}} + \mathbf{E}_{\text{rot}}, \quad (1)$$

where the irrotational and rotational components \mathbf{E}_{irr} and \mathbf{E}_{rot} obey²

$$\nabla \times \mathbf{E}_{\text{irr}} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{E}_{\text{rot}} = 0. \quad (2)$$

For the case that \mathbf{E} is the electric field, discuss the relation of the Helmholtz decomposition to use of the Coulomb gauge.³

2 Solution

The Helmholtz decomposition (1)-(2) is an artificial split of the vector field \mathbf{E} into two parts, which parts have interesting mathematical properties.

We recall that in electrodynamics the electric field \mathbf{E} and the magnetic field \mathbf{B} can be related to a scalar potential V and a vector potential \mathbf{A} according to

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad (3)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (4)$$

This results in another decomposition of the electric field \mathbf{E} which might be different from that of Helmholtz. Here, we explore the relation between these two decompositions.

We also recall that the potentials V and \mathbf{A} are not unique, but can be redefined in a systematic way such that the fields \mathbf{E} and \mathbf{B} are invariant under such redefinition. A particular choice of the potentials is called a choice of **gauge**, and the relations (3)-(4) are said to be **gauge invariant**.

Returning to Helmholtz' decomposition, we note that he also showed how

$$\mathbf{E}_{\text{irr}}(\mathbf{r}) = -\nabla \int \frac{\nabla' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad \text{and} \quad \mathbf{E}_{\text{rot}}(\mathbf{r}) = \nabla \times \int \frac{\nabla' \times \mathbf{E}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad (5)$$

¹The essence of this decomposition was anticipated by Stokes (1849) in secs. 5-6 of [2].

²The irrotational component is sometimes labeled “longitudinal” or “parallel,” and the rotational component is sometimes labeled “solenoidal” or “transverse.”

³Vector plane waves $\mathbf{E} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ do not vanish “suitably quickly” at infinity, so care is required in applying the Helmholtz decomposition $\mathbf{E}_{\text{irr}} = (\mathbf{E} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}$, $\mathbf{E}_{\text{rot}} = \mathbf{E} - \mathbf{E}_{\text{irr}}$ of this mathematically useful, but physically unrealistic class of fields. See, for example, sec. 2.4.2 of [3].

where $R = |\mathbf{r} - \mathbf{r}'|$. Time does not appear in eq. (5), which indicates that the vector field \mathbf{E} at some point \mathbf{r} (and some time t) can be reconstructed from knowledge of its vector derivatives, $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$, over all space (at the same time t). The main historical significance of the Helmholtz decomposition (1) and (5) was in showing that Maxwell's equations, which give prescriptions for the vector derivatives $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$, are mathematically sufficient to determine the field \mathbf{E} . Since $\nabla \cdot \mathbf{E} = \rho_{\text{total}}/\epsilon_0$ and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, the fields \mathbf{E}_{irr} and \mathbf{E}_{rot} involve instantaneous action at a distance and should not be regarded as physically real. This illustrates how gauge invariance is necessary, but not sufficient, for electromagnetic fields to correspond to "reality".^{4,5}

The Helmholtz decomposition (1) and (5) can be rewritten as

$$\mathbf{E} = -\nabla V + \nabla \times \mathbf{F}, \quad (6)$$

where

$$V(\mathbf{r}) = \int \frac{\nabla' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad \text{and} \quad \mathbf{F}(\mathbf{r}) = \int \frac{\nabla' \times \mathbf{E}(\mathbf{r}')}{4\pi R} d\text{Vol}'. \quad (7)$$

It is consistent with usual nomenclature to call V a scalar potential and \mathbf{F} a vector potential. That is, Helmholtz decomposition lends itself to an interpretation of fields as related to derivatives of potentials.

When the vector field \mathbf{E} is the electric field, it also obeys Maxwell's equations, two of which are (in SI units and for media where the permittivity is ϵ_0)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8)$$

where ρ is the electric charge density and \mathbf{B} is the magnetic field.

If we insert these physics relations into eq. (7), we find

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 R} d\text{Vol}', \quad (9)$$

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial}{\partial t} \int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}'. \quad (10)$$

The scalar potential (9) is calculated from the instantaneous charge density, which is exactly the prescription (35) of the Coulomb gauge. That is, Helmholtz + Maxwell implies use of the Coulomb-gauge prescription for the scalar potential.

However, eq. (10) for the vector potential \mathbf{F} does not appear to be that of the usual procedures associated with the Coulomb gauge. Comparing eqs. (6)-(7) and (10), we see that we can introduce another vector potential \mathbf{A} which obeys

$$\nabla \times \mathbf{F} = -\frac{\partial \mathbf{A}}{\partial t}, \quad (11)$$

such that

$$\mathbf{A}(\mathbf{r}) = \nabla \times \int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}', \quad (12)$$

⁴See Appendix A for an application of the Helmholtz decomposition to Hertzian dipole radiation

⁵The Helmholtz decomposition leads to interesting interpretations of the momentum and angular momentum associated with electromagnetic fields [3].

and

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad (3)$$

which is the usual way the electric field is related to a scalar potential V and a vector potential \mathbf{A} . Note also that eq. (12) obeys the Coulomb gauge condition (31) that $\nabla \cdot \mathbf{A} = 0$.

Thus, the Helmholtz decomposition (1) and (5) of the electric field \mathbf{E} is equivalent to the decomposition (3) in terms of a scalar and a vector potential, provided those potentials are calculated in the Coulomb gauge.⁶

Using various vector calculus identities, we have

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \nabla \times \int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}' = \int \nabla \frac{1}{R} \times \frac{\mathbf{B}(\mathbf{r}')}{4\pi} d\text{Vol}' = - \int \nabla' \frac{1}{R} \times \frac{\mathbf{B}(\mathbf{r}')}{4\pi} d\text{Vol}' \\ &= \int \frac{\nabla' \times \mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}' + \int \nabla' \times \frac{\mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}' = \int \frac{\nabla' \times \mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}' + \oint d\text{Area}' \times \frac{\mathbf{B}(\mathbf{r}')}{4\pi R} \\ &= \int \frac{\nabla' \times \mathbf{B}(\mathbf{r}')}{4\pi R} d\text{Vol}', \end{aligned} \quad (13)$$

provided \mathbf{B} vanishes at infinity. In view of the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, we recognize eq. (13) as the Helmholtz decomposition $\mathbf{B} = \nabla \times \mathbf{A}$ for the magnetic field.

We can go further by invoking the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (14)$$

where \mathbf{J} is the current density vector, the medium is assumed to have permeability μ_0 , and c is the speed of light, so that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d\text{Vol}' + \frac{\partial}{\partial t} \int \frac{\mathbf{E}(\mathbf{r}')}{4\pi c^2 R} d\text{Vol}'. \quad (15)$$

This is not a useful prescription for calculation of the vector potential, because the second term of eq. (15) requires us to know $\mathbf{E}(\mathbf{r}')/c^2$ to be able to calculate $\mathbf{E}(\mathbf{r})$.⁷ But, c^2 is a big number, so \mathbf{E}/c^2 is only a “small” correction, and perhaps can be ignored.⁸ If we do so, then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d\text{Vol}', \quad (16)$$

which is the usual instantaneous prescription for the vector potential due to **steady** currents. Thus, it appears that practical use of the Helmholtz decomposition + Maxwell’s equations is largely limited to quasistatic situations, where eqs. (9) and (16) are sufficiently accurate.

Of course, we exclude wave propagation and radiation in this approximation. We can include radiation and wave propagation if we now invoke the usual prescription, eqs. (36)-(37)

⁶The fields \mathbf{E}_{irr} and \mathbf{E}_{rot} can be deduced from vector and scalar potentials in any gauge, but only in the Coulomb gauge is $\mathbf{A}_{\text{irr}}^{(C)} = 0$ such that the Helmholtz decomposition has the simple form $\mathbf{E}_{\text{irr}} = -\nabla V^{(C)}$ and $\mathbf{E}_{\text{rot}} = -\partial \mathbf{A}^{(C)}/\partial t$.

⁷Using the Helmholtz decomposition for \mathbf{E} in eq. (15) permits us to proceed without knowing \mathbf{E} , provided we know the charge density ρ and the time derivative $\partial \mathbf{B}/\partial t$, which is no improvement conceptually.

⁸See [4] for an argument that the second integral of eq. (15) vanishes in the quasistatic approximation.

of Appendix B, for the vector potential in the Coulomb gauge. However, this prescription does not follow very readily from the Helmholtz decomposition, which is an instantaneous calculation.

Note that in the case of practical interest when the time dependence of the charges and currents is purely sinusoidal at angular frequency ω , i.e., $e^{-i\omega t}$, the Lorenz gauge condition (30) becomes

$$V = -\frac{ic}{k}\nabla \cdot \mathbf{A}. \quad (17)$$

In this case it suffices to calculate only the vector potential \mathbf{A} , and then deduce the scalar potential V , as well as the fields \mathbf{E} and \mathbf{B} , from \mathbf{A} .

However, neither the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ nor the Lorenz gauge condition (30) suffices, in general, for a prescription in which only the scalar potential V is calculated, and then \mathbf{A} , \mathbf{E} and \mathbf{B} are deduced from this. Recall that the Helmholtz decomposition tells us how the vector field \mathbf{A} can be reconstructed from knowledge of both $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$. The gauge conditions tell us only $\nabla \cdot \mathbf{A}$, and we lack a prescription for $\nabla \times \mathbf{A}$ in terms of V .

[In 1 dimension, $\nabla \times \mathbf{A} = 0$, so in 1-dimensional problems we can deduce everything from the scalar potential V plus the gauge condition. But life in 3 dimensions is more complicated!]

Appendix A: Helmholtz Decomposition of Hertzian Dipole Radiation

The electric and magnetic fields of an ideal, point Hertzian electric dipole can be written (in Gaussian units) as

$$\mathbf{E} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{\cos(kr - \omega t)}{r} + p [3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \hat{\mathbf{p}}] \left[\frac{\cos(kr - \omega t)}{r^3} + \frac{k \sin(kr - \omega t)}{r^2} \right], \quad (18)$$

$$\mathbf{B} = \mathbf{B}_{\text{rot}} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \left[\frac{\cos(kr - \omega t)}{r} - \frac{\sin(kr - \omega t)}{kr^2} \right], \quad (19)$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, $\mathbf{p} = p \cos \omega t \hat{\mathbf{p}}$ is the electric dipole moment vector, $\omega = 2\pi f$ is the angular frequency, $k = \omega/c = 2\pi/\lambda$ is the wave number and c is the speed of light [5, 6].

The irrotational part of the electric field is the instantaneous field of the electric dipole,

$$\mathbf{E}_{\text{irr}} = p [3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \hat{\mathbf{p}}] \frac{\cos \omega t}{r^3}. \quad (20)$$

Thus, the rotational part of the electric field is

$$\begin{aligned} \mathbf{E}_{\text{rot}} = \mathbf{E} - \mathbf{E}_{\text{irr}} &= k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{\cos(kr - \omega t)}{r} \\ &+ p [3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \hat{\mathbf{p}}] \left[\frac{\cos(kr - \omega t) - \cos \omega t}{r^3} + \frac{k \sin(kr - \omega t)}{r^2} \right]. \end{aligned} \quad (21)$$

Both fields \mathbf{E}_{irr} and \mathbf{E}_{rot} have instantaneous terms.

The flow of energy in the electromagnetic field is described by the Poynting vector \mathbf{S} , so the Helmholtz decomposition leads us to write

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = \frac{c}{4\pi} \mathbf{E}_{\text{irr}} \times \mathbf{B}_{\text{rot}} + \frac{c}{4\pi} \mathbf{E}_{\text{rot}} \times \mathbf{B}_{\text{rot}}. \quad (22)$$

Using eqs. (19)-(20), we have that

$$\mathbf{S}_1 = \frac{ck^2p^2}{4\pi} [(3 \cos^2 \theta - 1) \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{p}}] \cos \omega t \left[\frac{\cos(kr - \omega t)}{r^4} - \frac{\sin(kr - \omega t)}{kr^5} \right], \quad (23)$$

where θ is the angle between vectors \mathbf{r} and \mathbf{p} . Similarly,

$$\begin{aligned} \mathbf{S}_2 = & \frac{c}{4\pi} \left\{ k^4 p^2 \sin^2 \theta \hat{\mathbf{r}} \left[\frac{\cos^2(kr - \omega t)}{r^2} - \frac{\cos(kr - \omega t) \sin(kr - \omega t)}{kr^3} \right] \right. \\ & + k^2 p^2 [(3 \cos^2 \theta - 1) \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{p}}] \left[\frac{\cos^2(kr - \omega t) - \sin^2(kr - \omega t)}{r^4} \right. \\ & + \cos(kr - \omega t) \sin(kr - \omega t) \left(\frac{k}{r^3} - \frac{1}{kr^5} \right) \\ & \left. \left. - \cos \omega t \left(\frac{\cos(kr - \omega t)}{r^4} - \frac{\sin(kr - \omega t)}{kr^5} \right) \right] \right\}. \quad (24) \end{aligned}$$

Neither \mathbf{S}_1 nor \mathbf{S}_2 describes the flow of energy at an identifiable speed, so the Helmholtz decomposition, which is based on present source terms, does not seem well suited to a general characterization of the flow of energy in electromagnetic fields.

We can restrict our attention to the region very close to the source, where $kr \ll 1$ and we have

$$\mathbf{S}_1(kr \ll 1) = \frac{ck^2p^2}{4\pi} [(3 \cos^2 \theta - 1) \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{p}}] \left(\frac{\cos^2 \omega t}{r^4} + \frac{\cos \omega t \sin \omega t}{kr^5} \right), \quad (25)$$

and

$$\begin{aligned} \mathbf{S}_2(kr \ll 1) = & \frac{c}{4\pi} \left[k^4 p^2 \sin^2 \theta \hat{\mathbf{r}} \left(\frac{\cos^2 \omega t}{r^2} + \frac{\cos \omega t \sin \omega t}{kr^3} \right) \right. \\ & \left. + k^2 p^2 [(3 \cos^2 \theta - 1) \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{p}}] \left(\frac{k \cos \omega t \sin \omega t}{r^3} - \frac{\sin^2 \omega t}{r^4} \right) \right]. \quad (26) \end{aligned}$$

Here, the separation of the total Poynting vector \mathbf{S} into \mathbf{S}_1 and \mathbf{S}_2 is cleaner than for large kr , but, to this author, this separation is still not associated with any crisp physical insight.

We can also consider only the time average of eqs. (23)-(24),

$$\langle \mathbf{S}_1 \rangle = \frac{ck^2p^2}{8\pi} [(3 \cos^2 \theta - 1) \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{p}}] \left(\frac{\cos kr}{r^4} - \frac{\sin kr}{kr^5} \right), \quad (27)$$

and

$$\langle \mathbf{S}_2 \rangle = \frac{c}{8\pi} \frac{k^4 p^2 \sin^2 \theta}{r^2} \hat{\mathbf{r}} - \langle \mathbf{S}_1 \rangle. \quad (28)$$

Again, there seems to be little physical insight associate with this decomposition.

Appendix B: Coulomb Gauge

The relations

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (29)$$

between the electric and magnetic fields \mathbf{E} and \mathbf{B} and the potentials V and \mathbf{A} permits various conventions (gauges) for the potentials. One popular choice is the Lorenz gauge [7],

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \quad (\text{Lorenz}). \quad (30)$$

In situations with steady charge and current distributions (electrostatics and magneto-statics), $\partial V/\partial t = 0$, so the condition (30) reduces to

$$\nabla \cdot \mathbf{A} = 0 \quad (\text{Coulomb}). \quad (31)$$

Even in time-dependent situations it is possible to define the vector potential to obey eq. (31), which has come to be called the **Coulomb gauge** condition.

We restrict our discussion to media for which the dielectric permittivity is ϵ_0 and the magnetic permeability is μ_0 . Then, using eq. (29) in the Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ leads to

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0}, \quad (32)$$

and the Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \partial \mathbf{E}/\partial t$ leads to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right). \quad (33)$$

Thus, in the Coulomb gauge (31), eq. (32) becomes Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad (34)$$

which has the formal solution

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{R} d\text{Vol}' \quad (\text{Coulomb}), \quad (35)$$

where $R = |\mathbf{r} - \mathbf{r}'|$, in which changes in the charge distribution ρ instantaneously affect the potential V at any distance.

It is possible to choose gauges for the electromagnetic potentials such that some of their components appear to propagate at any specified velocity v [9, 10]. One can also choose that the scalar potential has no time dependence, such that all time dependence of the electric field is associated with that of the vector potential [11].

For completeness, a formal solution for the vector potential in the Coulomb gauge is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_t(\mathbf{r}', t' = t - R/c)}{R} d\text{Vol}' \quad (\text{Coulomb}), \quad (36)$$

where the transverse current density is defined by

$$\mathbf{J}_t(\mathbf{r}, t) = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{r}', t)}{R} d\text{Vol}' \quad (37)$$

Unless the geometry of the problem is such that the transverse current density \mathbf{J}_t is easy to calculate, use of the Coulomb gauge is technically messier than the use of the Lorenz gauge, in which case the (retarded) potentials are given by are the retarded potentials

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t' = t - R/c)}{R} d\text{Vol}' \quad (\text{Lorenz}), \quad (38)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t' = t - R/c)}{R} d\text{Vol}' \quad (\text{Lorenz}), \quad (39)$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

Analysis of circuits is often performed in the quasistatic approximation that effects of wave propagation and radiation can be neglected. In this case, the speed of light is taken to be infinite, so that the Lorenz gauge condition (30) is equivalent to the Coulomb gauge condition (31), and the potentials are calculated from the instantaneous values of the charge and current distributions. As a consequence, gauge conditions are seldom mentioned in “ordinary” circuit analysis.

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