

# Lewin's Circuit Paradox

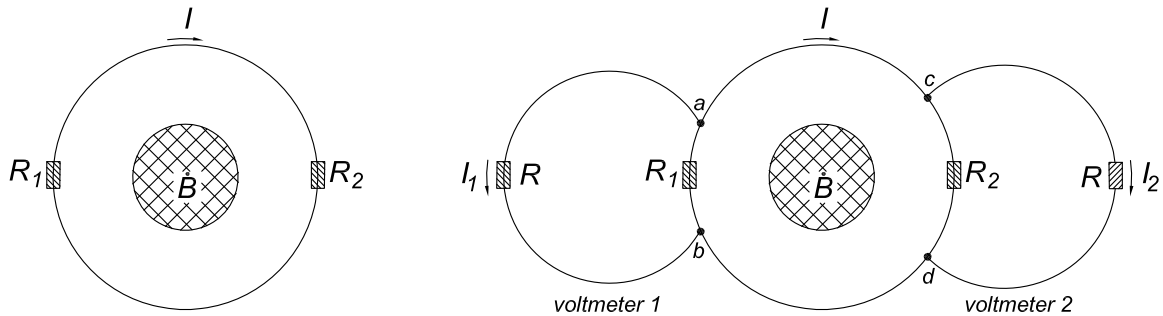
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## 1 Problem

W. Lewin of MIT has given an example of a deceptively simple circuit with somewhat paradoxical behavior.<sup>1</sup> As sketched below, a loop containing resistors  $R_1$  and  $R_2$  surrounds a solenoid magnet that is excited by a time-dependent current.



This circuit is probed by two voltmeters as sketched in the righthand figure above. Each voltmeter consists of a resistor  $R \gg R_1, R_2$  and an ammeter that measures the currents  $I_i$  and reports the “voltage”  $V_{\text{meter},i} = I_i R$ . The “positive” leads of the voltmeters are connected to points  $a$  and  $c$ , such that the directions of currents  $I_1$  and  $I_2$  are as shown when positive.

The paradox is that  $V_{\text{meter}1}$  does not equal  $V_{\text{meter}2}$ , which is particularly surprising in the case that points  $a$  and  $c$  are the same, and points  $b$  and  $d$  are the same.

Discuss.

## 2 Solution

This problem appears to have a long history in pedagogic lore, some of which can be traced in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. A position sensor based on the principle of this problem is described in [17].

### 2.1 Analysis via Kirchhoff's Circuit Law

A circuit diagram for the system is shown on the top of the next page. The solenoid magnet is the primary of a transformer that is excited by a time-dependent current  $I_p$  which is assumed to be known. The loop with resistors  $R_1$  and  $R_2$  is the secondary of the transformer. The (small) self inductance of the secondary is  $L$ , and  $M \gg L$  is the mutual inductance between the primary and secondary.

<sup>1</sup><http://www.youtube.com/watch?v=eqj1-qRy71w&NR=1>  
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Kirchhoff's circuit equation for the secondary loop is

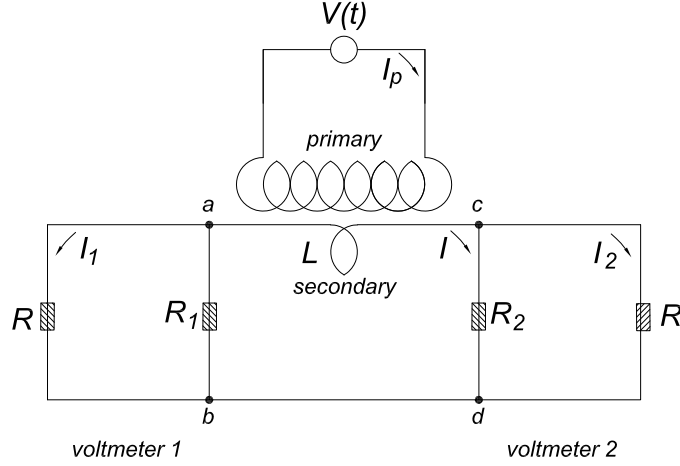
$$0 = (I + I_1)R_1 + (I - I_2)R_2 + L\dot{I} + M\dot{I}_p. \quad (1)$$

We restrict the analysis to low frequencies such that the term  $L\dot{I}$  is negligible. Since  $\dot{I}_p$  is known, we define

$$\mathcal{E} = -M\dot{I}_p = -\dot{\Phi}_p, \quad (2)$$

where  $\Phi_p$  is the magnetic flux from the primary through the secondary loop. Then, eq. (1) can be written as

$$\mathcal{E} = (R_1 + R_2)I + R_1I_1 - R_2I_2. \quad (3)$$



Similarly, Kirchhoff's equations for the voltmeter loops are

$$0 = I_1R + (I_1 + I)R_1 \approx R_1I + RI_1, \quad (4)$$

$$0 = I_2R + (I_2 - I)R_2 \approx -R_2I + RI_2. \quad (5)$$

Solving the three simultaneous linear equations (3)-(5) for the currents, we find

$$I \approx \frac{\mathcal{E}}{R_1 + R_2}, \quad I_1 \approx -\frac{\mathcal{E}R_1}{R(R_1 + R_2)}, \quad I_2 \approx \frac{\mathcal{E}R_2}{R(R_1 + R_2)}. \quad (6)$$

The meter readings are therefore,

$$V_{\text{meter1}} = I_1R \approx -\frac{\mathcal{E}R_1}{(R_1 + R_2)}, \quad V_{\text{meter2}} = I_2R \approx \frac{\mathcal{E}R_2}{(R_1 + R_2)}. \quad (7)$$

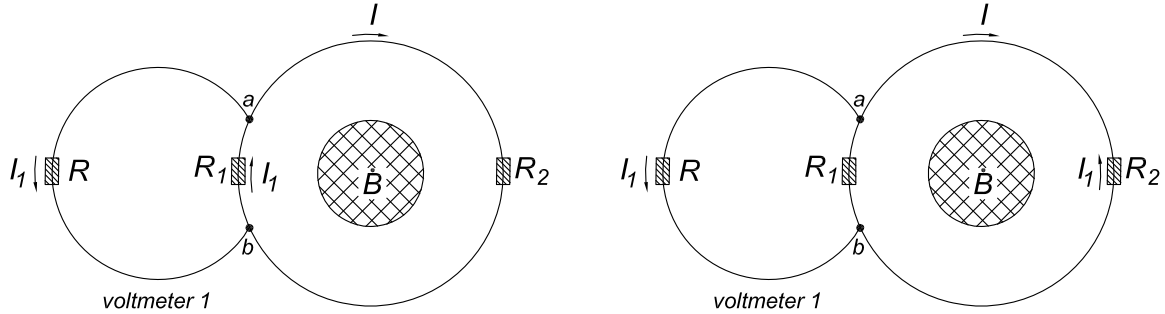
The meter readings do not depend on the locations of points  $a$ ,  $b$ ,  $c$  and  $d$ . But, if both meters are outside the secondary loop and their leads both attached to that loop from its "left" side, both meters would read  $-\mathcal{E}R_1/(R_1 + R_2)$ , while if all leads were attached from the "right" side they would both read  $\mathcal{E}R_2/(R_1 + R_2)$ .<sup>2</sup> But when one meter is on the left and the other is on the right, their readings have opposite signs, and also obey  $|V_{\text{meter1}}| + |V_{\text{meter2}}| = \mathcal{E}$ .

These results were validated by experiment during Lewin's lecture demonstration.

<sup>2</sup>See Fig. 6 of [10].

### 2.1.1 Which Way Does Current $I_1$ Go Around the Secondary Loop?

In the preceding analysis we tacitly assumed that the current  $I_1$  flows only on the left side of the secondary loop between points  $a$  and  $b$ , as shown on the left below.



But it could be that the current flowed on the right side of the secondary loop, as shown on the right above. In this case Kirchhoff's law for the secondary loop is

$$\mathcal{E} = IR_1 + (I - I_1)R_2 = (R_1 + R_2)I - R_2I_1, \quad (8)$$

while Kirchhoff's law for the voltmeter loop is

$$-\mathcal{E} = I_1R + (I_1 - I)R_2 \approx -R_2I + RI_1, \quad (9)$$

noting that now all the magnetic flux in the solenoid is linked by the voltmeter loop, but since the sense of current  $I_1$  is defined to be opposite to that of current  $I$  the effective  $\mathcal{EMF}$  in the voltmeter loop is  $-\mathcal{E}$  rather than  $\mathcal{E}$ . Solving eqs. (8)-(9) for the currents we find

$$I = \frac{\mathcal{E}}{R_1 + R_2}, \quad I_1 = -\frac{\mathcal{E}R_1}{R(R_1 + R_2)}, \quad (10)$$

as found in eq. (6) assuming that current  $I_1$  flowed on the left side of the secondary loop. Hence, the value of the current  $I_1$  does not depend on whether it flowed on the left or on the right side of the primary loop, and the circuit analysis based on Kirchhoff's law cannot determine the partitioning of current  $I_1$  between the left and right sides.

However, the amount of heat dissipated in resistors  $R_1$  and  $R_2$  depends on the partitioning of current  $I_1$  between them. We suppose that the partition minimizes the heat dissipation. Defining  $f$  to be the fraction of current  $I_1$  that passes through resistor  $R_1$ , the power dissipated in the two resistors is

$$P = (I + fI_1)^2R_1 + [I - (1 - f)I_1]^2R_2 = 2fII_1(R_1 - R_2) + f^2I_1^2(R_1 + R_2) + \text{constant}. \quad (11)$$

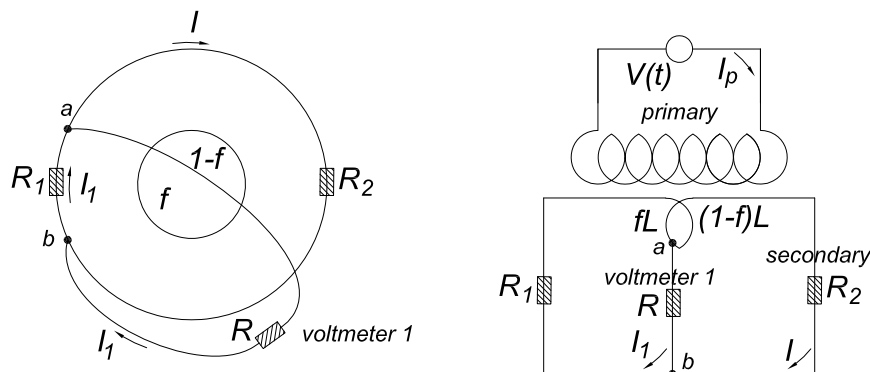
On minimizing the power  $P$  we find that

$$f = -\frac{I}{I_1} = \frac{R}{R_1} \gg 1. \quad (12)$$

Since  $f$  cannot be larger than 1, we infer that  $f = 1$ , and all the current  $I_1$  flows through resistor  $R_1$  as initially assumed.

### 2.1.2 The Voltmeter Leads Pass Through the Interior of the Secondary Loop

Suppose the voltmeter leads cross the interior of the secondary loop, such that a fraction  $f$  of the magnetic flux of the solenoid passes through the voltmeter loop, as shown in the sketches below.



As discussed in sec. 2.1.1, it suffices to suppose that all of current  $I_1$  passes through resistor  $R_1$ . Then, Kirchhoff's law for the secondary loop is

$$\mathcal{E} = (I + I_1)R_1 + IR_2 = (R_1 + R_2)I + R_1I_1, \quad (13)$$

while Kirchhoff's law for the voltmeter loop is

$$f\mathcal{E} = I_1R + (I_1 + I)R_1 \approx R_1I + RI_1, \quad (14)$$

noting that the sense of current  $I_1$  is the same as that of current  $I$ , such that the  $\mathcal{EMF}$  which drives the voltmeter loop is  $f\mathcal{E}$ . Solving eqs. (13)-(14) for the currents we find

$$I = \frac{\mathcal{E}}{R_1 + R_2}, \quad I_1 = -\frac{\mathcal{E}[R_1 - f(R_1 + R_2)]}{R(R_1 + R_2)}. \quad (15)$$

The limiting cases for the current  $I_1$  are

$$I_1(f = 0) = -\frac{\mathcal{E}R_1}{R(R_1 + R_2)}, \quad I_1(f = 1) = \frac{\mathcal{E}R_2}{R(R_1 + R_2)}, \quad (16)$$

which correspond to the previous results for the voltmeter on the “left” and on the “right”, respectively. Thus, there is a continuum of possible readings of the voltmeter between the “left” and “right” readings, depending on the routing of the voltmeter leads.

## 2.2 Scalar and Vector Potentials

The fact that the two meters give different readings when  $a = c$  and  $b = d$  (in sec. 2.1) indicates that the meter readings are not simply related to the electric scalar potential  $V$  at those points, as would be the case for a DC circuit.

In DC circuit analysis, ideal conductors (such as we have tacitly assume all wires in the system to be) are equipotentials. However, in time-dependent circuit analysis the proper assumption is that the electric field tangential to the wires is zero (in the limit of perfectly conducting wires).

In time-dependent situations the electric field  $\mathbf{E}$  is related to both the scalar potentials  $V$  as well as to the vector potential  $\mathbf{A}$  according to

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \quad (17)$$

Supposing that the secondary loop is circular, its tangential component  $E_\phi$  is given

$$E_\phi = -\frac{1}{r} \frac{\partial V}{\partial \phi} - \frac{\partial A_\phi}{\partial t}. \quad (18)$$

Since  $\mathbf{B} = \nabla \times \mathbf{A}$ , we have from Stoke's theorem that  $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area} = \Phi$ , so the azimuthal component  $A_\phi$  if the vector potential is given by

$$A_\phi = \frac{\Phi}{2\pi r}, \quad (19)$$

where  $r$  is the radius of the secondary loop. In the present example the magnetic flux  $\Phi$  through the secondary loop has contributions from the magnetic fields due to all four currents  $I_p$ ,  $I$ ,  $I_1$  and  $I_2$ . Of these only the contribution  $\Phi_p$  from the primary current  $I_p$  is significant.

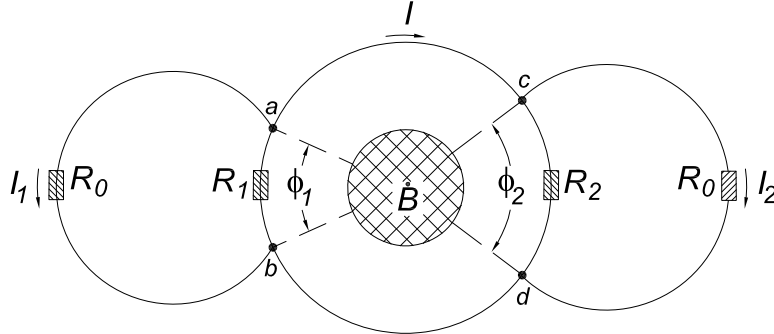
Applying this to wire segments in the central loop, for which  $E_\phi = 0$ , we have that

$$\frac{\partial V}{\partial \phi} = -r \frac{\partial A_\phi}{\partial t} = -\frac{\dot{\Phi}}{2\pi} \approx -\frac{\dot{\Phi}_p}{2\pi} = \frac{\mathcal{E}}{2\pi}, \quad (20)$$

recalling eq. (2), and so the scalar potential along a wire segment has the form

$$V(\phi) = V_0 + \frac{\mathcal{E}\phi}{2\pi} \quad (21)$$

where  $\phi$  increases for counterclockwise movement around the loop.



The voltage drops across resistors  $R_1$  and  $R_2$  are

$$\Delta V_1 \approx IR_1 \approx \frac{\mathcal{E}R_1}{(R_1 + R_2)}, \quad \Delta V_2 \approx IR_2 \approx \frac{\mathcal{E}R_2}{(R_1 + R_2)}, \quad (22)$$

if the azimuthal extent of the resistors is negligible, and we now move in a clockwise sense around the loop. In this convention the voltage drop along each of the wire segments of the secondary loop are  $-\mathcal{E}/2$ , so the total voltage drop around the loop is zero, as expected for

the scalar potential. Finally, the voltage drops between the points where the voltmeters are attached to the central loop are

$$V_a - V_b \approx \frac{\mathcal{E}\phi_1}{2\pi} - \frac{\mathcal{E}R_1}{(R_1 + R_2)}, \quad V_c - V_d \approx -\frac{\mathcal{E}\phi_2}{2\pi} + \frac{\mathcal{E}R_2}{(R_1 + R_2)}. \quad (23)$$

Only if the meter leads are connected directly to the ends of resistors  $R_1$  and  $R_2$  (as would be good practice) do the meter readings equal the voltage differences between the tips of the leads.<sup>3</sup>

*For additional examples of the relation of voltmeter readings to the scalar and vector potentials in time-varying situations, see [18].*

### 2.2.1 Secondary Loop of Resistive Wire

In an early statement [4] of the present problem, resistors 1 and 2 were not localized objects. Rather, the secondary loop was made of a resistive wire of total resistance  $R_0$ . Then, the current  $I$  in the loop would be  $\mathcal{E}/R_0$ . In this case, the  $IR$  drop between points  $a$  and  $b$  that subtend angle  $\phi$  is  $IR_0\phi/2\pi = \mathcal{E}\phi/2\pi = V_a - V_b$ . That is, in this version the  $IR$  drop equals the difference in the scalar potential between two points, which is perhaps less instructive than the case with localized resistors.

### 2.2.2 On the “Reality” of the Vector Potential

The example of a long solenoid (or toroid) with “zero” magnetic field, but nonzero vector potential, outside the coil, is often used to argue that the vector potential (or at least differences in the vector potential) should be considered as “real” (*i.e.*, having measurable effect). The best known of the arguments in the quantum example of Aharonov and Bohm [19], although this independently restates an earlier argument by Ehrenberg and Siday [20]. A related argument for classical electrodynamics has been given by Konopinski [21]; see also [22, 23].

The classical argument is weaker, being that the electric field outside the solenoid with a time-varying current is due to the magnetic field inside the solenoid according to Faraday’s law, but the magnetic field is “zero” outside the solenoid, so this effect appears to be action at a distance. Since field theory was developed with the goal of eliminating action at a distance, it seems that the “local” result,

$$\mathbf{E}_{\text{induced}} = -\frac{\partial \mathbf{A}}{\partial t}, \quad (24)$$

implies that we should consider the vector potential to be “real”. However, as noted by Lodge in 1889 [1], the magnetic field outside the solenoid is not strictly zero, and we can

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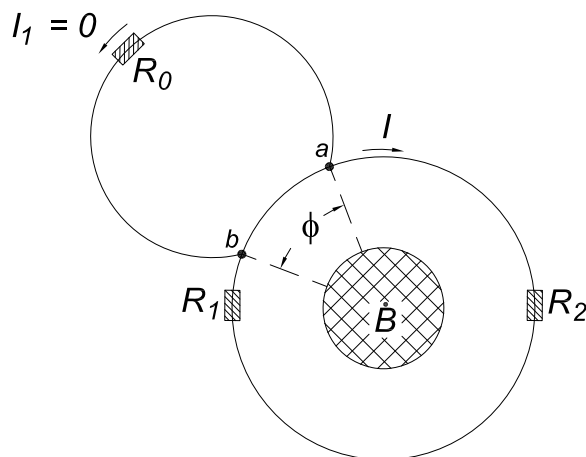
<sup>3</sup>If the meter leads cross the region of magnetic flux from the solenoid, intercepting fraction  $f$  of that flux as in sec. 2.1.2, there is always a set of points  $\{a, b\}$  such that the meter reading equals  $V_a - V_b$ . In particular, if  $f = 1/2$  the desired points are halfway between the two resistors, as noted in [9]. However, in these cases the meter leads cannot be short.

argue that the weak time-varying magnetic field outside the solenoid “creates” the induced electric field there.<sup>4</sup>

The quantum argument seems stronger, in that even for a static magnetic field in the solenoid, there is a detectable effect on the trajectories of electrons that pass outside the solenoid [19, 20]. Independent of the skepticism expressed in articles such as [26], the author notes that the supposedly “real” vector potential in the examples of Lewin, and of Aharonov and Bohm, is the gauge-invariant rotational part of the vector potential,<sup>5</sup> which is the total vector potential in the Coulomb gauge. This vector potential includes terms that depend on the instantaneous current distribution throughout the Universe, *i.e.*, it incorporates action at a distance. The author’s attitude is that any quantity which involves action at a distance is not physically “real”,<sup>6</sup> and hence even the gauge-invariant part of the vector potential should not be regarded as “real”.

### 2.3 Comments

Suppose a voltmeter were connected to two points on the upper wire between resistors 1 and 2, as shown in the sketch below. The voltmeter loop is not coupled to the solenoid, so there is no  $\mathcal{EMF}$  in this loop, and hence  $I_1 = 0$ , and the meter reading would be  $V_{\text{meter}} = 0$ .



However, the difference between the scalar potential at points  $a$  and  $b$  is

$$V_a - V_b \approx \frac{\mathcal{E}\phi}{2\pi}, \quad (25)$$

where  $\phi$  is the azimuthal angle between the two points, recalling CEQ. (21). While the meter does not read the difference in the scalar potential between the tips of its leads, the result  $V_{\text{meter}} = 0$  is appealing in that we might naïvely expect the “voltage drop” to be zero between points along a good/perfect conductor.

This leads some people to argue that the term “voltage drop” should not be defined as the (unique) difference in the scalar potential  $V$  between two points (as done in this note),

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<sup>4</sup>In greater detail, weak radiation fields exist outside a time-varying solenoid (and toroid), which must be considered in a full classical description. See, for example, [22, 24, 25].

<sup>5</sup>See, for example, sec. 2.1 of [27].

<sup>6</sup>The wave function of nonrelativistic quantum mechanics is such a quantity, and thus not “real” in the author’s view.

but rather this term should have only the operational meaning as the value measured by a voltmeter when connected to those two points. While possibly appealing in examples such as the present, this usage renders the concept of “voltage drop” to be more a property of the voltmeter (and the routing of its leads) than of the circuit it probes. See [18] for further discussion.

In the present example,

$$V_{\text{meter}} = \int_a^b \mathbf{E} \cdot d\mathbf{l}. \quad (26)$$

While some people designate the integral  $\int_a^b \mathbf{E} \cdot d\mathbf{l}$  as a “voltage drop”, we advocate calling this the  $\mathcal{EMF}$  between points  $a$  and  $b$ , and that the “voltage drop” between points  $a$  and  $b$  be reserved to mean simply  $V_a - V_b$ , the difference in the scalar potential between the two points.

## References

- [1] O. Lodge, *On an Electrostatic Field Produced by Varying Magnetic Induction*, Phil. Mag. **27**, 469 (1889),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/lodge\\_pm\\_27\\_116\\_89.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/lodge_pm_27_116_89.pdf)
- [2] J. Slepian, *Energy Flow In Electric Systems – the Vi Energy-Flow Postulate*, Trans. Am. IEE **61**, 835 (1942),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/slepian\\_taiee\\_61\\_835\\_42.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/slepian_taiee_61_835_42.pdf)
- [3] E.G. Cullwick, *P.D. vs. E.M.F.*, Students Quart. J. **30**, 140 (1960),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/cullwick\\_sqj\\_30\\_140\\_60.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/cullwick_sqj_30_140_60.pdf)
- [4] J.W. Buchta, *Electromotive Force and Faraday’s Law of Electromagnetic Induction*, Phys. Teacher **1**, 133 (1963),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/buchta\\_pt\\_1\\_133\\_63.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/buchta_pt_1_133_63.pdf)
- [5] M. Philips, *Electromotive Force and the Law of Induction*, Phys. Teacher **1**, 155 (1963),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/philips\\_pt\\_1\\_155\\_63.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/philips_pt_1_155_63.pdf)
- [6] D.R. Moorcroft, *Faraday’s Law – Demonstration of a Teaser*, Am. J. Phys. **37**, 221 (1969), [http://puhep1.princeton.edu/~mcdonald/examples/EM/moorcroft\\_ajp\\_37\\_221\\_69.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/moorcroft_ajp_37_221_69.pdf)
- [7] D.R. Moorcroft, *Faraday’s Law, Potential and Voltage – Demonstration of a Teaser*, Am. J. Phys. **38**, 376 (1970),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/moorcroft\\_ajp\\_38\\_376.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/moorcroft_ajp_38_376.pdf)
- [8] W. Klein, *Discovering Induction*, Am. J. Phys. **49**, 603 (1981),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/klein\\_ajp\\_49\\_603\\_81.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/klein_ajp_49_603_81.pdf)
- [9] F. Reif, *Generalized Ohm’s law, potential difference, and voltage measurements*, Am. J. Phys. **50**, 1048 (1982),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/reif\\_ajp\\_50\\_1048\\_82.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/reif_ajp_50_1048_82.pdf)

- [10] R.H. Romer, *What do voltmeters measure?: Faraday's law in a multiply connected region*, Am. J. Phys. **50**, 1089 (1982),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/romer\\_ajp\\_50\\_1089\\_82.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/romer_ajp_50_1089_82.pdf)
- [11] P.C. Peters, *The role of induced emf's in simple circuits*, Am. J. Phys. **52**, 208 (1983),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/peters\\_ajp\\_52\\_208\\_83.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/peters_ajp_52_208_83.pdf)
- [12] W. Gough and J.P.G. Richards, **Electromagnetic or electromagnetic induction?**, Eur. J. Phys. **7**, 195 (1986),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/gough\\_ejp\\_7\\_195\\_86.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/gough_ejp_7_195_86.pdf)
- [13] J. Roche, *Explaining electromagnetic induction: a critical re-examination*, Phys. Ed. **22**, 91 (1987),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/roche\\_pe\\_22\\_91\\_87.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/roche_pe_22_91_87.pdf)
- [14] E. Lanzara and R. Zangara, *Potential difference measurements in the presence of a varying magnetic field*, Phys. Ed. **30**, 85 (1995),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/lanzara\\_pe\\_30\\_85\\_95.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/lanzara_pe_30_85_95.pdf)
- [15] D. Goldberg and O. Zik, *Topologically dependent voltage measurement*, Phys. Ed. **26**, 256 (1991),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/goldberg\\_pe\\_26\\_256\\_91.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/goldberg_pe_26_256_91.pdf)
- [16] D. Iencinella and G. Matteucci, *An introduction to the vector potential*, Eur. J. Phys. **25**, 249 (2004),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/iencinella\\_ejp\\_25\\_249\\_04.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/iencinella_ejp_25_249_04.pdf)
- [17] Maxim Integrated Products, *Position Sensor Exploits Faraday's Law*, App. Note 2238 (July 21, 2003), [http://puhep1.princeton.edu/~mcdonald/examples/EM/maxim\\_2238\\_03.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/maxim_2238_03.pdf)
- [18] K.T. McDonald, *What Does an AC Voltmeter Measure?* (March 16, 2008),  
<http://puhep1.princeton.edu/~mcdonald/examples/voltage.pdf>
- [19] Y. Aharonov and D. Bohm, *Significance of the Electromagnetic Potentials in the Quantum Theory*, Phys. Rev. **115**, 485 (1959),  
[http://puhep1.princeton.edu/~mcdonald/examples/QM/aharonov\\_pr\\_115\\_485\\_59.pdf](http://puhep1.princeton.edu/~mcdonald/examples/QM/aharonov_pr_115_485_59.pdf)
- [20] W. Ehrenberg and R.E. Siday, *Significance of the Electromagnetic Potentials in the Quantum Theory*, Proc. Phys. Soc. B **62**, 8 (1949),  
[http://puhep1.princeton.edu/~mcdonald/examples/QM/ehrenberg\\_pps\\_b62\\_8\\_49.pdf](http://puhep1.princeton.edu/~mcdonald/examples/QM/ehrenberg_pps_b62_8_49.pdf)
- [21] E.J. Konopinski, *What the electromagnetic vector potentials describes*, Am. J. Phys. **46**, 499 (1978),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/konopinski\\_ajp\\_46\\_499\\_78.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/konopinski_ajp_46_499_78.pdf)
- [22] B. Batell and A. Ferstl, *Electrically induced magnetic fields; a consistent approach*, Am. J. Phys. **71**, 925 (2003),  
[http://puhep1.princeton.edu/~mcdonald/examples/EM/batell\\_ajp\\_71\\_925\\_03.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/batell_ajp_71_925_03.pdf)

- [23] G. Rousseaux, R. Kofman, and O. Minazzoli, *The Maxwell-Lodge effect: significance of electromagnetic potentials in the classical theory*, Eur. Phys. J. D **49**, 249 (2008), [http://puhep1.princeton.edu/~mcdonald/examples/EM/rousseau\\_epjd\\_49\\_249\\_08.pdf](http://puhep1.princeton.edu/~mcdonald/examples/EM/rousseau_epjd_49_249_08.pdf)
- [24] K.T. McDonald, *The Fields Outside a Long Solenoid with a Time-Dependent Current* (Dec. 6, 1996), <http://puhep1.princeton.edu/~mcdonald/examples/solenoid.pdf>
- [25] K.T. McDonald, *Electromagnetic Fields of a Small Helical Toroidal Antenna* (Dec. 8, 2008), <http://puhep1.princeton.edu/~mcdonald/examples/cwhta.pdf>
- [26] T.H. Boyer, *Misinterpretation of the Aharonov-Bohm Effect*, Am. J. Phys. **40**, 56 (1972), [http://puhep1.princeton.edu/~mcdonald/examples/QM/boyer\\_ajp\\_40\\_56\\_72.pdf](http://puhep1.princeton.edu/~mcdonald/examples/QM/boyer_ajp_40_56_72.pdf)
- [27] K.T. McDonald, *Orbital and Spin Angular Momentum of Electromagnetic Fields* (Mar. 12, 2009), <http://puhep1.princeton.edu/~mcdonald/examples/spin.pdf>