The Levitron$^{\text{TM}}$

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1 Problem

A science toy marketed under the name Levitron$^{\text{TM}}$ [1] consists of a bar magnet of mass m and magnetic dipole moment $\boldsymbol{\mu}$ that is levitated above a static magnetic field which is circularly symmetric about the vertical (z) axis (such as that of a loop of current perpendicular to the z-axis). The magnet is also spinning with angular velocity $\boldsymbol{\omega}$ about its symmetry axis, which axis is parallel to $\boldsymbol{\mu}$.

Deduce conditions on the derivatives of the magnetic field such that the center of mass motion of the magnet is stable. You may assume that ω is large enough that the rotational motion is stable, and that the equilibrium point lies on the symmetry axis of the magnetic field.

For the example of a magnet levitated antiparallel to the field of a loop of radius a that carries a steady current, find the range of equilibrium heights z_0 above the plane of the loop for which the motion is stable.

2 Solution

The complex history of the invention, development and marketing of the Levitron$^{\text{TM}}$ toy can be viewed at [2]. The commercial toy uses a permanent magnet base, rather than an electromagnet as analyzed here. The case of a permanent-magnet base has been extensively discussed by Berry [3], who derives the magnetic field from a scalar potential, which technique is more compact for fields produced by a permanent magnet than that presented below. If the magnetic field is due to a uniformly magnetized disk of radius a, then the stability region obeys

$$\frac{a}{2} < z_0 < \frac{a}{\sqrt{2.5}}\,, \tag{1}$$

as also found in eq. (33) for a loop coil. For a square permanent magnet with a hole in its center, as used in the commercial Levitron$^{\text{TM}}$, the region of stability is even smaller.

See also [4].

2.1 Conditions for Stability

To discuss the center of mass motion, we construct a potential and require that the second spatial derivatives be positive when the first derivatives vanish.

The gravitational potential energy is just mgz, taking the z-axis as vertically upwards. The potential energy of a magnetic dipole $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is $-\boldsymbol{\mu}\cdot\mathbf{B}$; the moment tends to align itself parallel to the magnetic field. Hence

$$U(r,z) = mgz - \boldsymbol{\mu}\cdot\mathbf{B}(r,z)\,. \tag{2}$$

For a simple circularly symmetric field, $\mathbf{B}(r, z)$, the equilibrium points will be on the z-axis of symmetry. (We ignore more complex fields that might lead to a ring of equilibrium points.) Then, the condition that $(0, z_0)$ be an equilibrium point is

$$F_z = -\frac{\partial U(0,z_0)}{\partial z} = 0 = -mg + \boldsymbol{\mu}\cdot\frac{\partial \mathbf{B}(0,z_0)}{\partial z}\,, \tag{3}$$

$$F_r = -\frac{\partial U(0,z_0)}{\partial r} = 0 = \boldsymbol{\mu}\cdot\frac{\partial \mathbf{B}(0,z_0)}{\partial r}\,. \tag{4}$$

The conditions that this equilibrium be stable are

$$\frac{\partial^2 U(0,z_0)}{\partial z^2} = -\boldsymbol{\mu}\cdot\frac{\partial^2 \mathbf{B}(0,z_0)}{\partial z^2} > 0\,, \tag{5}$$

$$\frac{\partial^2 U(0,z_0)}{\partial r^2} = -\boldsymbol{\mu}\cdot\frac{\partial^2 \mathbf{B}(0,z_0)}{\partial r^2} > 0\,. \tag{6}$$

Prior to more detailed analysis, we can learn several things from eqs. (3)-(6). Recalling (or rederiving, as is done below) that a cylindrically symmetric magnetic field has a radial component B_r that grows linearly with radius near the axis, while its axial component B_z drops off quadratically with radius, eq. (4) tells us that the magnetic dipole moment must be either parallel or antiparallel to the (axial) magnetic field at the equilibrium point. We desire the equilibrium point to be above the source of the magnetic field, so the device can operate on a tabletop. Then magnitude of the magnetic field will decrease with increasing height, so if the axial magnetic field is positive, then $\partial B_z(0, z_0)/\partial z$ will be negative, and eq. (3) tells us that the magnetic dipole must be antiparallel to field to obtain an upward force to balance that of gravity. Likewise, if $B_z < 0$, then $\partial B_z(0, z_0)/\partial z > 0$, and again the magnetic moment must be antiparallel to the field.

Since magnetic dipoles prefer to be parallel to an applied magnetic field, there must be some mechanism to insure that if the dipole is initially antiparallel it will remain so. Spinning the dipole rapidly about its axis has this effect.

Assuming that the spin keeps the dipole antiparallel to the local magnetic field vector (and not simply antiparallel to the symmetry axis), we can rewrite the stability conditions (5)-(6) as

$$\frac{\partial^2 B(0,z_0)}{\partial z^2} = \frac{\partial^2 B_z(0,z_0)}{\partial z^2} > 0\,, \tag{7}$$

$$\frac{\partial^2 B(0,z_0)}{\partial r^2} = \frac{\partial^2 B_z(0,z_0)}{\partial r^2} + \frac{1}{B_0}\left(\frac{\partial B_r(0,z_0)}{\partial r}\right)^2 > 0\,, \tag{8}$$

using $B = \sqrt{B_r^2 + B_z^2}$, and defining $B_0 = B_z(0, z_0)$. In arriving at (6), we note that since $B_r(0, z) = 0$, then $\partial B_r(0, z)/\partial z = 0$ also.

The presence of the second term in eq. (8) is crucial for stability, since $\partial^2 B_z(0, z_0)/\partial r^2 = -(\partial^2 B_z(0, z_0)/\partial z^2)/2$ for axially symmetric magentic fields (see eq. (23). Hence, a key aspect of stability is that when the dipole moves off the symmetry axis, its magnetic moment aligns itself with the local magnetic field. The spin of the dipole enforces this condition, but only for a range of spin angular velocities. If the spin is too little, the nutations of the spin will provide insufficient alignment of the dipole with the field [3], and if the spin is too large the precession frequency (12) will be less than the oscillation frequency, and the dipole cannot stay aligned with the local magnetic field [4].

The axial field due to a bounded source of magnetic field will have, in general, a profile something like a bell-curve. Near the source, the second derivative of the axial field will be negative, but at greater distances it becomes positive. So condition (6) will lead to a minimum height for stability.

The stability condition (8) is the subtlest. The transverse profile of the axial component of the field will also have a shape like a bell-curve, so the first term of eq. (8) is negative. Without the second, positive term, stability would not be possible. That term arises only if the dipole tips so as to remain aligned with the field as it moves off the symmetry axis, as required by the physics of a spinning dipole. Even so, it is not evident that the second term is larger than the first. Indeed, in the more detailed argument below we will find that condition (8) is satisfied only close to the source, but that there is a small range of z_0 for which both conditions (7) and (8) are satisfied.

2.2 Spin Precession

The motion of the spinning dipole about its center of mass is described by the torque equation

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\mu} \times \mathbf{B}\,, \tag{9}$$

where $\mathbf{L}$ is the angular momentum. In the limit of large spin angular velocity ω, we may approximate

$$\mathbf{L} = I\boldsymbol{\omega} = I\omega\frac{\boldsymbol{\mu}}{\mu}\,, \tag{10}$$

where I is the moment of inertia about the symmetry axis (which is parallel to $\boldsymbol{\mu}$). Then we can rewrite the torque equation as

$$\frac{d\boldsymbol{\mu}}{dt} = -\frac{\mu\mathbf{B}}{I\omega} \times \boldsymbol{\mu}\,. \tag{11}$$

Hence, we see that the motion of the magnet relative to its center of mass consists of precession about the local direction of the magnetic field at angular velocity

$$\Omega = -\frac{\mu B}{I\omega}\,. \tag{12}$$

In particular, we see that

$$\boldsymbol{\mu} \cdot \mathbf{B} = \text{ const } = \mu B \cos\theta_0\,, \tag{13}$$

where θ_0 is the (constant) angle between $\boldsymbol{\mu}$ and $\mathbf{B}$. We have seen that condition (4) requires that $\cos\theta_0 = -1$.

2.3 Evaluation of the Field Derivatives

To complete the problem, we express the magnitude of the field B in terms of only its z-component, $B_z(0, z)$. The approach is to use Maxwell's equations, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\times\mathbf{B} = 0$, to relate B_r to B_z. From the above, we see that we will use only the first and second derivatives of B, so it suffices to use a series expansion to second order in r and z. Say,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 + B_3 r + B_4 r^2 + B_5 r(z - z_0)\,, \tag{14}$$

and

$$B_r(r, z) = C_0 + C_1(z - z_0) + C_2(z - z_0)^2 + C_3 r + C_4 r^2 + C_5 r(z - z_0)\,. \tag{15}$$

In cylindrical coordinates we have

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{r}\frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0\,, \tag{16}$$

and

$$(\boldsymbol{\nabla}\times\mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0\,. \tag{17}$$

From eq. (16),

$$\begin{gathered}[C_0 + C_1(z - z_0) + C_2(z - z_0)^2]\,/r + \\ 2C_3 + 3C_4 r + 2C_5(z - z_0) + \\ B_1 + 2B_2(z - z_0) + B_5 r = 0\,,\end{gathered} \tag{18}$$

and so

$$C_0 = C_1 = C_2 = 0, \quad C_3 = -\frac{B_1}{2}, \quad C_4 = -\frac{B_5}{3}, \quad C_5 = -B_2\,. \tag{19}$$

That is,

$$B_r(r, z) = -\frac{B_1 r}{2} - \frac{B_5 r^2}{3} - B_2 r(z - z_0)\,. \tag{20}$$

Then, from eq. (17)

$$-B_2 r - B_3 - 2B_4 r - B_5(z - z_0) = 0\,, \tag{21}$$

and hence,

$$B_3 = B_5 = 0, \quad B_4 = -\frac{B_2}{2}\,. \tag{22}$$

Altogether,

$$B_z(r, z) = B_0 + B_1(z - z_0) + B_2(z - z_0)^2 - \frac{B_2 r^2}{2}\,, \tag{23}$$

and

$$B_r(r, z) = -\frac{B_1 r}{2} - B_2 r(z - z_0)\,, \tag{24}$$

accurate to second order in r and z.

With $\cos\theta_0 = -1$, the equilibrium condition (3) is now

$$-\frac{mg}{\mu} = \frac{\partial B(0, z_0)}{\partial z} = \frac{\partial B_z(0, z_0)}{\partial z} = B_1\,. \tag{25}$$

The conditions (7)-(8) on the second derivatives of the potential now lead to

$$\frac{\partial^2 B_z(0,z_0)}{\partial z^2} = 2B_2 > 0\,, \tag{26}$$

and

$$\frac{\partial^2 B_z(0,z_0)}{\partial r^2} + \frac{1}{B_0}\left(\frac{\partial B_r(0,z_0)}{\partial r}\right)^2 = -B_2 + \frac{B_1^2}{4B_0} > 0\,. \tag{27}$$

2.4 Example of a Current Loop

If may be of interest to consider the example of a loop of current. Let a be the radius of the loop and A be the maximum magnetic field, which occurs at (0,0). The field along the z-axis is readily found to be

$$B_z(0,z) = \frac{Aa^3}{(a^2+z^2)^{3/2}}\,. \tag{28}$$

Then,

$$\frac{\partial B_z}{\partial z} = -\frac{3Aa^3 z}{(a^2+z^2)^{5/2}}\,, \tag{29}$$

so the equilibrium position z_0 is related by

$$B_0 = \frac{Aa^3}{(a^2+z_0^2)^{3/2}}\,, \tag{30}$$

and

$$B_1 = \frac{\partial B_z(0,z_0)}{\partial z} = -\frac{3Aa^3 z_0}{(a^2+z_0^2)^{5/2}} = -\frac{mg}{\mu}\,. \tag{31}$$

Further,

$$B_2 = \frac{1}{2}\frac{\partial^2 B_z(0,z_0)}{\partial z^2} = \frac{3Aa^3(4z_0^2-a^2)}{2(a^2+z_0^2)^{7/2}} = \frac{3B_0(4z_0^2-a^2)}{2(a^2+z_0^2)^2}\,. \tag{32}$$

The requirement (26) for stable equilibrium that $B_2 > 0$ is satisfied so long as $z_0 > a/2$. The second requirement, (27), for stability, $B_1^2/4B_0 > B_2$, is satisfied when $z_0 < a/\sqrt{2.5}$. Together we must have

$$\frac{a}{2} < z_0 < \frac{a}{\sqrt{2.5}}\,, \tag{33}$$

which holds for suitable choices of m, μ, a and A.

References

[1] The Levitron™ is marketed by Fascinations Inc, 19224 Des Moines Way South, Suite 100, Seattle, WA 98148, http://www.levitron.com

[2] M. and K. Sherlock, *The Hidden History of the Levitron*, http://www.amasci.com/maglev/lev/expose.html

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