How Much of Magnetic Energy Is Kinetic Energy?

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1 Problem

How much of the "magnetic" energy stored in a current-carrying inductor is due to the kinetic energy of the moving charges?

Consider the example of a toroidal coil of major radius a and minor radius $b \ll a$ that carries current I in its N turns, which current is due to electrons of charge $-e$ and (rest) mass m.

2 Solution

The "magnetic" energy U stored in an in inductor can be written as

$$U = \frac{1}{2} L I^2 = \int \frac{B^2}{8\pi}\, d\mathrm{Vol}, \tag{1}$$

where L is the self inductance, and the latter form assumes all media have unit (relative) permeability and is expressed in Gaussian units.

For the example of a toroidal inductor, the magnetic field B_0 along its circular axis follows from Ampère's law as

$$B_0 = \frac{2NI}{ac}, \tag{2}$$

where c is the speed of light in vacuum. Thus, the stored "magnetic" energy is

$$U \approx \frac{B_0^2}{8\pi} 2\pi^2 a b^2 = \frac{N^2 I^2 b^2}{ac^2} \qquad \left(\text{and hence } \ L \approx \frac{2N^2 b^2}{ac^2}\right). \tag{3}$$

Supposing that all conduction electrons have the same speed v, the current I is related to the number density n of conduction electrons per unit length along the (spiral) conductor according to

$$I = nev. \tag{4}$$

The total length of the conductor is $2\pi N b$, so the kinetic energy T of the conduction electrons is

$$T = 2\pi N n b \frac{m v^2}{2} = \pi N b \frac{m I^2}{n e^2} = \frac{\pi N b I^2}{n c^2 r_0}, \tag{5}$$

where $r_0 = e^2/mc^2 \approx 3 \times 10^{-13}$ cm is the classical electron radius.

The ratio of the kinetic energy to the "magnetic" energy is

$$\frac{T}{U} \approx \frac{\pi a}{N n\, b\, r_0}. \tag{6}$$

Since only the product nv is determined by eq. (4), the result (6) is ambiguous.

To go further, we suppose that there is one conduction electron per atom in the copper conductor, such that the volume density of conduction electrons $n_e \approx 8 \times 10^{22}/\text{cm}^3$. Then, the linear number density is $n = \pi n_e d^2/4$, where $d \ll b$ is the diameter of the copper conductor. Equation (6) now becomes

$$\frac{T}{U} \approx \frac{4a}{N n_e\, b\, d^2\, r_0} \approx \frac{a}{6Nb\, d^2} 10^{-9}, \tag{7}$$

for a, b and d measured in cm.

We could also suppose that the N turns are tightly wound on the toroid, such that $Nd = 2\pi a$. Then,

$$\frac{T}{U} \approx \frac{1}{12\pi b\, d} 10^{-9}, \tag{8}$$

As an example, suppose $b = 1$ cm and $d = 1$ mm $= 0.1$ cm, for which $T/U \approx 3 \times 10^{-10}$.

3 Comments

This problem was of interest to Maxwell, who did not have a vision of currents as due to the motion of electrons. In sec. 551 of his *Treatise* [1], he writes

It appears, therefore, that a system containing an electric current is a seat of energy of some kind; and since we can form no conception of an electric current except as a kinetic phenomenon, its energy must be kinetic energy, that is to say, the energy which a moving body has in virtue of its motion.

We have already shewn that the electricity in the wire cannot be consider as the moving body in which we are to find the energy, for the energy of a moving body does not depend on anything external to itself, whereas the presence of other bodies near the current alters its energy.

We are therefore led to enquire where there may not be some motion going on in the space outside the wire, which is not occupied by the electric current, but in which the electromagnetic effects of the current are manifested.

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What I now propose to do is to examine the consequences of the assumption that the phenomena of the electric current are those of a moving system, the motion being communicated from one part of the system to another by forces, the nature and laws of which we do not yet even attempt to define, because we can eliminate those forces from the equations of motion by the method given by Lagrange for any connected system.

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I have chosen this method because I wish to shew that there are other ways of viewing the phenomena which appear to be more satisfactory, and at the same time more consistent ... than those which proceed on the hypothesis of direct action at a distance.

It appears that although Maxwell did not make a calculation like that of eq. (8) he correctly understood that the (ordinary) kinetic energy of "electricity" in the wire of an inductor can account for only a small part of the "magnetic" energy of that inductor. This reinforced his vision of some kind of active *æther* in and through which electromagnetic effects are transmitted. He also had come to realize that an abstract, mathematical characterization of that *æther* suffices, and that the earlier mechanical models of *æther* are of little/no enduring utility.

Maxwell returns to the question of what portion of the "magnetic" energy is actually ordinary kinetic energy in secs. 574-576 of the *Treatise* and considers several experiments, which at the time produced null results. This theme is reviewed in chap. 18 of an interesting book by Cullwick [2], who notes that with time all of the experiments considered by Maxwell did provide evidence for small amounts of ordinary momentum, energy and angular momentum associated with currents in circuits. However, Cullwick ends by making the striking (and unsupportable, as known even to Maxwell) hypothesis that all "magnetic" energy is due to the ordinary kinetic energy of the conduction electrons.

There remains an issue of terminology. Maxwell's definition seems very reasonable, that kinetic energy is *the energy which a moving body has in virtue of its motion*.

Moving charges are associated with a magnetic field, and energy is associated with this field. Can/should we therefore say that this energy is "kinetic" because it is due to the motion of the charges? Only a tiny fraction of "magnetic" energy is associated with the "ordinary kinetic energy" $\sum mv^2/2$ of the moving charges, so we would have to invent a new category of "extraordinary kinetic energy". This usage is not common, and I do not advocate it. On the whole, the common term "magnetic energy" seems descriptive enough, although it draws attention away from the tiny component of ordinary kinetic energy associated with magnetic phenomena.[1]

In his sec. 551, Maxwell was wrestling with the question of how the older, mechanical notion of the kinetic energy $mv^2/2$ of a moving mass should be extended into the larger domain of electrodynamics. Similar issues arose after 1905 in the context of Einstein's theory of special relativity, which which considers masses with high velocity, and also in the context of the his concept of photons of light. The attitude favored by this author is that the term "kinetic energy" should be restricted to its original meaning of $mv^2/2$, rather than following Maxwell's suggestion that any energy of a body/system "in virtue of its motion" be called "kinetic energy". In this view, a photon does not possess kinetic energy simply because it has no (rest) mass. Similarly, in this view a fast moving mass with total energy $mc^2/\sqrt{1 - v^2/c^2} = \gamma mc^2 = mc^2 + mv^2/2 + mv^4/8c^2 + \cdots$ still has "kinetic energy" $mv^2/2$, although one is tempted to speak of its "relativistic kinetic energy" as $(\gamma - 1)mc^2 = mv^2/2 + mv^4/8c^2 + \cdots$. And, in this view, the magnetic energy of a circuit at rest is to be considered as part of the rest energy of that system, rather than as a "kinetic energy".[2] In sum, the view of this author is that the antiquated term "kinetic energy" should mainly be used in its historical context of classical mechanics, and not used when discussing the energy of electromagnetic fields, or of photons, and also not used in relativistic discussions.

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[1] A separate issue is the terminology for the tiny amount of ordinary momentum associated with currents in circuits. In some situations the term "hidden mechanical momentum" is used, but this usage has severely restricted applicability. See, for example, [3].

[2] In sec. 2 of [4], Maxwell allowed that energy stored in electromagnetic fields might be either "actual" (kinetic) or "potential". It is consistent to say that a magnetostatic field stores only "potential" energy.

References

[1] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Clarendon Press, 1891; reprinted 1954 by Dover, and 2007 by Merchant Books).

[2] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, London, 1959).

[3] K.T. McDonald, *"Hidden" Momentum of a Steady Current Distribution in a System at "Rest"* (Apr. 21, 2009), http://puhep1.princeton.edu/~mcdonald/examples/current.pdf

[4] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://puhep1.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf