Maxwell's Objection to Lorenz' Retarded Potentials

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(October 26, 2009; updated June 27, 2010)

1 Problem

Maxwell seems to have considered the great paper of L. Lorenz on retarded potentials [1] (published simultaneously in 1867 with a paper written in 1858 by B. Riemann on the same theme [2])[1] as insufficiently supportive of his vision of a dynamical theory of the electromagnetic field [4], whereas the present attitude is that Riemann and Lorenz made important contributions to the Maxwellian view.[2]

Maxwell made an objection [5] (p. 651) that if a pair of equal and opposite charges move collinearly, then the retarded potential experienced by the charge in front has smaller magnitude than that experienced by the charge in the rear because the former retarded distance is larger than the latter; hence, there must be a net electrical force on the system, which accelerates it without limit, providing an infinite source of free energy.[3]

Is Maxwell's objection valid?

This topic is discussed briefly on p. 671 of [6].

2 Solution

Maxwell's objection was based on a misunderstanding of a (now) famous subtle issue about the use of the retarded potentials for small charges. This issue is avoided by use of the relativistic transformation of the electromagnetic potentials, (see, for example, [8]), which is implicit in Maxwell's electrodynamics, but was not recognized as such until the efforts of Lorentz [9] and Einstein [10] in 1904-5. The electric field of a point charge along the direction of its velocity is the same in the rest frame of the charge and in a frame where the charge has velocity $\mathbf{v}$, contrary to Maxwell's overly hasty conclusion. That is, the total electrical force is zero on a pair of opposite charges that move the same velocity (no matter what the angle between the velocity vector and the line of centers of the charges).[4]

Returning to use of the retarded potentials, consider a localized charge density ρ that is in motion with velocity $\mathbf{v}$ where v is less than the speed of light c in the surrounding

[1] Lorenz developed a scalar retarded potential in 1861 when studying waves of elasticity [3].

[2] Maxwell did not actually argue that the concept of retarded potentials was wrong, but he seems to have distrusted it, as he appears never to have referred to it again.

[3] Maxwell's comment is reminiscent of a famous thought experiment of Galileo on why the acceleration of gravity must be independent of mass [7].

[4] Maxwell avoided discussion of two charges with noncollinear velocities, perhaps because of the ambiguity (first noted by Ampère in the 1820's) in extrapolating from the force law for pairs of current loops to that for pairs of moving charges. Extrapolation from the Biot-Savart force law leads to what is now called the Lorentz force law, which has the consequence that the total force is nonzero on a pair of charges which move with noncollinear velocities. Maxwell had only a partial understanding that this behavior is compatible with his theory in that the electromagnetic fields of carry momentum $\mathbf{P}_{\rm EM}$ such that $\mathbf{F}_{\rm total} = d\mathbf{P}_{\rm mechanical}/dt = -d\mathbf{P}_{\rm EM}/dt$, and the total momentum of the system is constant. See, for example, [11].

medium. Then the associated current density can be written as $\mathbf{J} = \rho\mathbf{v}$, and the retarded potentials are (in Gaussian units)

$$V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}', t' = t - R/c)}{R}\, d\text{Vol}', \qquad \mathbf{A}(\mathbf{r},t) = \int \frac{\mathbf{J}(\mathbf{r}', t' = t - R/c)}{cR}\, d\text{Vol}' = V\frac{\mathbf{v}}{c}, \quad (1)$$

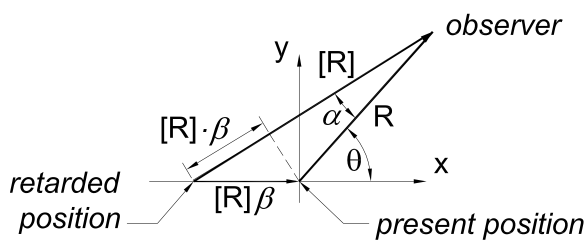
where R is the magnitude of the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. It is tempting to suppose that this can be simplified to read

$$V(\mathbf{r},t) \stackrel{?}{=} \frac{1}{[R]}\int \rho(\mathbf{r}')\, d\text{Vol}' = \frac{q}{[R]}, \qquad \mathbf{A}(\mathbf{r},t) = V\frac{\mathbf{v}}{c} = V\boldsymbol{\beta}, \tag{2}$$

where $q = \int \rho\, d\text{Vol}$ is the total charge of the small object, and the retarded distance vector $[\mathbf{R}]$ is related to the present distance $\mathbf{R}$ as in the figure below,

$$[\mathbf{R}] = \mathbf{R} + [R]\boldsymbol{\beta}, \tag{3}$$

where $\boldsymbol{\beta} = \mathbf{v}/c = v\,\hat{\mathbf{x}}/c$.



Thus,

$$[R]^2 = R^2 + [R]^2\beta^2 + 2R[R]\beta\cos\theta, \tag{4}$$

and so the retarded distance $[R]$ is related to the present distance R by

$$[R] = \gamma^2 R\left(\beta\cos\theta + \sqrt{1 - \beta^2\sin^2\theta}\right), \tag{5}$$

where θ is the angle between vectors $\mathbf{v}$ and $\mathbf{R}$, and $\gamma = 1/\sqrt{1-\beta^2}$.

If eq. (2) were valid, then the electric field,

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{6}$$

of the moving charge would indeed be fore/aft asymmetric as implied by Maxwell.

However, because the charge is moving the retarded integrals of eq. (1) actually imply that

$$V(\mathbf{r},t) = \frac{q}{[R] - [\mathbf{R}\cdot\boldsymbol{\beta}]}, \qquad \mathbf{A}(\mathbf{r},t) = V\frac{[\mathbf{v}]}{c} = V[\boldsymbol{\beta}], \tag{7}$$

where $[\mathbf{v}]$ is the velocity at the retarded time, as first deduced by Liénard [12] (1898) and Wiechert [13] (1900). Their argument is subtle (see, for example, sec. 10.3 of [14]; the first English exposition of this may have been on pp. 254-255 of [15]).

When the velocity $\mathbf{v}$ is constant in time, we have (referring to the figure above),

$$[R] - [\mathbf{R}\cdot\boldsymbol{\beta}] = R\cos\alpha = R\sqrt{1 - \beta^2\sin^2\theta}, \tag{8}$$

noting that

$$\frac{\sin\alpha}{[R]\beta} = \frac{\sin(\pi-\theta)}{[R]}\,. \tag{9}$$

Thus,

$$V(\mathbf{r},t) = \frac{q}{R\sqrt{1-\beta^2\sin^2\theta}} = \frac{q}{\sqrt{x^2+(1-\beta^2)y^2}}\,, \qquad \mathbf{A}(\mathbf{r},t) = V\boldsymbol{\beta} = V\beta\,\hat{\mathbf{x}}, \tag{10}$$

at the moment when the moving charge is at the origin with velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, and the observer is at $(x,y,0)$. Since the charge rather than the observer is moving, $dx/dt = -v$, and the components of the electric field are

$$\begin{aligned}
E_x &= -\frac{\partial V}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = -\frac{\partial V}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial x}\frac{dx}{dt} = -(1-\beta^2)\frac{\partial V}{\partial x} = \frac{qx}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,,\\
E_y &= \frac{qy}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,,\\
E_z &= 0,
\end{aligned} \tag{11}$$

so that the electric field is along the present vector $\mathbf{R}$ and is fore/aft symmetric,

$$\mathbf{E} = \frac{q\mathbf{R}}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,. \tag{12}$$

The magnetic field at the observer has only a z-component,

$$B_z = -\frac{\partial A_x}{\partial y} = -\beta\frac{\partial V}{\partial y} = -\beta E_y = -\left|\boldsymbol{\beta}\times\mathbf{E}\right|, \qquad \text{and so} \qquad \mathbf{B} = \boldsymbol{\beta}\times\mathbf{E}. \tag{13}$$

The results (12)-(13) were first derived for $v \ll c$ by J.J. Thomson [16, 17] (1881), two years after Maxwell's death, and by Heaviside [18] (1889) for any $v < c$, which showed that the electric field at a given distance from the charge is fore/aft symmetric, although not isotropic.[5] Neither of them used the retarded potentials in their calculations.

The first successful application of the retarded potentials was made in 1883 by Fitzgerald [21, 22], who apparently reinvented them in a derivation of the radiation from an oscillating current loop (magnetic dipole).

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