Doubly Negative Metamaterials

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

In 1967 Veselago considered the possibility of materials with (relative) permittivity ϵ and (relative) permeability μ both negative [1]. Building on the long history of artifical dielectrics for high-frequency applications (see, for example, [2, 3, 4]), in 1999 Pendry *et al.* proposed a technique for fabrication of metamaterials with both negative ϵ and negative μ within a certain frequency range [5]. Such materials were first realized in the laboratory in 2000 [6].

An idealized model of the frequency dependence of the permittivity and the permeability (both assumed to be linear and isotropic) is that

$$\epsilon(\omega) = 1 \pm \frac{\omega_{\epsilon,p}^2}{\omega^2 - \omega_{\epsilon,0}^2 + i\Gamma_\epsilon \omega} \approx \frac{\omega^2 - \omega_{\epsilon,1}^2}{\omega^2 - \omega_{\epsilon,0}^2}\,, \tag{1}$$

and

$$\mu(\omega) = 1 \pm \frac{\omega_{\mu,p}^2}{\omega^2 - \omega_{\mu,0}^2 + i\Gamma_\mu \omega} \approx \frac{\omega^2 - \omega_{\mu,1}^2}{\omega^2 - \omega_{\mu,0}^2}\,, \tag{2}$$

where the time dependence is assumed to have the form $e^{-i\omega t}$,

$$\omega_1 = \sqrt{\omega_0^2 \pm \omega_p^2}\,, \tag{3}$$

ω_p is equivalent to the plasma frequency of the medium, and ω_0 is a resonant frequency of the medium that is associated with a small damping coefficient Γ which we neglect by restricting the analysis to frequencies not too close to ω_0 or ω_1. In passive metamaterials the $\pm$ sign is negative, while for an active metamaterial [7] with an inverted population the sign can be positive. *In general, there is no relation between the frequencies* $\omega_{\epsilon,j}$ *and* $\omega_{\mu,j}$*, but it suffices to suppose that either* $\omega_{\epsilon,0} = \omega_{\mu,0}$, $\omega_{\epsilon,1} = \omega_{\mu,1}$ *or* $\omega_{\epsilon,0} = \omega_{\mu,1}$, $\omega_{\epsilon,1} = \omega_{\mu,0}$.

(a) Discuss the relation between the phase and group velocities, $\mathbf{v}_p$ and $\mathbf{v}_g$, in a metamaterial for waves of the form $e^{i(kx-\omega t)}$, where for negligible damping we can write (see chap. 6, sec. 2 of [8] or sec. 85 of [9])

$$\mathbf{v}_g = \frac{d\omega}{dk}\hat{\mathbf{x}} = \frac{\hat{\mathbf{x}}}{dk/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{d[\omega n]/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{n + \omega\ dn/d\omega}\,. \tag{4}$$

with $n = \pm ck/\omega$ being the index of refraction, and c is the speed of light in vacuum. For what range of material parameters, if any, can the phase and group velocity be in opposite directions and/or the index n be considered as negative?

(b) By consideration of the phases of planes waves at the interface between an ordinary transparent medium and a metamaterial, deduce the form of Snell's law for waves incident on the interface at angle θ_i from within the ordinary medium, and refracted into the metamaterial at angle θ_t to the normal.

2 Solution

2.1 Phase and Group Velocity

In a medium free of external charge and current, that can be characterized by linear, isotropic (relative) permittivity ϵ and permeability μ, Maxwell's equations are (in Gaussian units)

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \tag{5}$$

and the constitutive relations are

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}. \tag{6}$$

These can be combined to give a wave equation for, say, the magnetic field $\mathbf{H}$,

$$\nabla^2\mathbf{H} = \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}. \tag{7}$$

In general we consider plane waves of the form

$$\mathbf{H} = \mathbf{H}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{8}$$

where $\mathbf{H}_0$ and $\mathbf{k}$ can be complex functions of the angular frequency ω (but not of time). The approximations of the models (1)-(2) for metamaterials permit imply that the wave vector $\mathbf{k}$ is purely real. In this case, eqs. (7) and (8) combine to give the dispersion relation

$$\mathbf{k}^2 \equiv k^2 = \frac{\epsilon\mu\omega^2}{c^2} \equiv \frac{n^2\omega^2}{c^2} \tag{9}$$

where the scalar wave number k is defined to be positive, and n is the index of refraction. For wave propagation, both ϵ and μ must be positive, or both negative. Then,

$$k = \frac{\sqrt{\epsilon\mu}\,\omega}{c} = \pm\frac{n\omega}{c}, \tag{10}$$

where the index n could be either positive or negative,

$$n = \pm\sqrt{\epsilon\mu}. \tag{11}$$

We write the wave vector as $\mathbf{k} = k\,\hat{\mathbf{k}}$ with k positive, so the waveform (8) becomes

$$\mathbf{H} = \mathbf{H}_0\, e^{i(k\,\hat{\mathbf{k}}\cdot\mathbf{x}-\omega t)}, \tag{12}$$

such that we identify the phase velocity $\mathbf{v}_p$ as

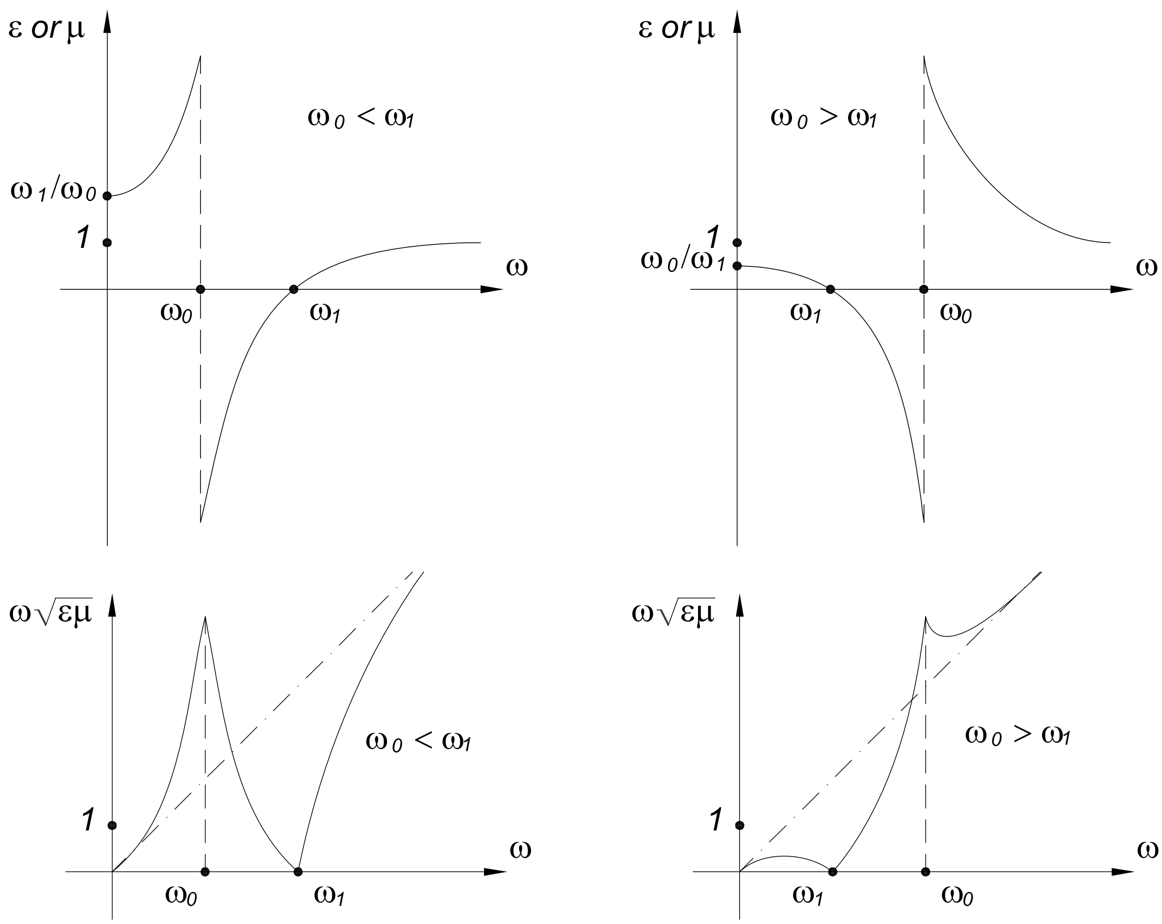
$$\mathbf{v}_p = \frac{\omega}{k}\,\hat{\mathbf{k}} = \frac{c\,\hat{\mathbf{x}}}{\sqrt{\epsilon\mu}} = \pm\frac{c}{n}\,\hat{\mathbf{k}}. \tag{13}$$

The phase-velocity vector $\mathbf{v}_p$ is in the direction of the wave vector $\mathbf{k} = k\,\hat{\mathbf{k}}$.

Turning to the group velocity, we combining eqs. (4) and (10) to write the group velocity of a packet of waves of the form (8) as[1]

$$\mathbf{v}_g = \frac{1}{dk/d\omega}\,\hat{\mathbf{x}} = \frac{c\,\hat{\mathbf{x}}}{d[\omega\sqrt{\epsilon\mu}]/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{d[\omega n]/d\omega}\,. \tag{14}$$

For passive metamaterials the model forms (1)-(2) hold for $\omega_0 < \omega_1$, as sketched on the left in the two figures below. In this case $d(\omega\sqrt{\epsilon\mu})/d\omega$ is negative for $\omega_0 < \omega < \omega_1$ where ϵ and μ are both negative. Thus, the phase and group velocities, eqs. (13) and (14), have the opposite signs in a doubly negative, passive metamaterial [10].



We can use the notation $n = -\sqrt{\epsilon\mu}$ together with $k = -n\omega/c$ to write the waveform (8) as

$$\mathbf{H} = \mathbf{H}_0 e^{i(n\omega x/c - \omega t)}, \tag{15}$$

for the case of a wave with group velocity in the $+x$-direction and phase velocity in the $-x$-direction. This convention is often summarized by saying that a passive, doubly negative metamaterial has a negative index of refraction.[2]

[1] Care must be taken to avoid the mathematical oddity that one might write $df/d\omega = d\sqrt{(-f)^2}/d\omega = 2(1/2)\sqrt{(-f)/(-f)}\ d(-f)/d\omega = -df/d\omega$, which is a variant on the theme that while $1 = e^{2i\pi}$ we cannot say that $1 = \sqrt{1} = \sqrt{e^{2i\pi}} = e^{i\pi} = -1$.

[2] Of course, it is also consistent to write $n = \sqrt{\epsilon\mu}$ and $k = n\omega/c$. Conversely, any medium could be defined to have a negative index of refraction so long as we also write $k = -n\omega/c$.

On the other hand, if active metamaterials can be constructed with behavior as shown in the right figures above, then for $\omega_1 < \omega < \omega_0$, where both ϵ and μ are negative, $d(\omega\sqrt{\epsilon\mu})/d\omega$ is positive, and the group and phase velocities are in the same direction. In this case it would be best to write simply $n = \sqrt{\epsilon\mu}$.

2.1.1 Energy Density

If a medium could have negative ϵ and negative μ at zero frequency, then the static energy density,

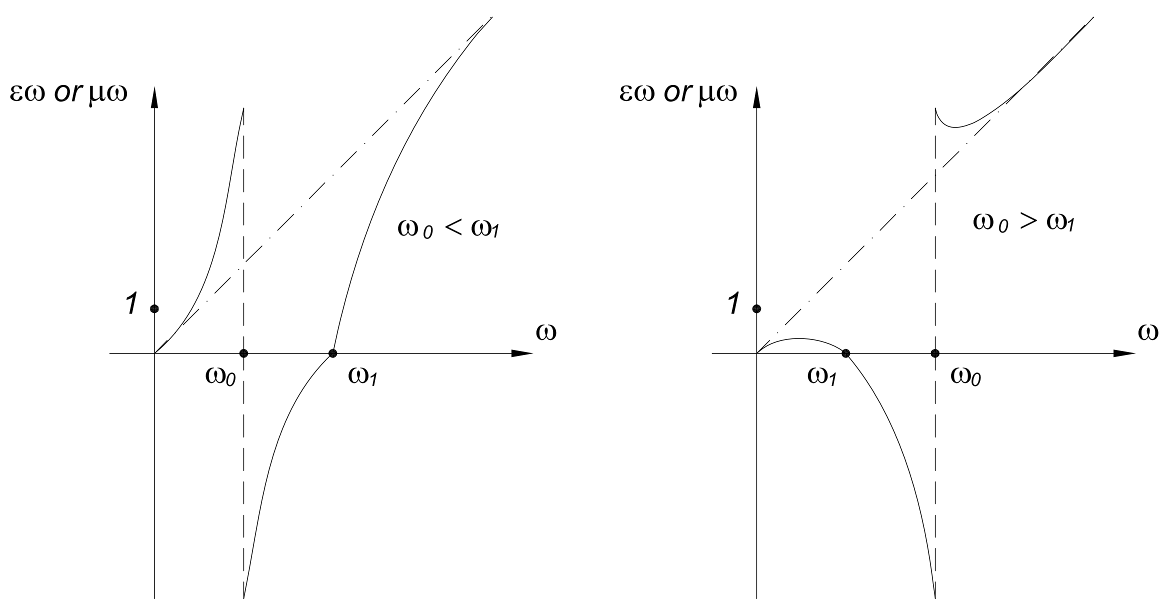
$$u = \frac{\epsilon E^2 + \mu H^2}{8\pi}, \tag{16}$$

would be negative, which is unreasonable for a passive material.

When the permittivity and permeability have frequency dependence, the energy density takes on a more complicated form (due to Brillouin (1932), reviewed in chap. IV of [8]; see also sec. 80 of [9]),

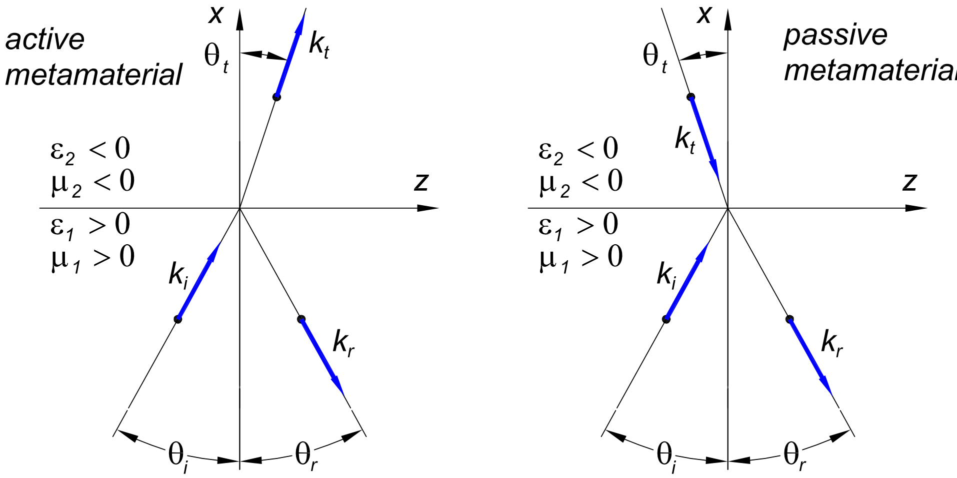
$$\langle u_\omega \rangle = \frac{1}{8\pi}\left(\frac{d(\epsilon\omega)}{d\omega}\left\langle E^2\right\rangle + \frac{d(\mu\omega)}{d\omega}\left\langle H^2\right\rangle\right). \tag{17}$$

The products $\epsilon\omega$ and $\mu\omega$ are illustrated for the models (1)-(2) in the figure below. We see that for the passive metamaterial ($\omega_0 < \omega_1$) the energy density is positive (as was anticipated by Veselago [1]). But for an active metamaterial the energy density is formally negative in the region $\omega_1 < \omega < \omega_0$ of doubly negative ϵ and μ. Such behavior is nonclassical, but occurs in gain media [11].



2.2 Snell's Law

We consider the case of an ordinary medium with $\epsilon_1 > 0$, $\mu_1 > 0$ at $x < 0$, and a doubly negative metamaterial with $\epsilon_2 < 0$, $\mu_2 < 0$ at $x > 0$, as sketched in the figure on the next page.



We also consider plane waves of the form $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. One condition on the incident, reflected and transmitted waves is that the phase of all three waves must be the same at any point $(0, y, z)$ on the interface $x = 0$. This implies that

$$k_i \sin\theta_i = k_r \sin\theta_r = \pm k_t \sin\theta_t, \tag{18}$$

where for the moment we suppose that the transmitted wave is "ordinary" in the sense that the transmitted ray which passes through the origin lies in the first quadrant of the x-z plane, as shown in the left figure above for an active metamaterial. Then, the upper (lower) sign in eq. (18) holds for an active (passive) metamaterial.

Since the incident and reflected waves are in the same medium the magnitudes of the wave numbers k_i and k_r are the same (for waves of a given frequency ω),

$$k_i = k_r = \frac{\sqrt{\epsilon_1\mu_1}\,\omega}{c} = \frac{n_1\omega}{c}\,, \tag{19}$$

and hence

$$\theta_r = \theta_i. \tag{20}$$

If we designate k_t to be a positive quantity,

$$k_t = \frac{\sqrt{\epsilon_2\mu_2}\,\omega}{c} = \pm\frac{n_2\omega}{c}\,, \tag{21}$$

where the index of refraction,

$$n_2 = \pm\sqrt{\epsilon_2\mu_2}, \tag{22}$$

can be considered as negative for a passive metamaterial, as discussed in sec. 2.1. Then, eq. (18) also indicates that

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{23}$$

where n_2, and hence θ_t as well, are negative for passive metamaterials. In the latter case, the transmitted ray which passes through the origin lies in the second quadrant of the x-z

plane, as shown in the right figure above. This behavior is often called negative refraction.[3,4]

A Appendix: "Left-Handed" Metamaterials

The fourth Maxwell equation (5) for a plane wave of form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ indicates that

$$\mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B} = \frac{\mu\omega}{c}\mathbf{H} = \pm\sqrt{\frac{\mu}{\epsilon}}k\mathbf{H}, \qquad \text{or} \qquad \mathbf{H} = \pm\sqrt{\frac{\epsilon}{\mu}}\,\hat{\mathbf{k}}\times\mathbf{E}, \qquad \mathbf{B} = \sqrt{\epsilon\mu}\,\hat{\mathbf{k}}\times\mathbf{E}, \quad (24)$$

using eq. (10), where the upper sign holds for ordinary media and active metamaterials, while the lower sign holds for passive metamaterials. This result has led to the characterization of passive metamaterials as left-handed media, in contrast to ordinary media, and active metamaterials, for which $\hat{\mathbf{k}}$, $\mathbf{E}$ and $\mathbf{H}$ form a right-handed triad.[5] According to the discussion in sec. 2.1, only left-handed media should be characterized as having a negative index of refraction.

Acknowledgment

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References

[1] V.G. Veselago, *The Electrodynamics of Substances with Simultaneously Negative Values of ϵ and μ*, Sov. Phys. Uspekhi **10**, 5 (1968),
http://puhep1.princeton.edu/~mcdonald/examples/EM/vesalago_spu_10_5_68.pdf

[2] W.E. Kock, *Metal-Lens Antennas*, Proc. IRE **34**, 828 (1946),
http://puhep1.princeton.edu/~mcdonald/examples/EM/kock_pire_34_828_46.pdf

[3] J. Brown, *Artificial Dielectrics Having Refractive Indices Less Than Unity*, Proc. IEE **100**, 51 (1953),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brown_piee_100_51_53.pdf

[4] W. Rotman, *Plasma Simulation by Artificial Dielectrics and Parallel-Plate Media*, IRE Trans. Ant. Prop. **10**, 82 (1962),
http://puhep1.princeton.edu/~mcdonald/examples/EM/rotman_iretap_10_82_62.pdf

[3] Thus, we have the option of retaining Snell's law in the form $n_1 \sin\theta_i = n_2 \sin\theta_t$ for medium 2 being a passive metamaterial by considering n_2 and θ_t as both negative, with θ_t defined to be positive in the first quadrant of the x-z plane. Alternatively, we could consider n_2 to be positive, and modify Snell's law to be $n_1 \sin\theta_i = -n_2 \sin\theta_t$, where again θ_t is defined to be positive in the first quadrant of the x-z plane. And yet another alternative is to write Snell's law as $n_1 \sin\theta_i = n_2 \sin\theta_t$ with n_2 positive, but to define θ_t to be positive in the second quadrant of the x-z plane. Among these three alternatives the first is perhaps the most pleasing.

[4] Negative refraction in the sense considered here can also be occur for materials with positive $Re(n)$ if the losses are sufficiently high ($Im(n)$ large) [12, 13].

[5] Note that $\hat{\mathbf{k}}$, $\mathbf{E}$ and $\mathbf{B}$ form a right-handed triad in all materials.

[5] J.B. Pendry *et al.*, *Magnetism from Conductors and Enhanced Nonlinear Phenomena*, IEEE Trans. Micro. Theor. Tech. **47**, 2075 (1999),
http://puhep1.princeton.edu/~mcdonald/examples/EM/pendry_ieeetmtt_47_2075_99.pdf

[6] D.R. Smith *et al.*, *Composite Medium with Simultaneously Negative Permeability and Permittivity*, Phys. Rev. Lett. **85**, 4184 (2000),
http://puhep1.princeton.edu/~mcdonald/examples/EM/smith_prl_84_4184_00.pdf

[7] S.A. Ramakrishna and J.B. Pendry, *Removal of absorption and increase in resolution in a near-field lens via optical gain*, Phys. Rev. B **67**, 201101 (2003),
http://puhep1.princeton.edu/~mcdonald/examples/EM/ramakrishna_prb_67_201101_03.pdf

[8] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960).

[9] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Butterworth-Heinemann, Oxford, 1984).

[10] D.R. Smith and N. Kroll, *Negative Refractive Index in Left-Handed Materials*, Phys. Rev. Lett. **85**, 2933 (2000),
http://puhep1.princeton.edu/~mcdonald/examples/EM/smith_prl_85_2933_00.pdf

[11] K.T. McDonald, *Negative Group Velocity* (June 23, 2000),
http://puhep1.princeton.edu/~mcdonald/examples/negativegroupvelocity.pdf

[12] M. Sanz *et al.*, *Transmission measurements in wedge-shaped absorbing samples: An experiment for observing negative refraction*, Phys. Rev. E **67**, 067601 (2003),
http://puhep1.princeton.edu/~mcdonald/examples/EM/sanz_pre_67_067601_03.pdf

[13] R.C. Hansen, *Negative Refraction Without Negative Index*, IEEE Trans. Ant. Prop. **56**, 402 (2008),
http://puhep1.princeton.edu/~mcdonald/examples/EM/hansen_ieeetap_56_402_08.pdf