Flow of Energy and Momentum in the Near Zone of a Hertzian Dipole

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 11, 2007)

1 Problem

Discuss the density and flow of energy and momentum in the electromagnetic fields of an idealized Hertzian (point) oscillating electric dipole.[1]

2 Solution

The electric and magnetic fields of an ideal, point Hertzian electric dipole of moment $\mathbf{m}\cos\omega t$ can be written (in Gaussian units) as

$$\mathbf{E} = k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \times \hat{\mathbf{r}} \frac{\cos(kr - \omega t)}{r} + m[3(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}] \left[\frac{\cos(kr - \omega t)}{r^3} + \frac{k \sin(kr - \omega t)}{r^2} \right], \quad (1)$$

$$\mathbf{H} = k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \left[\frac{\cos(kr - \omega t)}{r} - \frac{\sin(kr - \omega t)}{kr^2} \right], \quad (2)$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, $\mathbf{m} = m\,\hat{\mathbf{m}}$ is the peak electric dipole moment vector,[2] $\omega = 2\pi f$ is the angular frequency, $k = \omega/c = 2\pi/\lambda$ is the wave number and c is the speed of light [1, 2].

In a spherical coordinate system (r, θ, ϕ) for which the z-axis is along the direction of the dipole $\mathbf{m}$ and θ is the angle between $\mathbf{m}$ and $\mathbf{r}$, the fields are

$$\mathbf{E} = -k^2 m \sin\theta\, \hat{\boldsymbol{\theta}} \frac{\cos(kr - \omega t)}{r} + m(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \left[\frac{\cos(kr - \omega t)}{r^3} + \frac{k \sin(kr - \omega t)}{r^2} \right], \quad (3)$$

$$\mathbf{H} = -k^2 m \sin\theta\, \hat{\boldsymbol{\phi}} \left[\frac{\cos(kr - \omega t)}{r} - \frac{\sin(kr - \omega t)}{kr^2} \right], \quad (4)$$

In the far zone, $r \gg \lambda$, only the parts of these fields that vary as $1/r$ are significant:

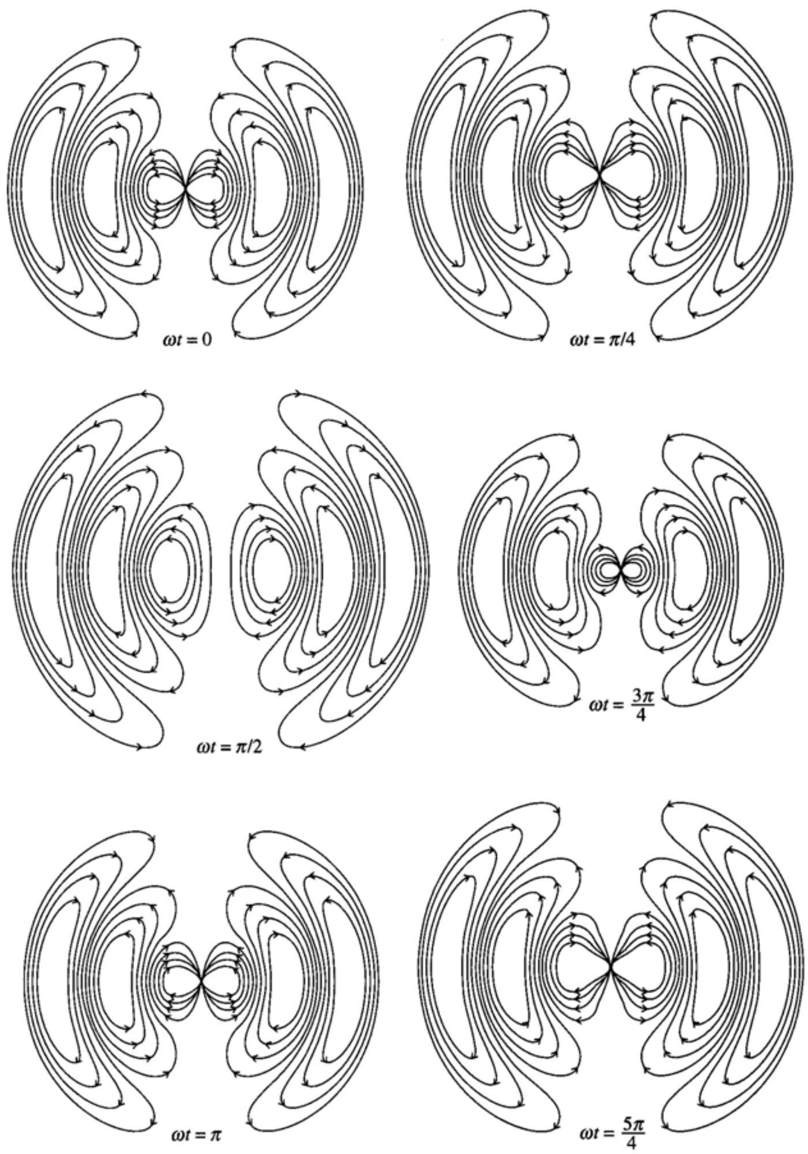
$$\mathbf{E}_{\rm far} = -k^2 \sin\theta m\, \hat{\boldsymbol{\theta}} \frac{\cos(kr - \omega t)}{r}, \qquad \mathbf{H}_{\rm far} = -k^2 m \sin\theta\, \hat{\boldsymbol{\phi}} \frac{\cos(kr - \omega t)}{r}. \quad (5)$$

[1] This problem is an extension of
http://puhep1.princeton.edu/~mcdonald/examples/nearzone.pdf
Some additional details, including discussion of the scalar and vector potentials of a Hertzian dipole in both the Coulomb and Lorenz gauges, are given in prob. 2 of
http://puhep1.princeton.edu/~mcdonald/examples/ph501set8.pdf
Some consideration of Hertz vectors and scalars is given in the Appendix of
http://puhep1.princeton.edu/~mcdonald/examples/smallloop.pdf

[2] We reserve the symbol **p** for the momentum density of the electromagnetic field.

These fields are often called the radiation fields. Note that even in the near zone, $r \lesssim \lambda$, terms of the form (5) are present, and we say that the radiation fields exist in the near zone as well as in the far zone.

Of course, in the near zone of the dipole the radiation fields are smaller that the other components of **E** and **H**. The most prominent feature of the fields in the near zone is that the electric field looks a lot like that of an electrostatic dipole, as shown in the figure below. Because field patterns that look like radiation are discernable only for $r \gtrsim \lambda$, there may be an impression that the radiation is created at some distance from an antenna, rather than at the antenna itself.



2.1 Energy Density

According to Maxwell the density of energy stored in the electromagnetic field (in vacuum) is

$$u = \frac{E^2 + H^2}{8\pi}. \tag{6}$$

For the fields (3)-(4), we find

$$\begin{aligned} u &= \frac{m^2}{4\pi}\left\{\left[\frac{k^4 \sin^2\theta}{r^2} + \frac{3\cos^2\theta + 1}{2r^6}\right]\cos^2(kr - \omega t) - \frac{k^2}{r^4}\cos 2(kr - \omega t)\right. \\ &\quad \left. - \left[\frac{k^3 \sin^2\theta}{r^3} - \frac{k(3\cos^2\theta + 1)}{2r^5}\right]\sin 2(kr - \omega t) + \frac{k^2\cos^2\theta}{r^4}\right\}. \end{aligned} \tag{7}$$

The first term of eq. (7) is the energy associated with the radiation fields, while the second term is the electrostatic energy of the dipole $\mathbf{m}$ multiplied by the wave function $\cos^2(kr-\omega t)$. The other four terms are due to interference between the radiation fields and the quasistatic field of the oscillating dipole. Rewriting eq. (7) in terms of its Fourier components, we have

$$\begin{aligned} u &= \frac{m^2}{8\pi}\left\{\frac{k^4\sin^2\theta}{r^2} + \frac{2k^2\cos^2\theta}{r^4} + \frac{3\cos^2\theta + 1}{2r^6}\right. \\ &\quad + \left[\frac{k^4\sin^2\theta}{r^2} - \frac{2k^2}{r^4} + \frac{3\cos^2\theta + 1}{2r^6}\right]\cos 2(kr - \omega t) \\ &\quad \left. - \left[\frac{2k^3\sin^2\theta}{r^3} - \frac{k(3\cos^2\theta + 1)}{r^5}\right]\sin 2(kr - \omega t)\right\}. \end{aligned} \tag{8}$$

It is of interest to record the total energy density,

$$U(t) = \int u(\mathbf{r}, t)\, d\text{Vol} = 2\pi \int_0^\infty r^2\, dr \int_{-1}^{1} d\cos\theta\, u(\mathbf{r}, t). \tag{9}$$

The part of the energy density (8) associated with radiation varies as $1/r^2$, so the integral of this energy density is infinite (since the forms (1)-(2) tacitly assumed that the antenna has been radiating "forever"). Hence, we obtain a finite result only if we restrict our attention to the energy U_{near} in the "near zone" of the antenna, which corresponds to the terms in eq. (8) that vary more quickly than $1/r^2$. Furthermore, we obtain a finite result only if we consider the energy in the region outside some small, but finite, radius $a \ll \lambda$. Thus,

$$\begin{aligned} U_{\text{near}} &= \frac{m^2}{4}\int_a^\infty r^2\, dr \int_{-1}^{1} d\cos\theta \left\{\frac{2k^2\cos^2\theta}{r^4} + \frac{3\cos^2\theta + 1}{2r^6}\right. \\ &\quad + \left[-\frac{2k^2}{r^4} + \frac{3\cos^2\theta + 1}{2r^6}\right](\cos 2kr\cos 2\omega t + \sin 2kr \sin 2\omega t) \\ &\quad \left. - \left[\frac{2k^3\sin^2\theta}{r^3} - \frac{k(3\cos^2\theta + 1)}{r^5}\right](\sin 2kr\cos 2\omega t - \cos 2kr\sin 2\omega t)\right\} \\ &= \frac{m^2}{4}\int_a^\infty dr\left\{\frac{4k^2}{3r^2} + \frac{2}{r^4} + \left[-\frac{4k^2}{r^2} + \frac{2}{r^4}\right](\cos 2kr\cos 2\omega t + \sin 2kr\sin 2\omega t)\right. \end{aligned}$$

$$
- \left[\frac{8k^3}{3r} - \frac{4k}{r^3}\right] (\sin 2kr \cos 2\omega t - \cos 2kr \sin 2\omega t)\Bigg\}
$$

$$
\approx \frac{m^2}{4}\frac{2}{3a^3}(1+\cos 2\omega t) = \frac{m^2 \cos^2 \omega t}{3a^3}. \tag{10}
$$

The near-field energy oscillates in time, which implies there is an oscillatory exchange of energy between the electromagnetic fields and the source that drives the dipole antenna.

2.2 Energy Flow

Since the radiated power comes from the antenna (from the power supply that drives the antenna), there must be a flow of energy out from the antenna into the surrounding space. The usual electrodynamic measure of energy flow is Poynting's vector [3],

$$
\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{11}
$$

When we use the fields (1)-(2) to calculate the Poynting vector we find six terms, some of which do not point along the radial vector $\hat{\mathbf{r}}$:

$$
\begin{aligned}
\mathbf{S} &= \frac{c}{4\pi}\Bigg\{k^4 m^2 \sin^2\theta\, \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} \left[\frac{\cos(kr-\omega t)}{r^2} - \frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right] \\
&\quad - k^2 m^2 (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \times \sin\theta\, \hat{\boldsymbol{\phi}} \Bigg[\frac{\cos^2(kr-\omega t) - \sin^2(kr-\omega t)}{r^4} \\
&\qquad + \cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3} - \frac{1}{kr^5}\right)\Bigg]\Bigg\} \\
&= \frac{cm^2}{4\pi}\Bigg\{\sin^2\theta\, \hat{\mathbf{r}} \left[\frac{k^4\cos^2(kr-\omega t)}{r^2} - \frac{k^2\cos 2(kr-\omega t)}{r^4} - \frac{k^3 \sin 2(kr-\omega t)}{r^3}\left(1 - \frac{1}{2k^2r^2}\right)\right] \\
&\qquad + \sin 2\theta\, \hat{\boldsymbol{\theta}} \left[\frac{k^2\cos 2(kr-\omega t)}{r^4} + \frac{k^3\sin 2(kr-\omega t)}{2r^3}\left(1 - \frac{1}{k^2r^2}\right)\right]\Bigg\}.
\end{aligned} \tag{12}
$$

Since the functions $\cos 2(kr - \omega t)$ and $\sin 2(kr - \omega t)$ can be both positive and negative, part of the energy flow is inwards at times, rather than outwards as expected for pure radiation. The presence of an inward flow of energy reinforces the conclusion of the previous section that there is an oscillatory exchange of energy between source and fields, rather than a simple flow of energy from the source to the fields.

However, we obtain a simple result if we consider only the time-averaged Poynting vector, $\langle \mathbf{S} \rangle$. Noting that $\langle \cos^2(kr - \omega t)\rangle = 1/2$ and $\langle \cos 2(kr - \omega t)\rangle = \langle \sin 2(kr - \omega t)\rangle = 0$, eq (12) leads to

$$
\langle \mathbf{S} \rangle = \frac{ck^4 m^2 \sin^2\theta}{8\pi r^2}\,\hat{\mathbf{r}}. \tag{13}
$$

The time-average Poynting vector is purely radially outwards, and falls off as $1/r^2$ at all radii, as expected for a flow of energy that originates in the oscillating point dipole. The time-average angular distribution $d\langle P \rangle / d\Omega$ of the radiated power is related to the Poynting vector by

$$
\frac{d\langle P \rangle}{d\Omega} = r^2\, \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle = \frac{ck^4 p^2 \sin^2\theta}{8\pi} = \frac{p^2\omega^4 \sin^2\theta}{8\pi c^3}, \tag{14}
$$

which is the expression usually derived for dipole radiation in the far zone. Here we see that this expression holds in the near zone as well.

We conclude that radiation, as measured by the time-averaged Poynting vector, exists in the near zone of an antenna as well as in the far zone.

We verify that the Poynting vector $\mathbf{S}$ *corresponds to the flow of energy such that the continuity equation,*

$$\boldsymbol{\nabla}\cdot\mathbf{S}+\frac{\partial u}{\partial t}=0, \tag{15}$$

is satisfied. From eq. (8) we have,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{cm^2}{4\pi}\left[\frac{k^5\sin^2\theta}{r^2}-\frac{2k^3}{r^4}+\frac{k(3\cos^2\theta+1)}{2r^6}\right]\sin 2(kr-\omega t)\\ &\quad+\frac{cm^2}{4\pi}\left[\frac{2k^4\sin^2\theta}{r^3}-\frac{k^2(3\cos^2\theta+1)}{r^5}\right]\cos 2(kr-\omega t),\end{aligned} \tag{16}$$

while from eq.(12) we find,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{S} &= \frac{1}{r^2}\frac{\partial}{\partial r}(rS_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_\theta)\\ &= \frac{cm^2}{4\pi}\left\{\sin^2\theta\left[-\frac{k^5\sin 2(kr-\omega t)}{r^2}+\frac{2k^3\sin 2(kr-\omega t)}{r^4}+\frac{2k^2\cos 2(kr-\omega t)}{r^5}\right.\right.\\ &\quad\left.-\frac{2k^4\cos 2(kr-\omega t)}{r^3}\left(1-\frac{1}{2k^2r^2}\right)+\frac{k^3\sin 2(kr-\omega t)}{r^4}\left(1-\frac{3}{2k^2r^2}\right)\right]\\ &\quad\left.+(3\cos^2\theta-1)\left[\frac{2k^2\cos 2(kr-\omega t)}{r^5}+\frac{k^3\sin 2(kr-\omega t)}{r^4}\left(1-\frac{1}{k^2r^2}\right)\right]\right\}\\ &= \frac{cm^2}{4\pi}\left[-\frac{k^5\sin^2\theta}{r^2}+\frac{2k^3}{r^4}-\frac{k(3\cos^2\theta+1)}{2r^6}\right]\sin 2(kr-\omega t)\\ &\quad-\frac{cm^2}{4\pi}\left[\frac{2k^4\sin^2\theta}{r^3}-\frac{k^2(3\cos^2\theta+1)}{r^5}\right]\cos 2(kr-\omega t)=-\frac{\partial u}{\partial t}.\end{aligned} \tag{17}$$

We note that the part of the energy density due only to the radiation fields,

$$u_{\rm rad}=\frac{k^4m^2\sin^2\theta}{4\pi r^2}\cos^2(kr-\omega t), \tag{18}$$

and the part of the Poynting vector due only to the radiation fields,

$$\mathbf{S}_{\rm rad}=\frac{ck^4m^2\sin^2\theta}{4\pi r^2}\cos^2(kr-\omega t)\,\hat{\mathbf{r}}, \tag{19}$$

obey the continuity equation

$$\boldsymbol{\nabla}\cdot\mathbf{S}_{\rm rad}+\frac{\partial u_{\rm rad}}{\partial t}=0. \tag{20}$$

We can identify part of the energy density as being due to the quasistatic electric dipole,

$$u_{\rm dipole}=\frac{m^2(3\cos^2\theta+1)}{8\pi r^6}\cos^2(kr-\omega t). \tag{21}$$

However, there is no magnetic field, and hence no Poynting vector, associated with the quasistatic electric dipole. Hence, conservation of energy in the field of the quasistatic electric dipole can be accounted for only when we consider the interference terms of the Poynting vector that involve the quasistatic dipole fields and the so-called intermediate-zone fields of eqs. (3)-(4) that vary as $1/r^2$.

2.3 Momentum Density

As noted by Abraham [4], the Poynting vector plays the dual role of describing the energy flow in the electromagnetic field as well as the density $\mathbf{p}$ of momentum stored in the field,

$$\mathbf{p} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{22}$$

where we restrict our attention to waves in vacuum.[3]

From eq. (12) we obtain the momentum density of the fields of the oscillating dipole,

$$\begin{aligned}\mathbf{p} \;=\; & \frac{m^2}{4\pi c}\left\{\sin^2\theta\,\hat{\mathbf{r}}\left[\frac{k^4\cos^2(kr-\omega t)}{r^2} - \frac{k^2\cos 2(kr-\omega t)}{r^4} - \frac{k^3\sin 2(kr-\omega t)}{r^3}\left(1-\frac{1}{2k^2r^2}\right)\right]\right. \\ & \left. +\sin 2\theta\,\hat{\boldsymbol{\theta}}\left[\frac{k^2\cos 2(kr-\omega t)}{r^4} + \frac{k^3\sin 2(kr-\omega t)}{2r^3}\left(1-\frac{1}{k^2r^2}\right)\right]\right\}. \end{aligned} \tag{23}$$

The part of the momentum density associated only with the radiation fields is

$$\mathbf{p}_{\rm rad} = \frac{\mathbf{S}_{\rm rad}}{c^2} = \frac{u_{\rm rad}}{c}\,\hat{\mathbf{r}} = \frac{k^4 m^2\sin^2\theta}{4\pi c r^2}\cos^2(kr-\omega t)\,\hat{\mathbf{r}}. \tag{24}$$

3 Momentum Flux

If we know the velocity with which the momentum is flowing, we can identify a momentum flux (tensor) $\mathsf{\Pi}$ as the product of the momentum density and its velocity. The radiation fields propagate radially with velocity c, so we identify

$$\mathsf{\Pi}_{{\rm rad},rr} = c p_{{\rm rad},r} = u_{\rm rad} = \frac{k^4 m^2\sin^2\theta}{4\pi r^2}\cos^2(kr-\omega t). \tag{25}$$

For a more general consideration of momentum flux, we note that the time rate of change of momentum density, $\partial\mathbf{p}/\partial t$, has dimensions of a force density. We recall that Maxwell described the forces associated with electromagnetic fields in terms of the stress tensor T, whose components are (for fields in vacuum)

$$\mathsf{T}_{ij} = \frac{E_iE_j + H_iH_j}{4\pi} - u\delta_{ij}, \tag{26}$$

where u is the energy density (6). The volume force $\mathbf{f}$ density associated with the fields is the divergence of the stress tensor,

$$\mathbf{f} = \boldsymbol{\nabla}\cdot\mathsf{T}, \tag{27}$$

[3]For waves in a dielectric medium one must also consider the momentum of the oscillating dipoles of that medium, which requires some care [5, 6].

and the equation of continuity for electromagnetic field momentum is

$$\frac{\partial \mathbf{p}}{\partial t} - \boldsymbol{\nabla} \cdot \mathsf{T} = 0. \tag{28}$$

This leads to the interpretation of

$$\mathsf{\Pi} = -\mathsf{T} = u\delta_{ij} - \frac{E_i E_j + H_i H_j}{4\pi} \tag{29}$$

as the momentum flux tensor.

The six distinct momentum flux components for the fields of the Hertzian dipole are

$$\begin{aligned}
\mathsf{\Pi}_{rr} &= \frac{m^2}{4\pi}\left\{\left[\frac{k^4\sin^2\theta}{r^2} + \frac{1-5\cos^2\theta}{2r^6}\right]\cos^2(kr-\omega t) + \frac{k^2\cos 2\theta}{r^4}\cos 2(kr-\omega t)\right. \\
&\quad \left. - \left[\frac{k^3\sin^2\theta}{r^3} - \frac{k(1-5\cos^2\theta)}{2r^5}\right]\sin 2(kr-\omega t) - \frac{k^2\cos^2\theta}{r^4}\right\}, && (30) \\
\mathsf{\Pi}_{r\theta} = \mathsf{\Pi}_{\theta r} &= -\frac{m^2\sin 2\theta}{4\pi}\left[\frac{k^2\sin 2(kr-\omega t)}{r^5} + \frac{\cos^2(kr-\omega t)}{r^6}\right], && (31) \\
\mathsf{\Pi}_{r\phi} = \mathsf{\Pi}_{\phi r} &= 0, && (32) \\
\mathsf{\Pi}_{\theta\theta} &= \frac{m^2}{8\pi}\left\{\frac{5\cos^2\theta - 1}{r^6}\cos^2(kr-\omega t) + \frac{k^2(1-3\cos^2\theta)}{r^4}\cos 2(kr-\omega t)\right. \\
&\quad \left. + \frac{k(5\cos^2\theta-1)}{r^5}\sin 2(kr-\omega t) + \frac{k^2(1+\cos^2\theta)}{r^4}\right\}, && (33) \\
\mathsf{\Pi}_{\theta\phi} = \mathsf{\Pi}_{\phi\theta} &= 0, && (34) \\
\mathsf{\Pi}_{\phi\phi} &= \frac{m^2}{8\pi}\left\{\frac{3\cos^2\theta + 1}{r^6}\cos^2(kr-\omega t) - \frac{k^2(1+\cos^2\theta)}{r^4}\cos 2(kr-\omega t)\right. \\
&\quad \left. + \frac{k(3\cos^2\theta+1)}{r^5}\sin 2(kr-\omega t) + \frac{k^2(3\cos^2\theta-1)}{r^4}\right\}. && (35) \\
& && (36)
\end{aligned}$$

The momentum flux due only to the radiation fields is the first term of $\mathsf{\Pi}_{rr}$, as anticipated in eq. (25). The remaining terms of the tensor $\mathsf{\Pi}$ describe the somewhat nontrivial flow of momentum in the near zone of the antenna.

For comparison, we record the momentum flux associated with a static electric dipole $\mathbf{m}$. Then, the magnetic field vanishes and the electric field is simply

$$\mathbf{E}_{\text{static}} = m\frac{3(\hat{\mathbf{m}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}}{r^3} = m\frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}. \tag{37}$$

The field energy density is

$$u_{\text{static}} = m^2\frac{3\cos^2\theta + 1}{8\pi r^6}, \tag{38}$$

and the components of the momentum-flux tensor are

$$\mathsf{\Pi}_{\text{static},rr} = m^2\frac{1-5\cos^2\theta}{8\pi r^6}, \tag{39}$$

$$\Pi_{\mathrm{static},r\theta} = \Pi_{\mathrm{static},\theta r} = -m^2 \frac{\sin 2\theta}{4\pi r^6}, \tag{40}$$

$$\Pi_{\mathrm{static},r\phi} = \Pi_{\mathrm{static},\phi r} = 0, \tag{41}$$

$$\Pi_{\mathrm{static},\theta\theta} = m^2 \frac{5\cos^2\theta - 1}{8\pi r^6}, \tag{42}$$

$$\Pi_{\mathrm{static},\theta\phi} = \Pi_{\mathrm{static},\phi\theta} = 0, \tag{43}$$

$$\Pi_{\mathrm{static},\phi\phi} = m^2 \frac{3\cos^2\theta + 1}{8\pi r^6}. \tag{44}$$

$$\tag{45}$$

4 Wave Velocities

4.1 Phase/Wavefront Velocity

Since the electromagnetic fields of a Hertzian dipole are not plane waves we cannot simply speak of a phase velocity. But, we can consider the velocity of a wavefront, which is the essence of the phase velocity. Since the fields of an oscillating electric dipole are transverse magnetic, it is clearest to consider wavefronts of the magnetic field, eq. (4), which vanish on spherical surfaces given by

$$\tan(kr - \omega t) = kr, \qquad i.e., \qquad t = \frac{r}{c} - \frac{1}{\omega}\tan^{-1} kr + \frac{n\pi}{\omega}. \tag{46}$$

We designate the radial velocity of these surfaces as the phase velocity,

$$v_p = \frac{dr}{dt} = \frac{1}{dt/dr} = c\left(1 + \frac{1}{k^2 r^2}\right) > c. \tag{47}$$

In the near zone this phase velocity exceeds the speed of light in vacuum, while it approaches the speed of light in the far zone.

4.2 Energy/Group Velocity

As discussed in eq. (17) of [7], we take the group velocity to be equal to the energy velocity,

$$v_g = \frac{\langle S \rangle}{\langle u \rangle} = \frac{c}{1 + \frac{\cot^2\theta}{k^2 r^2} + \frac{3\cos^2\theta + 1}{2k^4 r^4 \sin^2\theta}} < c, \tag{48}$$

recalling eqs. (8) and (13). The group velocity is less than the speed of light at all angles and all radii, and is extremely small along the direction of the dipole moment, where the strength of the radiation is very weak.

Note that $v_g v_p \neq c^2$.

References

[1] H. Hertz, *The Forces of Electrical Oscillations Treated According to Maxwell's Theory*, Weidemann's Ann. **36**, 1 (1889); reprinted in chap. 9 of H. Hertz, *Electric Waves* (Dover,

New York, 1962). A translation by O. Lodge appeared in Nature **39**, 402 (1889),
http://puhep1.princeton.edu/~mcdonald/examples/EM/hertz.pdf

[2] Sec. 9.2 of J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).

[3] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://puhep1.princeton.edu/~mcdonald/examples/EM/poynting_ptrsl_175_343_84.pdf

[4] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://puhep1.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[5] R. Peierls, *The momentum of light in a refracting medium*, Proc. Roy. Soc. London A **347**, 475 (1976),
http://puhep1.princeton.edu/~mcdonald/examples/EM/peierls_prsla_347_475_76.pdf

[6] D.F. Nelson, *Momentum, pseudomomentum and wave momentum: Toward resolving the Minkowski-Abraham controversy*, Phys. Rev. A **44**, 3985 (1991),
http://puhep1.princeton.edu/~mcdonald/examples/EM/nelson_pra_44_3985_91.pdf

[7] K.T. McDonanld, *Radiation from Hertzian Dipoles in a Uniaxial Anisotropic Medium* (Nov. 16, 2007), http://puhep1.princeton.edu/~mcdonald/examples/anisotropic.pdf