

Unipolar Induction via a Rotating Magnetized Cylinder

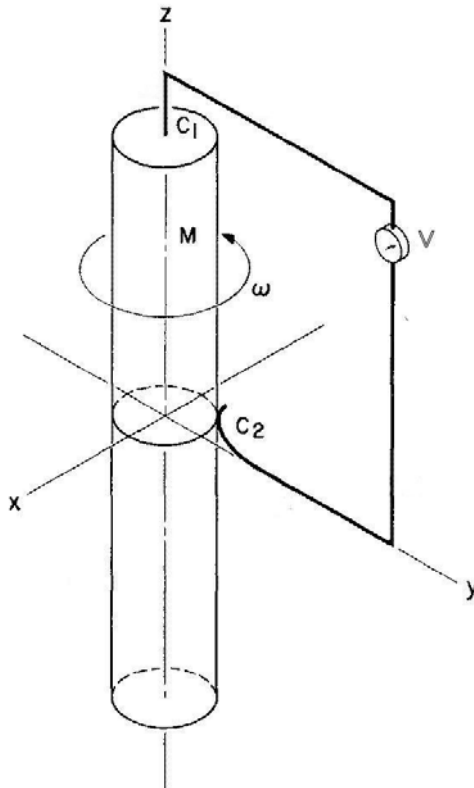
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1 Problem

A conducting cylinder of radius R with permanent magnetization density \mathbf{M}_0 parallel to its axis when at rest is rotated about that axis with angular velocity $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$ with respect to the lab frame. A voltmeter with very high internal resistance is connected to the rotating cylinder via wires with sliding contacts, one of which (C_1) is on the axis of the cylinder and the other (C_2) is on the circumference, as shown below.



Deduce the voltage V observed on the voltmeter by a lab-frame analysis as well as by an analysis in the rotating frame. You may assume that the velocity ωR is small compared to the speed of light c . Comment on the electric polarization density \mathbf{P} in the cylinder should it have (relative) permittivity ϵ that differs from unity.

This configuration of unipolar induction was first considered by Faraday in 1851 [1], who also considered the case of the magnetized cylinder at rest while the voltmeter and contact wires rotated around the axis of the cylinder.

2 Solution

2.1 Analysis Using a Comoving Inertial Frame

As discussed in [2, 3, 4, 5], the best approach to an understanding of lab-frame electrodynamics of a rotating system is via a comoving inertial frame corresponding to some point in the rotating system.

When the internal resistance of the voltmeter is high, negligible current flows through it. Hence, there are no significant conduction currents (and no net free charge density) in the problem, so that the magnetic field $\mathbf{H} = 0$ (and electric displacement $\mathbf{D} = 0$), and the magnetic field \mathbf{B} inside the cylinder is related by $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 4\pi M \hat{\mathbf{z}}$ (in Gaussian units), where \mathbf{M} is the magnetization of the rotating cylinder in the lab frame. However, it could be that this magnetization differs from \mathbf{M}_0 when the cylinder is at rest in the (inertial) lab frame.

We follow Minkowski [2] in arguing that the local magnetization at a point P in the rotating cylinder equals the rest value \mathbf{M}_0 according to an observer in the inertial frame that is instantaneously comoving with point P . That is $\mathbf{M}^* = \mathbf{M}_0$, where the superscript $*$ indicates quantities observed in the comoving inertial frame.

Similarly, we expect that the electric polarization \mathbf{P}^* near point P in the comoving inertial frame equals that of the magnetized cylinder in an inertial rest frame, namely $\mathbf{P}^* = 0$.

Writing \mathbf{v} as the velocity of point P in the lab frame, the field transformations to the comoving inertial frame are [6]

$$\begin{aligned} \mathbf{B}^* &= \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, & \mathbf{D}^* &= \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, & \mathbf{E}^* &= \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, & \mathbf{H}^* &= \mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}, \\ \mathbf{M}^* &= \mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}, & \mathbf{P}^* &= \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \end{aligned} \quad (1)$$

to order v/c . Hence, $\mathbf{D}^* = 0 = \mathbf{H}^*$ and $\mathbf{B}^* = 4\pi\mathbf{M}^* = 4\pi\mathbf{M}_0$. Then, from the relation $\mathbf{D}^* = \mathbf{E}^* + 4\pi\mathbf{P}^*$, we learn that $\mathbf{E}^* = 0$ also near point P .

We now know all six fields \mathbf{B}^* , \mathbf{D}^* , \mathbf{E}^* , \mathbf{H}^* , \mathbf{P}^* and \mathbf{M}^* near point P in the comoving inertial frame, so we can transform these back to the lab frame using the inverse of eq. (1),

$$\begin{aligned} \mathbf{B} &= \mathbf{B}^* + \frac{\mathbf{v}}{c} \times \mathbf{E}^*, & \mathbf{D} &= \mathbf{D}^* - \frac{\mathbf{v}}{c} \times \mathbf{H}^*, & \mathbf{E} &= \mathbf{E}^* - \frac{\mathbf{v}}{c} \times \mathbf{B}^*, & \mathbf{H} &= \mathbf{H}^* + \frac{\mathbf{v}}{c} \times \mathbf{D}^*, \\ \mathbf{M} &= \mathbf{M}^* - \frac{\mathbf{v}}{c} \times \mathbf{P}^*, & \mathbf{P} &= \mathbf{P}^* + \frac{\mathbf{v}}{c} \times \mathbf{M}^*. \end{aligned} \quad (2)$$

We now find the lab-frame fields inside the rotating cylinder to be

$$\mathbf{B} = 4\pi\mathbf{M}_0, \quad \mathbf{D} = 0, \quad \mathbf{E} = -\frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0, \quad \mathbf{H} = 0, \quad \mathbf{P} = \frac{\mathbf{v}}{c} \times \mathbf{M}_0, \quad \mathbf{M} = \mathbf{M}_0. \quad (3)$$

Writing $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ for the velocity of point P at position \mathbf{x} in the lab frame, the electric field inside the cylinder is, in cylindrical coordinates (r, ϕ, z) ,

$$\mathbf{E} = -\frac{4\pi\omega r M_0}{c} \hat{\mathbf{r}}, \quad (4)$$

so the voltage difference between a point on the axis of the cylinder and its outer surface is

$$V = \int_0^R \mathbf{E} \cdot d\mathbf{r} = -\frac{2\pi\omega R^2 M_0}{c}. \quad (5)$$

There is a nonzero electric field $\mathbf{E} = -4\pi\mathbf{P} = -\mathbf{v}/c \times 4\pi\mathbf{M}_0$ inside the rotating cylinder according to an observer in the lab frame, but the electric displacement \mathbf{D} in the lab frame is zero. Hence, the relation $\mathbf{D} = \epsilon\mathbf{E}$ does not hold in the lab frame for a rotating (dielectric) cylinder.

Again, we follow Minkowski [2] in noting that the constitutive equation

$$\mathbf{D}^* = \epsilon\mathbf{E}^* \quad (6)$$

holds in the comoving inertial frame in the vicinity of point P in cases where the electric field \mathbf{E}^* is nonzero in that frame. The present example involves a permanent magnet with magnetization \mathbf{M}_0 in the comoving inertial frame, so we do not say that $\mathbf{B}^* = \mu\mathbf{H}^*$, but simply that

$$\mathbf{B}^* = \mathbf{H}^* + 4\pi\mathbf{M}_0, \quad (7)$$

in case there were conduction currents present so that \mathbf{H}^* were nonzero. Expressing eqs. (6)-(7) in terms of lab-frame quantities we obtain the following form of the constitutive equations,

$$\mathbf{D} = \epsilon \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0 \right) - (\epsilon - 1) \frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}_0 - (\epsilon - 1) \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (8)$$

to order v/c in the lab frame for a rotating permanent magnet with permittivity ϵ . The lab-frame fields (3) are a particular examples of these relations.

There is no free charge in or on the cylinder, but there is a uniform bound charge density given by

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P} = \nabla \cdot \frac{\mathbf{E}}{4\pi} = \frac{1}{4\pi r} \frac{\partial(rE_r)}{\partial r} = -\frac{2\omega M_0}{c}. \quad (9)$$

Likewise there is a bound surface charge density on the circumferential surface of the cylinder given by

$$\sigma_{\text{bound}} = \mathbf{P} \cdot \hat{\mathbf{r}} = -\frac{\mathbf{E} \cdot \hat{\mathbf{r}}}{4\pi} = \frac{\omega R M_0}{c}. \quad (10)$$

The total bound charge is zero,

$$Q_{\text{bound}} = \int \rho_{\text{bound}} d\text{Vol} + \int \sigma_{\text{bound}} d\text{Area} = -\int_0^R \frac{2\omega M_0}{c} 2\pi h r dr + 2\pi R h \frac{\omega R M_0}{c} = 0. \quad (11)$$

The bound charge density (10) does not depend on the permittivity ϵ and is not related to molecular electric dipoles. Rather, the lab-frame electric polarization $\mathbf{P} = \mathbf{v}/c \times \mathbf{M}_0$ is an effect of the moving/rotating magnetization. See, for example, sec. 88 of [6] or sec. 18-6 of [7].

If the rotating cylinder has $\epsilon = 1$ and is immersed in a nonuniform external electric field $\mathbf{E}_{\text{ext}}(\mathbf{x})$, then the cylinder would experience a net force

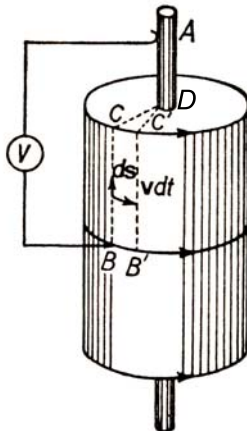
$$\mathbf{F} = \int \rho_{\text{bound}} \mathbf{E}_{\text{ext}} d\text{Vol} + \int \sigma_{\text{bound}} \mathbf{E}_{\text{ext}} d\text{Area}. \quad (12)$$

2.2 Analysis in the Lab Frame

The analysis in the lab frame follows Chap. E III of [6], where it is naively assumed that the magnetization of the rotating cylinder in the lab frame is \mathbf{M}_0 . See also [8]. Strictly speaking, the analysis of sec. 2.2 does not hold without having first made the arguments of sec. 2.1.

2.2.1 Analysis via Faraday's Law

The current in the circuit ABCDA in the figure below is negligible because of the high resistance of the voltmeter, so the resistive voltage drop in the circuit can be ignored.



Then, the reading V on the voltmeter equals the electromotive force around the circuit. According to Faraday's law, the electromotive force in the lab frame is given by

$$V = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c} \frac{d\Phi_B}{dt}, \quad (13)$$

in Gaussian units, where $\Phi_B = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area}$ is the magnetic flux linked by the loop ABCDA. Since the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 4\pi M_0 \hat{\mathbf{z}}$ inside the cylinder has no azimuthal component in this example, it might seem that $\Phi_B = 0$ and hence, $V = 0$. However, because the cylinder is rotating, we can argue that the portion BCD of the loop deforms into BB'C'D during time dt where the arc BB' has length $v dt = \omega R dt$. Hence, the flux through the loop increases by amount $\omega R^2 B dt/2$ during time dt , and the voltage according to eq. (1) is

$$V = -\frac{\omega R^2 B}{2c} = -\frac{2\pi\omega R^2 M_0}{c}. \quad (14)$$

The negative sign of the voltage (14) means that it is higher at points B and C than at points A and D. We infer that there is an inward radial electric field E_r inside the cylinder such that $V = \int_0^R E_r dr$, and hence

$$\mathbf{E}_{\text{in}} = -\frac{\omega R B}{c} \hat{\mathbf{r}} = -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0, \quad (15)$$

as found in eq. (3).

2.2.2 Analysis Using the Lorentz Force Law

Another well-known analysis in the lab frame notes that the conduction electrons in the rotating cylinder have no radial motion since no current flows in the circuit. These electrons, of charge e , experience a Lorentz force

$$F = e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (16)$$

which must equal the centripetal force $-m\omega^2 \mathbf{r}$ required for the charges to undergo uniform circular motion. The usual approximation is that this centripetal force is negligible on the scale of $e\mathbf{E}$, so that the electric field inside the rotating cylinder must be

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0, \quad (17)$$

as found previously, if we again assume that $\mathbf{B} = 4\pi\mathbf{M}_0$,

2.3 Analysis in the Rotating Frame

The principles of electrodynamics in a rotating frame are summarized in the Appendix.

We cannot assume without question that the magnetization of the cylinder is \mathbf{M}_0 according to an observer at rest in the rotating frame. The best strategy is to use the comoving analysis of sec. 2.1 to identify the fields in the lab frame, and then use the transformations (24)-(25) to find the fields in the rotating frame, which we designate with a $'$:

$$\mathbf{B}' = 4\pi\mathbf{M}_0, \quad \mathbf{D}' = 0, \quad \mathbf{E}' = 0, \quad \mathbf{H}' = 0, \quad \mathbf{P}' = 0, \quad \mathbf{M}' = \mathbf{M}_0. \quad (18)$$

In the spirit of sec. 2.2, we might have naively assumed these results to be obvious.

We can now consider Maxwell's equations (30)-(33) for \mathbf{D}' and \mathbf{H}' in the rotating frame. In the present example there are no free sources for \mathbf{D}' or \mathbf{H}' , and also no "other" sources according to eqs. (34)-(35). Thus, it is consistent with Maxwell's equations in the rotating frame that $\mathbf{D}' = 0 = \mathbf{H}'$. Then, using $\mathbf{P}' = 0$ and $\mathbf{M}' = \mathbf{M}_0$ we have that $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Alternatively, we can consider Maxwell's equations (36)-(37) for \mathbf{E}' and \mathbf{B}' . On examining the extensive list (38)-(41) of possible sources in the rotating frame, we see that the eqs. (36)-(37) reduce to

$$\nabla' \cdot \mathbf{E}' = 0, \quad \nabla' \times \mathbf{B}' = \nabla' \times 4\pi\mathbf{M}_0, \quad (19)$$

so that we again find $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Transforming the fields from the rotating frame back to the lab frame we again obtain the results of eq. (3).¹

Although the electric polarization, $\mathbf{P}' = 0$, vanishes in the rotating frame (since this could only be due to a moving magnetization in this example), the bound charge density (26) is nonzero,

$$\rho'_{\text{bound}} = -\nabla' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \nabla' \times \mathbf{M}' = -\frac{2\omega M_0}{c} = \rho_{\text{bound}}, \quad (20)$$

¹In particular, we find it completely consistent to use the transformation $\mathbf{P} = \mathbf{P}' + \mathbf{v}/c \times \mathbf{M}'$ from the rotating frame to the lab frame, despite a claim to the contrary in [12].

recalling eq. (9). Similarly, there is a bound surface charge density on the outer circumference of the cylinder in the rotating frame given by

$$\sigma'_{\text{bound}} = \frac{\omega R M'}{c} = \frac{\omega R M_0}{c} = \sigma_{\text{bound}}, \quad (21)$$

recalling eq. (10).

A Summary of the Principles of Electrodynamics in a Rotating Frame

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.^{2,3}

The (cylindrical) coordinate transformation is

$$r' = r, \quad \phi' = \phi - \omega t, \quad z' = z, \quad t' = t, \quad (22)$$

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of charge and current density are

$$\rho' = \rho, \quad \mathbf{J}' = \mathbf{J} - \rho \mathbf{v}, \quad (23)$$

where \mathbf{v} ($v \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame. The transformations of the electromagnetic fields are

$$\mathbf{B}' = \mathbf{B}, \quad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{H}' = \mathbf{H}. \quad (24)$$

The transformations of the electric and magnetic polarizations are

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \quad \mathbf{M}' = \mathbf{M}, \quad (25)$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities $\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$ and $\mathbf{J}_{\text{bound}} = \partial\mathbf{P}/\partial t + c\nabla \times \mathbf{M}$ transform to

$$\rho'_{\text{bound}} = -\nabla' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \nabla' \times \mathbf{M}', \quad (26)$$

$$\mathbf{J}'_{\text{bound}} = \frac{\partial\mathbf{P}'}{\partial t'} + c\nabla' \times \mathbf{M}' + \mathbf{v}(\nabla' \cdot \mathbf{P}') + \frac{\mathbf{v}}{c} \times \frac{\partial\mathbf{M}'}{\partial t'} + \boldsymbol{\omega} \times \mathbf{P}' - \omega \frac{\partial\mathbf{P}'}{\partial\phi'}. \quad (27)$$

Force \mathbf{F} is invariant under the transformation (22). In particular, a charge q with velocity \mathbf{v}_q in the lab frame experiences a Lorentz force in the rotating frame given by

$$\mathbf{F}' = q \left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c} \times \mathbf{B}' \right) = q \left(\mathbf{E} + \frac{\mathbf{v}_q}{c} \times \mathbf{B} \right) = \mathbf{F}, \quad (28)$$

²This Appendix is from sec. 2.2.5 of [5].

³This case is discussed most thoroughly by Ridgely [10, 11], but primarily for the interesting limit of steady charge and current distributions.

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density \mathbf{f}' on charge and current densities in the rotating frame is

$$\mathbf{f}' = \rho' \mathbf{E}' + \frac{\mathbf{J}'}{c} \times \mathbf{B}' = (\rho'_{\text{free}} + \rho'_{\text{bound}}) \mathbf{E}' + \frac{\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{bound}}}{c} \times \mathbf{B}'. \quad (29)$$

Maxwell's equations in the rotating frame can be written

$$\nabla' \cdot \mathbf{B}' = 0, \quad (30)$$

$$\nabla' \cdot \mathbf{D}' = 4\pi \rho'_{\text{free,total}} = 4\pi (\rho'_{\text{free}} + \rho'_{\text{other}}), \quad (31)$$

$$\nabla' \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial ct'} = 0, \quad (32)$$

$$\nabla' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial ct'} = \frac{4\pi}{c} \mathbf{J}'_{\text{free,total}} = \frac{4\pi}{c} (\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{other}}), \quad (33)$$

where $\rho'_{\text{free}} = \rho_{\text{free}}$ and $\mathbf{J}'_{\text{free}} = \mathbf{J}_{\text{free}} - \rho_{\text{free}} \mathbf{v}$ are the free charge and current densities, and the “other” charge and current densities that appear to an observer in the rotating frame are

$$\rho'_{\text{other}} = -\frac{\mathbf{v} \cdot \mathbf{J}'_{\text{free}}}{c^2} + \frac{\boldsymbol{\omega} \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'}, \quad (34)$$

$$\mathbf{J}'_{\text{other}} = \rho'_{\text{free}} \mathbf{v} + \boldsymbol{\omega} \times \frac{\mathbf{D}'}{4\pi} - \frac{\boldsymbol{\omega}}{4\pi} \frac{\partial \mathbf{D}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'}. \quad (35)$$

The “other” charge and current distributions are sometimes called “fictitious” [9], but we find this term ambiguous. For an example with an “other” charge density $\boldsymbol{\omega} \cdot \mathbf{H}'/2\pi c$ in the rotating frame, see [13].

Maxwell's equations can also be expressed only in terms of the fields \mathbf{E}' and \mathbf{B}' and charge and current densities associated with free charges as well as with electric and magnetic polarization:

$$\nabla' \cdot \mathbf{E}' = 4\pi \rho'_{\text{total}}, \quad (36)$$

and

$$\nabla' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c} \mathbf{J}'_{\text{total}}, \quad (37)$$

where

$$\begin{aligned} \rho'_{\text{total}} &= \rho'_{\text{free}} - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}'_{\text{free}} - \nabla' \cdot \mathbf{P}' + \frac{\boldsymbol{\omega} \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'} \\ &= \rho'_{\text{free,total}} - \nabla' \cdot \mathbf{P}' \\ &= \rho'_{\text{free}} + \rho'_{\text{bound}} + \rho'_{\text{more}}, \end{aligned} \quad (38)$$

$$\rho'_{\text{more}} = -\frac{\mathbf{v}}{c^2} \cdot \left(\mathbf{J}'_{\text{free}} + \frac{\partial \mathbf{P}'}{\partial t'} + c \nabla' \times \mathbf{M}' \right) + \frac{\boldsymbol{\omega} \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{E}'}{\partial ct'}, \quad (39)$$

$$\begin{aligned} \mathbf{J}'_{\text{total}} &= \mathbf{J}'_{\text{free}} + \frac{\partial \mathbf{P}'}{\partial t'} + c \nabla' \times \mathbf{M}' + \rho'_{\text{free}} \mathbf{v} + \boldsymbol{\omega} \times \frac{\mathbf{D}'}{4\pi} - \frac{\boldsymbol{\omega}}{4\pi} \frac{\partial \mathbf{D}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'} \\ &= \mathbf{J}'_{\text{free,total}} + \frac{\partial \mathbf{P}'}{\partial t'} + c \nabla' \times \mathbf{M}' \end{aligned}$$

$$= \mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{bound}} + \mathbf{J}'_{\text{more}}, \quad (40)$$

$$\begin{aligned} \mathbf{J}'_{\text{more}} = & \mathbf{v} \left(\rho'_{\text{free}} - \nabla' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \nabla' \times \mathbf{M}' \right) \\ & + \boldsymbol{\omega} \times \frac{\mathbf{E}'}{4\pi} - \frac{\boldsymbol{\omega}}{4\pi} \frac{\partial \mathbf{E}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{B}'}{\partial t'}. \end{aligned} \quad (41)$$

The contribution of the polarization densities to the source terms in Maxwell's equations in much more complex in the rotating frame than in the lab frame. Because of the "other" source terms that depend on the fields in the rotating frame, Maxwell's equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for linear isotropic media at rest in the rotating frame are

$$\mathbf{D}' = \epsilon \mathbf{E}', \quad \mathbf{B}' = \mu \mathbf{H}' - (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{E}', \quad (42)$$

in the rotating frame, and

$$\mathbf{D} = \epsilon \mathbf{E} + (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H} - (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (43)$$

in the lab frame. The lab-frame constitutive equations (43) are the same as for a nonrotating medium that moves with constant velocity \mathbf{v} with respect to the lab frame.

We can also write the constitutive equations (42) for a linear isotropic medium in terms of the fields \mathbf{B}' , \mathbf{E}' , \mathbf{P}' and \mathbf{M}' by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that

$$\begin{aligned} \mathbf{P}' &= \frac{\epsilon - 1}{4\pi} \mathbf{E}', \\ \mathbf{M}' &= \left(1 - \frac{1}{\mu}\right) \frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right) \frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right) \frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)} \frac{\mathbf{v}}{c} \times \mathbf{P}'. \end{aligned} \quad (44)$$

Similarly, the constitutive equations (43) in the lab frame can be written to order v/c as

$$\begin{aligned} \mathbf{P} &= \frac{\epsilon - 1}{4\pi} \mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right) \frac{\mathbf{v}}{c} \times \frac{\mathbf{B}}{4\pi} = \frac{\epsilon - 1}{4\pi} \mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1} \frac{\mathbf{v}}{c} \times \mathbf{M}, \\ \mathbf{M} &= \left(1 - \frac{1}{\mu}\right) \frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right) \frac{\mathbf{v}}{c} \times \frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right) \frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)} \frac{\mathbf{v}}{c} \times \mathbf{P}. \end{aligned} \quad (45)$$

Ohm's law for the conduction current \mathbf{J}_C has the same form for a medium with velocity \mathbf{u}' relative to the rotating frame as it does for a medium with velocity \mathbf{u} relative to the lab frame,

$$\mathbf{J}'_C = \sigma \left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}' \right) = \sigma \left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right) = \mathbf{J}_C, \quad (46)$$

where σ is the electric conductivity of a medium at rest.

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