

# A Mechanical Model That Exhibits a Gravitational Critical Radius

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## 1 Problem

A popular model at science museums (and also a science toy [1]) that illustrates how curvature can be associated with gravity consists of a surface of revolution  $r = -k/z$  with  $z < 0$  about a vertical axis  $z$ . The curvature of the surface, combined with the vertical force of Earth's gravity, leads to an inward horizontal acceleration of  $kg/r^2$  for a particle that slides freely on the surface in a circular, horizontal orbit.

Consider the motion of a particle that slides freely on an arbitrary surface of revolution,  $r = r(z) \geq 0$ , defined by a continuous and differentiable function on some interval of  $z$ . The surface may have a nonzero minimum radius  $R$  at which the slope  $dr/dz$  is infinite. Discuss the character of oscillations of the particle about circular orbits to deduce a condition that there be a critical radius  $r_{\text{crit}} > R$ , below which the orbits are unstable. That is, the motion of a particle with  $r < r_{\text{crit}}$  rapidly leads to excursions to the minimum radius  $R$ , after which the particle falls off the surface.

Give one or more examples of analytic functions  $r(z)$  that exhibit a critical radius as defined above. These examples provide a mechanical analogy as to how departures of gravitational curvature from that associated with a  $1/r^2$  force can lead to a characteristic radius inside which all motion tends toward a singularity.

## 2 Solution

We work in a cylindrical coordinate system  $(r, \theta, z)$  with the  $z$  axis vertical. It suffices to consider a particle of unit mass.

In the absence of friction, there is no torque on a particle about the  $z$  axis, so the angular momentum component  $J = r^2\dot{\theta}$  about that axis is a constant of the motion, where  $\dot{\phantom{x}}$  indicates differentiation with respect to time.

For motion on a surface of revolution  $r = r(z)$ , we have  $\dot{r} = r'\dot{z}$ , where  $'$  indicates differentiation with respect to  $z$ . Hence, the kinetic energy can be written

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) = \frac{1}{2}[\dot{z}^2(1 + r'^2) + r^2\dot{\theta}^2]. \quad (1)$$

The potential energy is  $V = gz$ . Using Lagrange's method, the equation of motion associated with the  $z$  coordinate is

$$\ddot{z}(1 + r'^2) + \dot{z}^2 r' r'' = -g + \frac{J^2 r'}{r^3}. \quad (2)$$

For a circular orbit at radius  $r_0$ , we have

$$r_0^3 = \frac{J^2 r'_0}{g}. \quad (3)$$

We write  $\dot{\theta}_0 = \Omega$ , so that  $J = r_0^2 \Omega$ .

For a perturbation about this orbit of the form

$$z = z_0 + \epsilon \sin \omega t, \quad (4)$$

we have, to order  $\epsilon$ ,

$$\begin{aligned} r(z) &\approx r(z_0) + r'(z_0)(z - z_0) \\ &= r_0 + \epsilon r'_0 \sin \omega t, \end{aligned} \quad (5)$$

$$r' \approx r'_0 + \epsilon r''_0 \sin \omega t, \quad (6)$$

$$\frac{1}{r^3} \approx \frac{1}{r_0^3} \left( 1 - 3\epsilon \sin \omega t \frac{r'_0}{r_0} \right). \quad (7)$$

Inserting (4-7) into (2) and keeping terms only to order  $\epsilon$ , we obtain

$$-\epsilon \omega^2 \sin \omega t (1 + r_0'^2) \approx -g + \frac{J^2}{r_0^3} \left( r'_0 - 3\epsilon \sin \omega t \frac{r_0'^2}{r_0} + \epsilon \sin \omega t r_0'' \right). \quad (8)$$

From the zeroeth-order terms we recover (3), and from the order- $\epsilon$  terms we find that

$$\omega^2 = \Omega^2 \frac{3r_0'^2 - r_0 r_0''}{1 + r_0'^2}. \quad (9)$$

The orbit is unstable when  $\omega^2 < 0$ , *i.e.*, when

$$r_0 r_0'' > 3r_0'^2. \quad (10)$$

This condition has the interesting geometrical interpretation (noted by a referee) that the orbit is unstable wherever

$$(1/r^2)'' < 0, \quad (11)$$

*i.e.*, where the function  $1/r^2$  is concave inwards.

For example, if  $r = -k/z$ , then  $1/r^2 = z^2/k^2$  is concave outwards,  $\omega^2 = J^2/(k^2 + r_0^4)$ , and there is no regime of instability.

We give three examples of surfaces of revolution that satisfy condition (11).

First, the hyperboloid of revolution defined by

$$r^2 - z^2 = R^2, \quad (12)$$

where  $R$  is a constant. Here,  $r'_0 = z_0/r_0$ ,  $r''_0 = R^2/r_0^3$ , and

$$\omega^2 = \Omega^2 \frac{3z_0^2 - R^2}{2z_0^2 + R^2} = \Omega^2 \frac{3r_0^2 - 4R^2}{2r_0^2 - R^2}. \quad (13)$$

The orbits are unstable for

$$z_0 < \sqrt{3}R, \quad (14)$$

or equivalently, for

$$r_0 < \frac{2\sqrt{3}}{3}R = 1.1547R \equiv r_{\text{crit}}. \quad (15)$$

As  $r_0$  approaches  $R$ , the instability growth time approaches an orbital period.

Another example is the Gaussian surface of revolution,

$$r^2 = R^2 e^{z^2}, \quad (16)$$

which has a minimum radius  $R$ , and a critical radius  $r_{\text{crit}} = R\sqrt[4]{e} = 1.28R$ .

Our final example is the surface

$$r = -\frac{k}{z\sqrt{1-z^2}}, \quad (-1 < z < 0), \quad (17)$$

which has a minimum radius of  $R = 2k$ , approaches the surface  $r = -k/z$  at large  $r$  (small  $z$ ), and has a critical radius of  $r_{\text{crit}} = 6k/\sqrt{5} = 1.34R$ .

These examples arise in a  $2 + 1$  geometry with curved space but flat time. As such, they are not fully analagous to black holes in  $3 + 1$  geometry with both curved space and curved time. Still, they provide a glimpse as to how a particle in curved spacetime can undergo considerably more complex motion than in flat spacetime.

### 3 Acknowledgement

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### References

- [1] The Vortex(tm) Miniature Wishing Well, Divnick International, Inc., 321 S. Alexander Road, Miamisburg, OH 45342, <http://www.divnick.com/>