Does the Moon Always Fall Towards the Sun?

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1 Problem

It is claimed in [1] that the Moon always falls towards the Sun? Can this be so?

It suffices to suppose that the Earth's orbit about the Sun, and the Moon's orbit about the Earth, are both circular.

2 Solution

The term "fall" apparently means different things to different people. The usual convention is that an objecting is "falling down" if its velocity is "downwards." However, some people consider that an object is "falling down" if its acceleration is "downwards" even when its velocity is "upwards."

In cylindrical coordinates (r, ϕ, z) the position vector $\mathbf{r} = r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}$ has velocity vector

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{z}}, \tag{1}$$

where $\dot{r} = dr/dt$, *etc.*, and acceleration vector

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\,\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\boldsymbol{\phi}} + \ddot{z}\,\hat{\mathbf{z}}, \tag{2}$$

Taking the Earth's and Moon's orbits to be circular relative to the Sun and Earth, with angular velocities Ω and ω, respectively, the distance r from the Sun to the Moon obeys

$$r^2 = r_1^2 + r_2^2 + 2r_1r_2\cos[(\omega - \Omega)t], \tag{3}$$

where r_1 is the radius of the Earth's orbit, $r_2 < r_1$ is the radius of the Moon's orbit, and the Moon is at its maximal distance from the Sun at time $t = 0$. The time derivative of eq. (3) is

$$r\dot{r} = -(\omega - \Omega)r_1r_2\sin[(\omega - \Omega)t]. \tag{4}$$

Thus, the radial velocity $\dot{r}\,\hat{\mathbf{r}}$ oscillates in sign and by the usual convention the Moon does not always fall towards the Sun.

2.1 Radial Acceleration

The acceleration $\mathbf{a}$ of the Moon (whose position vector is $\mathbf{r}$) follows from Newton's law of gravitation (ignoring effects of other planets and stars) as

$$\mathbf{a} = -\frac{GM_{\rm Sun}}{r^2}\,\hat{\mathbf{r}} + GM_{\rm Earth}\frac{\mathbf{r}_{\rm Earth} - \mathbf{r}}{|\mathbf{r}_{\rm Earth} - \mathbf{r}|^3}\,. \tag{5}$$

Clearly the radial component of the acceleration is negative when the Moon is farthest from the Sun. When the Moon is closest to the Sun,

$$\begin{aligned} a_r &= -\frac{GM_{\rm Sun}}{(r_1 - r_2)^2} + \frac{GM_{\rm Earth}}{r_2^2} = -\frac{GM_{\rm Sun}}{(r_1 - r_2^2} \left(1 - \frac{M_{\rm Earth}}{M_{\rm Sun}} \frac{(r_1 - r_2)^2}{r_2^2} \right) \\ &\approx -\frac{GM_{\rm Sun}}{(r_1 - r_2)^2} \left(1 - \frac{3 \times 10^{-6} \cdot (1.5 \times 10^7)^2}{(3.84 \times 10^5)^2} \right) \approx -0.995 \frac{GM_{\rm Sun}}{(r_1 - r_2)^2} < 0. \end{aligned} \tag{6}$$

Hence, the radial acceleration of the Moon is always negative, and some people [1] therefore characterize the Moon as always "falling" towards the Sun.

2.1.1 $\ddot{r}$

The time derivative of eq. (4) is

$$r\ddot{r} + \dot{r}^2 = -(\omega - \Omega)^2 r_1 r_2 \cos[(\omega - \Omega)t]. \tag{7}$$

Combining eqs. (4) and (7), we find

$$\ddot{r} = -(\omega - \Omega)^2 \frac{r_1 r_2}{r^3} \left\{ r^2 \cos[(\omega - \Omega)t] + r_1 r_2 \sin^2[(\omega - \Omega)t] \right\}. \tag{8}$$

When the Moon is closest to the Sun, $\cos[(\omega - \Omega)t] = -1$, $\sin[(\omega - \Omega)t] = 0$, and

$$\ddot{r} = (\omega - \Omega)^2 \frac{r_1 r_2}{r} = (\omega - \Omega)^2 \frac{r_1 r_2}{r_1 - r_2}. \tag{9}$$

That is, the second derivative $\ddot{r}$ of the Moon with respect to the Sun cannot always be negative for any values of r_1, r_2, Ω or ω. However, the radial acceleration $a_r = \ddot{r} - r\dot{\phi}^2$ happens to be always negative for the Moon (which is the only moon in the Solar system with this behavior).

References

[1] Kevin Brown, *The Moon Always Falls Toward the Sun*,
http://www.mathpages.com/home/kmath405/kmath405.htm
The `mathpages` *website includes no contact information as to its author, who appears to desire to be effectively anonymous.*