# Magnetic Force on a Permeable Wire

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# 1 Problem

What is the force per unit length on a wire of radius $a$ and (relative) permeability $\mu'$ when it carries uniform current density $\mathbf{J} = I\,\hat{\mathbf{z}}/\pi a^2$ and is placed along the $z$ axis in a magnetic field whose form is $\mathbf{B}_i = B_0\,\hat{\mathbf{x}} + B_1[(x/a)\,\hat{\mathbf{x}} - (y/a)\,\hat{\mathbf{y}}]$ before the wire is placed in that field? The medium surrounding the wire is a nonconducting liquid with relative permeability $\mu \neq 1$.

The form of the initial magnetic field has been chosen so that there will be both a $\mathbf{J} \times \mathbf{B}$ force associated with the uniform field $\mathbf{B}_0 = B_0\,\hat{\mathbf{x}}$, as well as a force due to the interaction of the induced magnetization with the nonuniform field $\mathbf{B}_1 = B_1[(x/a)\,\hat{\mathbf{x}} - (y/a)\,\hat{\mathbf{y}}]$.

# 2 Solution

This old problem [1] was recently reconsidered by Casperson [2], who reports an experimental result that appears to disagree with the theory he presents. The impression was given that no straightforward theory exists for this problem, which this note hopes to correct by presenting three "standard" solutions that are in agreement. We also explore use of the somewhat hybrid methods for calculating forces on magnetic media advocated by the so-called Coulomb Committee [3, 4, 5, 6] and obtain success only if those methods are revised in an important way. An overall perspective on these issues is given in [7].

A variant on this problem has practical import to high-energy physicists such as the present author, who consider it to be experimentally confirmed (and continually reconfirmed) for over 50 years [8] that high-energy particles of charge $q$ and velocity $v$ obey the Lorentz force law of the form

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B}) \tag{1}$$

(in Gaussian units) even when in a permeable medium where the magnetic fields are related by $\mathbf{B} = \mu\mathbf{H} \gg \mathbf{H}$ .

A simplifying principle is that no object can exert a net force on itself (Newton's first law). In particular, the fields that are set up or modified when the wire is added to the problem cannot result in a net force on the wire. Hence, the force on the wire can be calculated via the interaction of the current and magnetization in the wire with only the fields present before the wire was added to the problem, as emphasized in sec. 35 of [9] and at greater length in [10]. This important result is explicitly contained in the original formulation of the Biot-Savart force law (for media of unit permeability), that the magnetic force on circuit $a$ is due to effects caused by some other circuit $b$,

$$\mathbf{F}_a = \frac{I_a I_b}{c^2}\oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2} = \frac{I_a}{c}\oint_a d\mathbf{l}_a \times \mathbf{B}_b, \qquad \text{where} \qquad \mathbf{B}_b(a) = \frac{I_b}{c}\oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2}. \tag{2}$$

Thus, we expect the simple result that the force per unit length on the wire due to the initial uniform field $\mathbf{B}_0$ is

$$\mathbf{F} = \frac{1}{c}\mathbf{I} \times \mathbf{B_0}, \tag{3}$$

where $\mathbf{I} = \pi a^2 \mathbf{J}$, which is the macroscopic equivalent of the Lorentz force law (1).

Strictly speaking, eq. (3) describes the force on the conduction electrons, and not that on the lattice of positive ions through which the electrons flow. The force (3) results in a slight rearrangement of the distribution of the conduction electrons and positive lattice ions so that a transverse electric field is generated that acts on the lattice to provide the force experienced by an observer who holds the wire at rest. See [11] for further discussion.

The initial magnetic field $B_i$ also induces magnetization $\mathbf{M}$ in the wire, and an additional force results if the magnetic field is nonuniform at the position of the wire. According to the preceding argument, it should be possible to calculate this force as an interaction between the initial magnetic field and some representation of the induced magnetization in terms of bound currents or fictitious magnetic poles

Nonetheless, it is also desirable to have a method for calculating the magnetic force that uses the total fields in the problem, including those generated by the wire. Two successful approaches are to use the Maxwell stress tensor (sec. 2.2), and the bulk force density of Helmholtz (sec. 2.3), both of which confirm the result (3).

In sec. 3 we calculate the force on a permeable, current-carrying wire by combining the Biot-Savart force law force the volume force density $\mathbf{f}$,

$$\mathbf{f} = \frac{1}{c}\mathbf{J} \times \mathbf{B}_i, \tag{4}$$

with terms due to either magnetization currents or fictitious magnetic pole densities. As expected from the preceding argument, this approach is successful if only the initial magnetic fields are used in the force law. However, it turns out that when using the method of magnetization currents, the initial magnetic field to be used in the Biot-Savart law is $\mathbf{H}_i$ rather than $\mathbf{B}_i$.

For these calculations, the total magnetic fields $\mathbf{B}$ and $\mathbf{H}$ and the induced magnetization density $\mathbf{M}$ in the permeable media will first be deduced in sec. 2.1.

## 2.1 The Fields B, H and M

We adopt a coordinate system in which the axis of the wire is the $z$ axis with the conduction current density being

$$\mathbf{J}_{\rm cond} = \frac{I}{\pi a^2}\,\hat{\mathbf{z}} \tag{5}$$

inside the wire of radius $a$.

Because we are dealing with magnetic media with nonzero magnetization $\mathbf{M}$, both the magnetic fields $\mathbf{H}$ and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = \mu\mathbf{H}$ are of utility. The initial external field is

$$\mathbf{H}_i = H_0\,\hat{\mathbf{x}} + H_1\left(\frac{x}{a}\,\hat{\mathbf{x}} - \frac{y}{a}\,\hat{\mathbf{y}}\right), \qquad \mathbf{B}_i = \mu\mathbf{H}_i, \tag{6}$$

where $\mu$ is the permeability of the medium surrounding the wire. When the wire is placed into this medium, we expect a force in the $+y$ direction according to the Biot-Savart law (4), and a magnetization force in the $+x$ direction due to the nonuniform field $\mathbf{H}_i$ that increases with $x$.

In addition to the rectangular coordinate system $(x, y, z)$, we will work in a cylindrical coordinate system $(r, \theta, z)$. The usual transformation of the units vectors between these two coordinate systems are

$$\hat{\mathbf{x}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}, \qquad \hat{\mathbf{y}} = \sin\theta\ \hat{\mathbf{r}} + \cos\theta\ \hat{\boldsymbol{\theta}}, \tag{7}$$

and

$$\hat{\mathbf{r}} = \cos\theta\ \hat{\mathbf{x}} + \sin\theta\ \hat{\mathbf{y}}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{\mathbf{x}} + \cos\theta\ \hat{\mathbf{y}}. \tag{8}$$

The current density (5) causes magnetic field $\mathbf{H}_{\rm cond}$ according to Ampère's law, $\boldsymbol{\nabla}\times\mathbf{H} = (4\pi/c)\mathbf{J}_{\rm cond}$,

$$\mathbf{H}_{\rm cond} = \frac{2I}{c}\ \hat{\boldsymbol{\theta}} \begin{cases} \frac{r}{a^2} & (r < a), \\ \frac{1}{r} & (r > a). \end{cases} \tag{9}$$

The part of the field $\mathbf{H}$ not due to $\mathbf{J}_{\rm cond}$ we label as $\mathbf{H}_{\rm ind}$ (for induced), which then obeys $\boldsymbol{\nabla}\times\mathbf{H}_{\rm ind} = 0$. Hence, we may deduce this part of the field as $\mathbf{H}_{\rm ind} = -\boldsymbol{\nabla}\phi_{\rm ind}$ from a scalar potential $\phi_{\rm ind}$ that obeys Laplace's equation, $\nabla^2\phi_{\rm ind} = 0$.

The external field (6) can be regarded as due to the scalar potential

$$\phi_i = -H_0 x - \frac{H_1}{2}\frac{x^2 - y^2}{a} = -H_0 r\cos\theta - \frac{H_1}{2}\frac{r^2}{a}\cos 2\theta. \tag{10}$$

The external field induces additional terms in the scalar potential that also vary as $\cos\theta$ or $\cos 2\theta$, since these are two of the set of orthogonal functions in which the scalar potential $\phi_{\rm ind}(r, \theta)$ can be expanded. In particular, we can write

$$\phi_{\rm ind} = \begin{cases} -H_0 r\cos\theta - \frac{H_1}{2}\frac{r^2}{a}\cos 2\theta + A_0\frac{r}{a}\cos\theta + \frac{A_1}{2}\frac{r^2}{a^2}\cos 2\theta & (r < a), \\ -H_0 r\cos\theta - \frac{H_1}{2}\frac{r^2}{a}\cos 2\theta + A_0\frac{a}{r}\cos\theta + \frac{A_1}{2}\frac{a^2}{r^2}\cos 2\theta & (r > a), \end{cases} \tag{11}$$

which is continuous at $r = a$. The induced fields obey the additional matching condition that the radial component $B_r = \mu H_r$ of the magnetic field is continuous at $r = a$ (since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$). As we have different permeabilities $\mu'$ for $r < a$ and $\mu$ for $r > a$, the condition is that

$$\mu\frac{\partial\phi_{\rm ind}(r = a^+)}{\partial r} = \mu'\frac{\partial\phi_{\rm ind}(r = a^-)}{\partial r}, \tag{12}$$

and hence,

$$\begin{aligned} &\mu\left(-H_0\cos\theta - H_1\cos 2\theta - \frac{A_0}{a}\cos\theta - \frac{A_1}{a}\cos 2\theta\right) \\ &= \mu'\left(-H_0\cos\theta - H_1\cos 2\theta + \frac{A_0}{a}\cos\theta + \frac{A_1}{a}\cos 2\theta\right). \end{aligned} \tag{13}$$

The equality holds separately for the coefficients of the orthogonal functions $\cos\theta$ and $\cos 2\theta$, so that

$$A_{0,1} = \frac{\mu' - \mu}{\mu' + \mu} a H_{0,1}, \tag{14}$$

$$\phi_{\rm ind} = \begin{cases} -\frac{2\mu}{\mu'+\mu}\left(H_0 r\cos\theta + \frac{H_1}{2}\frac{r^2}{a}\cos 2\theta\right) & (r < a), \\ -H_0\left(r - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r}\right)\cos\theta - \frac{H_1}{2}\left(\frac{r^2}{a} - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^2}\right)\cos 2\theta & (r > a), \end{cases} \tag{15}$$

and

$$\begin{aligned} \mathbf{H}_{\rm ind} &= -\frac{\partial\phi_{\rm ind}}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\phi_{\rm ind}}{\partial\theta}\hat{\boldsymbol{\theta}} \\ &= \begin{cases} \frac{2\mu}{\mu'+\mu}\left(H_0\cos\theta + H_1\frac{r}{a}\cos 2\theta\right)\hat{\mathbf{r}} - \frac{2\mu}{\mu'+\mu}\left(H_0\sin\theta + H_1\frac{r}{a}\sin 2\theta\right)\hat{\boldsymbol{\theta}} & (r < a), \\ \left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\right)\cos\theta + H_1\left(\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\right)\cos 2\theta\right]\hat{\mathbf{r}} & \\ \quad - \left[H_0\left(1 - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\right)\sin\theta + H_1\left(\frac{r}{a} - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\right)\sin 2\theta\right]\hat{\boldsymbol{\theta}} & (r > a), \end{cases} \\ &= \begin{cases} \frac{2\mu}{\mu'+\mu}\left(H_0 + H_1\frac{x}{a}\right)\hat{\mathbf{x}} - \frac{2\mu}{\mu'+\mu}H_1\frac{y}{a}\,\hat{\mathbf{y}} & (r < a), \\ \left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right) + H_1\left(\frac{x}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right)\right]\hat{\mathbf{x}} & \\ \quad + \left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta + H_1\left(-\frac{y}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right)\right]\hat{\mathbf{y}} & (r > a). \end{cases} \end{aligned} \tag{16}$$

The total magnetic field is the sum of eqs. (9) and (16),

$$\mathbf{H} = \begin{cases} \frac{2\mu}{\mu'+\mu}\left(H_0\cos\theta + H_1\frac{r}{a}\cos 2\theta\right)\hat{\mathbf{r}} + \left[\frac{2Ir}{ca^2} - \frac{2\mu}{\mu'+\mu}\left(H_0\sin\theta + H_1\frac{r}{a}\sin 2\theta\right)\right]\hat{\boldsymbol{\theta}} & (r < a), \\ \left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\right)\cos\theta + H_1\left(\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\right)\cos 2\theta\right]\hat{\mathbf{r}} & \\ \quad + \left[\frac{2I}{cr} - H_0\left(1 - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\right)\sin\theta - H_1\left(\frac{r}{a} - \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\right)\sin 2\theta\right]\hat{\boldsymbol{\theta}} & (r > a), \end{cases} \tag{17}$$

$$= \begin{cases} \left[\frac{2\mu}{\mu'+\mu}\left(H_0 + H_1\frac{r}{a}\cos\theta\right) - \frac{2Ir}{ca^2}\sin\theta\right]\hat{\mathbf{x}} + \left(\frac{2Ir}{ca^2}\cos\theta - \frac{2\mu}{\mu'+\mu}H_1\frac{r}{a}\sin\theta\right)\hat{\mathbf{y}} & (r < a), \\ \left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right) + H_1\left(\frac{r}{a}\cos\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right) - \frac{2I}{cr}\sin\theta\right]\hat{\mathbf{x}} & \\ \quad + \left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta + H_1\left(-\frac{r}{a}\sin\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right) + \frac{2I}{cr}\cos\theta\right]\hat{\mathbf{y}} & (r > a). \end{cases} \tag{18}$$

Of course,

$$\mathbf{B} = \begin{cases} \mu'\mathbf{H} & (r < a), \\ \mu\mathbf{H} & (r > a). \end{cases} \tag{19}$$

These forms obey the matching conditions that $B_r$ and $H_\theta$ are continuous at the boundary $r = a$. Similarly, the magnetization is given by

$$\mathbf{M} = \begin{cases} \frac{\mu'-1}{4\pi}\mathbf{H} & (r < a), \\ \frac{\mu-1}{4\pi}\mathbf{H} & (r > a). \end{cases} \tag{20}$$

## 2.2 Calculation of the Force via the Maxwell Stress Tensor

We calculate the force on unit length of the wire by integrating the Maxwell stress tensor over a cylindrical surface of radius $r > a$, so that any effects at the surface $r = a$ are included. The surface element at radius $r$ is

$$d\mathbf{S} = r\ d\theta\ \hat{\mathbf{r}} = r\ d\theta\ (\cos\theta\ \hat{\mathbf{x}} + \sin\theta\ \hat{\mathbf{y}}). \tag{21}$$

In rectangular coordinates, and for $r > a$ where the permeability is $\mu$, the Maxwell stress tensor for the magnetic fields (ignoring magnetostriction) is [12]

$$T_{ij} = \frac{1}{4\pi}\left(B_i H_j - \frac{\delta_{ij}}{2}\mathbf{B}\cdot\mathbf{H}\right) = \frac{\mu}{4\pi}\left(H_i H_j - \frac{\delta_{ij}}{2}H^2\right) \tag{22}$$

We first calculate the $x$ component of the force which is not expected to depend on the current $I$, so we drop terms in $I^2$ and $IH$ that would eventually integrate to zero. Then,

$$\begin{aligned}
F_x &= \int (T_{xx}\, dS_x + T_{xy}\, dS_y) = \frac{\mu r}{8\pi}\int_0^{2\pi}(H_x^2 - H_y^2)\cos\theta\, d\theta + \frac{\mu r}{4\pi}\int_0^{2\pi} H_x H_y \sin\theta\, d\theta \\
&= \frac{\mu r}{8\pi}\int_0^{2\pi}\left\{\left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right) + H_1\left(\frac{r}{a}\cos\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right)\right]^2\right. \\
&\qquad \left. - \left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta + H_1\left(-\frac{r}{a}\sin\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right)\right]^2\right\}\cos\theta\, d\theta \\
&+ \frac{\mu r}{4\pi}\int_0^{2\pi}\left[H_0\left(1 + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right) H_1\left(\frac{r}{a}\cos\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right)\right]\cdot \\
&\qquad \left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta + H_1\left(-\frac{r}{a}\sin\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right)\right]\sin\theta\, d\theta \\
&= \frac{\mu r}{8\pi}\int_0^{2\pi}\left[H_0^2 + 2H_0H_1\left(\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a}{r}\right)\cos\theta + H_1^2\frac{r^2}{a^2}\cos^2\theta\right. \\
&\qquad + 2\frac{\mu'-\mu}{\mu'+\mu}\left((H_0^2 + H_1^2)\frac{a^2}{r^2}\cos 2\theta + H_0H_1\frac{a^3}{r^3}\cos 3\theta\right) \\
&\qquad \left. + \left(\frac{\mu'-\mu}{\mu'+\mu}\right)^2\left(H_0^2\frac{a^4}{r^4}\cos 4\theta + 2H_0H_1\frac{a^5}{r^5}\cos 5\theta + H_1^2\frac{a^6}{r^6}\cos 6\theta\right)\right]\cos\theta\, d\theta \\
&+ \frac{\mu r}{8\pi}\int_0^{2\pi}\left[2H_0H_1\left(-\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu}\frac{a}{r}\right)\sin\theta - H_1^2\frac{r^2}{a^2}\sin^2\theta\right. \\
&\qquad + 2\frac{\mu'-\mu}{\mu'+\mu}\left((H_0^2 + H_1^2)\frac{a^2}{r^2}\sin 2\theta + H_0H_1\frac{a^3}{r^3}\sin 3\theta\right) \\
&\qquad \left. + \left(\frac{\mu'-\mu}{\mu'+\mu}\right)^2\left(H_0^2\frac{a^4}{r^4}\sin 4\theta + 2H_0H_1\frac{a^5}{r^5}\sin 5\theta + H_1^2\frac{a^6}{r^6}\sin 6\theta\right)\right]\sin\theta\, d\theta \\
&= \frac{\mu H_0 H_1 r}{4\pi}\int_0^{2\pi}\left(\frac{r}{a}\cos 2\theta + \frac{\mu'-\mu}{\mu'+\mu}\frac{a}{r}\right)d\theta = \frac{\mu'-\mu}{\mu'+\mu}\frac{aB_0H_1}{2}\,.
\end{aligned} \tag{23}$$

This force is independent of the radius $r$ used in the calculation, so long as $r > a$, and vanishes if the wire has the same permeability as the surrounding medium.

We now calculate the $y$ component of the force, which is expected to exhibit the $\mathbf{J}\times\mathbf{B}$ force (3) proportional to $IH_0$, so we drop terms in $I^2$ and $H^2$ that eventually would integrate to zero. Then,

$$F_y = \int (T_{yx}\, dS_x + T_{yy}\, dS_y) = \frac{\mu r}{4\pi}\int_0^{2\pi} H_x H_y \cos\theta\, d\theta + \frac{\mu r}{8\pi}\int_0^{2\pi}(H_y^2 - H_x^2)\sin\theta\, d\theta$$

$$
\begin{aligned}
= \quad & \frac{\mu r}{4\pi}\int_0^{2\pi}\left[H_0\left(1+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right)+H_1\left(\frac{r}{a}\cos\theta+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right)-\frac{2I}{cr}\sin\theta\right]\cdot \\
& \qquad \left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta+H_1\left(-\frac{r}{a}\sin\theta+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right)+\frac{2I}{cr}\cos\theta\right]\cos\theta\ d\theta \\
+ \quad & \frac{\mu r}{8\pi}\int_0^{2\pi}\left\{\left[\frac{\mu'-\mu}{\mu'+\mu}H_0\frac{a^2}{r^2}\sin 2\theta+H_1\left(-\frac{r}{a}\sin\theta+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\sin 3\theta\right)+\frac{2I}{cr}\cos\theta\right]^2\right. \\
& \left. -\left[H_0\left(1+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right)+H_1\left(\frac{r}{a}\cos\theta\frac{\mu'-\mu}{\mu'+\mu}\frac{a^3}{r^3}\cos 3\theta\right)-\frac{2I}{cr}\sin\theta\right]^2\right\}\sin\theta\ d\theta \\
= \quad & \frac{\mu I}{2\pi c}\int_0^{2\pi}\left[H_0\left(1+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right)\cos^2\theta+\frac{\mu'-\mu}{\mu'+\mu}H_1\frac{a^3}{r^3}\cos 3\theta\cos^2\theta\right. \\
& \qquad \left. -\frac{H_0}{2}\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\sin^2 2\theta-\frac{\mu'-\mu}{\mu'+\mu}H_1\frac{a^3}{r^3}\sin 3\theta\sin\theta\cos\theta\right]d\theta \\
+ \quad & \frac{\mu I}{2\pi c}\int_0^{2\pi}\left[\frac{H_0}{2}\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\sin^2 2\theta+\frac{\mu'-\mu}{\mu'+\mu}H_1\frac{a^3}{r^3}\sin 3\theta\sin\theta\cos\theta\right. \\
& \qquad \left. +H_0\left(1+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right)\sin^2\theta-\frac{\mu'-\mu}{\mu'+\mu}H_1\frac{a^3}{r^3}\cos 3\theta\cos^2\theta\right]d\theta \\
= \quad & \frac{\mu I H_0}{2\pi c}\int_0^{2\pi}\left(1+\frac{\mu'-\mu}{\mu'+\mu}\frac{a^2}{r^2}\cos 2\theta\right)d\theta=\frac{\mu I H_0}{c}=\frac{IB_0}{c}. \qquad (24)
\end{aligned}
$$

The force is independent of the choice of the radius $r$, so long as $r > a$, is independent of the permeability $\mu'$ of the wire, and agrees with the simple expectation (3).

For the record, if we had integrated the stress tensor over a cylinder of radius $r < a$ the result would be $\mathbf{F} = 2\mu' I B_0 r^2\ \hat{\mathbf{y}}/a^2(\mu' + \mu)$. Since the limit of this as $r \to a$ does not equal the result for $r > a$, we infer that there are important effects at the interface $r = a$. The permeable liquid is presumably contained in a tank of some characteristic radial scale $b \gg a$, at whose surface additional magnetization forces will arise. We consider these forces as distinct from those at the interface $r = a$, and that only the latter are part of the forces on the wire.

Equation (24) was deduced in a similar manner in ref. [1].

## 2.3 Calculation Using the Bulk Force Density

An expression for a bulk force density $\mathbf{f}$ in magnetic media can be obtained by transformation of the surface integral of the stress tensor into a volume integral. See, for example, secs. 15 and 35 of [9]. The result, again ignoring magnetostriction, is

$$
\mathbf{f} = \frac{1}{c}\mathbf{J}_{\text{cond}} \times \mathbf{B} - \frac{H^2}{8\pi}\boldsymbol{\nabla}\mu, \qquad (25)
$$

which is due to Helmholtz [13].

In the present problem, $\nabla\mu = 0$ except across the surface $r = a$ that separates the wire of permeability $\mu'$ from the surrounding medium of permeability $\mu$. Hence, the $\nabla\mu$ term of the volume integral of eq. (25) becomes a surface integral on the cylinder $r = a$. However, care is required in this procedure when $H^2$ is not continous across this surface. Recalling that the tangential component $H_t$ and the normal component $B_n = \mu H_n$ of the magnetic fields are continuous across a boundary, it is preferable to write $H^2 = H_t^2 + H_n^2 = H_t^2 + B_n^2/\mu^2$. Then,[1]

$$\int H^2 \boldsymbol{\nabla}\mu \; d\text{Vol} = \int \left( H_t^2 + \frac{B_n^2}{\mu^2} \right) \frac{\partial \mu}{\partial n} \; \hat{\mathbf{n}} \; d\text{Vol} = \int \left( H_t^2 \frac{\partial \mu}{\partial n} - B_n^2 \frac{\partial (1/\mu)}{\partial n} \right) \hat{\mathbf{n}} \; d\text{Vol}, \tag{26}$$

and

$$\mathbf{F} = \int \mathbf{f} \; d\text{Vol} = \int \frac{1}{c} \mathbf{J}_{\text{cond}} \times \mathbf{B} \; d\text{Vol} - \frac{\mu - \mu'}{8\pi} \int H_t^2 \; \hat{\mathbf{n}} \; dS + \frac{1}{8\pi} \left( \frac{1}{\mu} - \frac{1}{\mu'} \right) \int B_n^2 \; \hat{\mathbf{n}} \; dS. \tag{27}$$

In contrast to the simple prescription given at the beginning of sec. 2, in this integral the magnetic field $\mathbf{B}$ is the field on the current element $\mathbf{J}_{\text{cond}} \; d\text{Vol}$ from all sources, including those in element $d\text{Vol}$. For the second and third terms of eq. (27) where $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, we recall eq. (8) that for the $x$ and $y$ components we need only the parts of $H_\theta^2$ and $B_r^2$ that vary as $\cos\theta$ and $\sin\theta$, respectively. From eq. (18) we find

$$H_\theta^2 = 4 B_0 H_1 \frac{\mu}{(\mu' + \mu)^2} \cos\theta - \frac{8}{\mu' + \mu} \frac{I B_0}{ca} \sin\theta + ..., \tag{28}$$

$$B_r^2 = 4 B_0 H_1 \frac{\mu \mu'^2}{(\mu' + \mu)^2} \cos\theta + ... \tag{29}$$

Thus,

$$\begin{aligned}
\mathbf{F} &= \frac{1}{c} \int_0^a r \; dr \int_0^{2\pi} d\theta \; \frac{I}{\pi a^2} \; \hat{\mathbf{z}} \times \\
&\qquad \mu' \left\{ \left[ \frac{2}{\mu' + \mu} \left( B_0 + B_1 \frac{r}{a} \cos\theta \right) - \frac{2Ir}{ca^2} \sin\theta \right] \hat{\mathbf{x}} + \left( \frac{2Ir}{ca^2} \cos\theta - \frac{2}{\mu' + \mu} B_1 \frac{r}{a} \sin\theta \right) \hat{\mathbf{y}} \right\} \\
&\quad - \frac{\mu - \mu'}{8\pi} \int_0^{2\pi} a \; d\theta \; \left( 4 B_0 H_1 \frac{\mu}{(\mu' + \mu)^2} \cos\theta \right) \cos\theta \; \hat{\mathbf{x}} \\
&\quad - \frac{\mu - \mu'}{8\pi} \int_0^{2\pi} a \; d\theta \; \left( - \frac{8}{\mu' + \mu} \frac{I B_0}{ca} \sin\theta \right) \sin\theta \; \hat{\mathbf{y}} \\
&\quad + \frac{\mu' - \mu}{8\pi \mu \mu'} \int_0^{2\pi} a \; d\theta \; \left( 4 B_0 H_1 \frac{\mu \mu'^2}{(\mu' + \mu)^2} \cos\theta \right) \cos\theta \; \hat{\mathbf{x}} \\
&= \frac{2\mu'}{\mu' + \mu} \frac{I B_0}{c} \; \hat{\mathbf{y}} - (\mu - \mu') \frac{a B_0 H_1}{2} \frac{\mu}{(\mu' + \mu)^2} \; \hat{\mathbf{x}} + \frac{\mu - \mu'}{\mu' + \mu} \frac{I B_0}{c} \; \hat{\mathbf{y}} - (\mu - \mu') \frac{a B_0 H_1}{2} \frac{\mu'}{(\mu' + \mu)^2} \; \hat{\mathbf{x}} \\
&= \frac{\mu' - \mu}{\mu' + \mu} \frac{a B_0 H_1}{2} \; \hat{\mathbf{x}} + \frac{I B_0}{c} \; \hat{\mathbf{y}}.
\end{aligned} \tag{30}$$

This agrees with the calculation of the previous section via the stress tensor.

[1] Thanks to J. Castro for pointing out this trick.

# 3 Use of Magnetization Currents or Fictitious Magnetic Poles

We now take a different approach to the calculation of the force on the wire, whereby the effects due to magnetization are included via either the magnetization current density $J_M = c\boldsymbol{\nabla} \times \mathbf{M}$ (and the associated surface current $K_M = c\,\mathbf{M} \times \hat{\mathbf{n}}$) or via the fictitious pole density $\rho_M = -\boldsymbol{\nabla} \cdot \mathbf{M}$ (and the associated surface pole density $\sigma_M = \mathbf{M} \cdot \hat{\mathbf{n}}$). The wire is considered to consist of filaments along the $z$ axis, and the total force is calculated as an integral over the force on the filaments in the spirit of the Biot-Savart force law.

The hope is that this approach would provide more intuitive explanations for the term $-(H^2/8\pi)\boldsymbol{\nabla}\mu$ in the bulk force expression (25). However, we achieve success only with prescriptions that differ in an important way from those advocated by Brown [5] on behalf of the Coulomb Committee.

## 3.1 Calculation Using Magnetization Currents

In the first version of this calculation, we include the so-called bound current density due to the bulk magnetization $\mathbf{M}$,

$$\mathbf{J}_M = c\boldsymbol{\nabla} \times \mathbf{M} = \frac{\mu' - 1}{4\pi/c}\boldsymbol{\nabla} \times \mathbf{H} = (\mu' - 1)\mathbf{J}_{\rm cond}, \tag{31}$$

since the magnetic field $\mathbf{H}$ is related to the conduction current density $\mathbf{J}_{\rm cond}$ by Ampère's law, $\boldsymbol{\nabla} \times \mathbf{H} = 4\pi\mathbf{J}_{\rm cond}/c$ in magnetostatics. Thus, the total current in the interior of the wire is

$$\mathbf{J}_{\rm total} = \mathbf{J}_M + \mathbf{J}_{\rm cond} = \mu'\mathbf{J}_{\rm cond}. \tag{32}$$

The current density $\mathbf{J}_{\rm total}$ is the one that should be used in the microscopic version of Ampère's law $\boldsymbol{\nabla} \times \mathbf{B} = (4\pi/c)\mathbf{J}_{\rm total}$, which leads to $\mathbf{B}(I) = 2\mu' I r\,\hat{\boldsymbol{\theta}}/ca^2 = \mu'\mathbf{H}(I)$ inside the wire. Since eq. (32) by itself also implies that $\mathbf{B}(I) = 2\mu' I\,\hat{\boldsymbol{\theta}}/cr$ outside the wire, we see that in addition to the volume magnetization current density $\mathbf{J}_M$, there must be surface currents at $r = a$. On the outer surface of the wire the current density is given by

$$\begin{aligned} c\,\mathbf{M}(r = a^-) \times \hat{\mathbf{r}} &= c\frac{\mu' - 1}{4\pi}H_\theta(r = a^-)\,\hat{\boldsymbol{\theta}} \times \hat{\mathbf{r}} \\ &= -\frac{\mu' - 1}{4\pi}\left[\frac{2I}{a} - \frac{2\mu}{\mu' + \mu}c(H_0 \sin\theta + H_1 \sin 2\theta)\right]\hat{\mathbf{z}}, \end{aligned} \tag{33}$$

while surface current density on the inner surface of the medium surrounding the wire is

$$c\,\mathbf{M}(r = a^+) \times (-\hat{\mathbf{r}}) = \frac{\mu - 1}{4\pi}\left[\frac{2I}{a} - \frac{2\mu}{\mu' + \mu}c(H_0 \sin\theta + H_1 \sin 2\theta)\right]\hat{\mathbf{z}}. \tag{34}$$

The total surface current density is thus,

$$\mathbf{K}_M = \frac{\mu - \mu'}{4\pi}\left[\frac{2I}{a} - \frac{2\mu}{\mu' + \mu}c(H_0 \sin\theta + H_1 \sin 2\theta)\right]\hat{\mathbf{z}}. \tag{35}$$

By adding the surface current (35) to the microscopic form of Ampère's law, we should be able to deduce that part of the magnetic field **B** not initially present. The first term of eq. (35) has no effect on **B** for $r < a$, while for $r > a$ it adds a piece $2(\mu - \mu')I/cr$, so the total magnetic field outside the wire due to current $I$ is $\mathbf{B}(I) = 2\mu I\ \hat{\boldsymbol{\theta}}/cr = \mu \mathbf{H}(\mathbf{I})$, as expected. The second term of eq. (35) contributes to the magnetic field both inside and outside the wire, and in principle provides an alternative method of calculating the fields summarized in eq. (16).

The force on a current-carrying filament is now to be calculated using an appropriate version of the Biot-Savart law. Having tried numerous possible variations, we only find agreement with eqs. (23) and (24) if the magnetic field we use is $\mathbf{H}_i$ and not $\mathbf{B}_i$, where $i$ means the initial fields before the wire was introduced into the permeable liquid. See sec. 3.3 for additional discussion. Apparently, this prescription is the one advocated by Lorentz [14]. However, Brown, in his eq. (1.3-4′) of [5], advocates the use of the initial field $\mathbf{B}_i$.

It is also important to use the initial field as given by eq. (6), and not the field that would hold if a cylindrical cavity of radius $a$ were introduced into the permeable liquid.

The magnetic force per unit length on the volume currents and surface currents is to be calculated as

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\rm total} \times \mathbf{H}_i\ d\text{Vol} + \frac{1}{c} \int \mathbf{K}_M \times \mathbf{H}_i(r = a)\ dS, \tag{36}$$

where $\mathbf{J}_{\rm total}$ is given by eqs. (5) and (32), and the surface current $\mathbf{K}_M$ is given by eq. (35). Then,

$$\begin{aligned}
\mathbf{F} &= \frac{1}{c} \int_0^a r\ dr \int_0^{2\pi} d\theta\ \frac{\mu' I}{\pi a^2}\ \hat{\mathbf{z}} \times \left[\left(H_0 + H_1 \frac{r}{a} \cos\theta\right) \hat{\mathbf{x}} - H_1 \frac{r}{a} \sin\theta \hat{\mathbf{y}}\right] \\
&\quad + \frac{1}{c} \int_0^{2\pi} a\ d\theta\ \frac{\mu - \mu'}{4\pi} \left[\frac{2I}{a} - \frac{2\mu}{\mu' + \mu} c(H_0 \sin\theta + H_1 \sin 2\theta)\right] \hat{\mathbf{z}} \times \\
&\qquad [(H_0 + H_1 \cos\theta)\ \hat{\mathbf{x}} - H_1 \sin\theta\ \hat{\mathbf{y}}] \\
&= \mu' \frac{I H_0}{c}\ \hat{\mathbf{y}} + \mu \frac{\mu' - \mu}{\mu' + \mu} \frac{a H_0 H_1}{2}\ \hat{\mathbf{x}} + (\mu - \mu') \frac{I H_0}{c}\ \hat{\mathbf{y}} = \mu \frac{\mu' - \mu}{\mu' + \mu} \frac{a H_0 H_1}{2}\ \hat{\mathbf{x}} + \frac{\mu I H_0}{c}\ \hat{\mathbf{y}} \\
&= \frac{\mu' - \mu}{\mu' + \mu} \frac{a B_0 H_1}{2}\ \hat{\mathbf{x}} + \frac{I B_0}{c}\ \hat{\mathbf{y}},
\end{aligned} \tag{37}$$

in agreement with eqs. (23) and (24). If we had used the field $\mathbf{B}_i$, the result would be $\mu$ times the above.

## 3.2 Calculation Using Fictitious Magnetic Poles

The forces on the magnetization of the media might also considered as due to a density of fictitious magnetic poles, rather than being due to currents $\mathbf{J}_M$ and $\mathbf{K}_M$. Some care is required to use this approach, since a true magnetic pole density $\rho_M$ would imply $\boldsymbol{\nabla} \cdot \mathbf{B} = 4\pi\rho_M$, and the bulk force density on these poles would be $\mathbf{F} = \rho_M \mathbf{B}$. However, in reality $0 = \boldsymbol{\nabla} \cdot \mathbf{B} = \boldsymbol{\nabla} \cdot (\mathbf{H} + 4\pi\mathbf{M})$, so we write

$$\boldsymbol{\nabla} \cdot \mathbf{H} = -4\pi \boldsymbol{\nabla} \cdot \mathbf{M} = 4\pi\rho_M, \tag{38}$$

and we identify $\rho_M = -\boldsymbol{\nabla} \cdot \mathbf{M}$ as the volume density of fictitious magnetic poles, following Poisson [15]. Inside linear magnetic media, such as those considered here, $\mathbf{B} = \mu' \mathbf{H}$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ together imply that $\rho_M = 0$. However, a surface density $\sigma_M$ of fictitious poles can exist on an interface between two media, and we see that Gauss' law for the field $\mathbf{H}$ implies that

$$\sigma_M = \frac{(\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}}}{4\pi}, \tag{39}$$

where unit normal $\hat{\mathbf{n}}$ points across the interface from medium 1 to medium 2. The surface pole density can also be written in terms of the magnetization $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$ as

$$\sigma_M = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \hat{\mathbf{n}}, \tag{40}$$

since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ insures that the normal component of $\mathbf{B}$ is continuous at the interface.

In the present problem, the density of fictitious magnetic poles on the surface $r = a$ is given by

$$\sigma_M = \frac{H_r(r = a^+) - H_r(r = a^-)}{4\pi} = \frac{1}{2\pi} \frac{\mu' - \mu}{\mu' + \mu} (H_0 \cos\theta + H_1 \cos 2\theta). \tag{41}$$

The force on the surface density of fictitious magnetic poles is

$$\mathbf{F} = \sigma_M \mathbf{B}(r = a), \tag{42}$$

assuming that the fictitious poles couple to $\mathbf{B}$ rather than to $\mathbf{H}$, following the advice of Thomson and Maxwell [16].[2]

The total force on the medium in this view is the sum of the force on the conduction current plus the force on the fictitious surface poles, where to avoid calculating a spurious force of the wire on itself we use the initial magnetic field $\mathbf{B}_i$,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\rm cond} \times \mathbf{B}_i \; d{\rm Vol} + \int \sigma_M \mathbf{B}_i(r = a) \; dS. \tag{43}$$

Then,

$$\begin{aligned} \mathbf{F} &= \frac{1}{c} \int_0^a r \; dr \int_0^{2\pi} d\theta \; \frac{I}{\pi a^2} \; \hat{\mathbf{z}} \times B_0 \; \hat{\mathbf{x}} \\ &\quad + \int_0^{2\pi} a \; d\theta \; \frac{1}{2\pi} \frac{\mu' - \mu}{\mu' + \mu} (H_0 \cos\theta + H_1 \cos 2\theta) \cdot [(B_0 + B_1 \cos\theta) \; \hat{\mathbf{x}} - B_1 \sin\theta \; \hat{\mathbf{y}}] \\ &= \frac{\mu' - \mu}{\mu' + \mu} \frac{a H_0 B_1}{2} \; \hat{\mathbf{x}} + \frac{I B_0}{c} \; \hat{\mathbf{y}}, \end{aligned} \tag{44}$$

in agreement with eqs. (23) and (24), since $H_0 B_1 = B_0 H_1$.

[2]We note that Brown in his eq. (1.3-3) of [5] recommends that the initial field $\mathbf{H}_i$ be used rather than $\mathbf{B}_i$ when using the method of fictitious magnetic poles. However, this would imply a force $1/\mu$ times that of eq. (42).

## 3.3 The Biot-Savart Force Law in a Permeable Medium

The results of secs. 3.1 and 3.2 show that care is needed when using the Biot-Savart force law in permeable media. We review this issue by starting with the simpler case that the wire and the surrounding liquid both have the same permeability $\mu \neq 1$. Then there is neither a surface current nor a fictitious pole density at the interface between the wire and the liquid. However, there remains the volume current density $\mathbf{J}_M = (\mu - 1)\mathbf{J}_{\rm cond}$, so that the total current is still $\mathbf{J}_{\rm total} = \mu \mathbf{J}_{\rm cond}$. Since the force on the wire is correctly calculated via the force law

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\rm cond} \times \mathbf{B}_i \, d{\rm Vol}, \tag{45}$$

using the conduction current, we see that if we wish to use the total current we must write

$$\mathbf{F} = \frac{1}{c} \int \frac{\mathbf{J}_{\rm total}}{\mu} \times \mathbf{B}_i \, d{\rm Vol} = \frac{1}{c} \int \mathbf{J}_{\rm total} \times \mathbf{H}_i \, d{\rm Vol}. \tag{46}$$

The other aspect of the analysis of Biot and Savart is the calculation of the magnetic field from the current density. The microscopic version of Ampère's law,

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_{\rm total}, \tag{47}$$

corresponds to the prescription that

$$\mathbf{B} = \frac{1}{c} \int \frac{\mathbf{J}_{\rm total} \times \hat{\mathbf{r}}}{r^2} \, d{\rm Vol} = \frac{\mu}{c} \int \frac{\mathbf{J}_{\rm cond} \times \hat{\mathbf{r}}}{r^2} \, d{\rm Vol} = \mu \mathbf{H}. \tag{48}$$

Hence, the macroscopic version of Ampère's law,

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\rm cond}, \tag{49}$$

corresponds to the prescription that

$$\mathbf{H} = \frac{1}{c} \int \frac{\mathbf{J}_{\rm cond} \times \hat{\mathbf{r}}}{r^2} \, d{\rm Vol}, \tag{50}$$

independent of the permeability.

The form of the eq. (2) for the force on circuit $a$ due to circuit $b$ supposing the wires and the surrounding media all have permeability $\mu$ is therefore

$$\mathbf{F}_a = \frac{I_a}{c} \oint_a d\mathbf{l}_a \times \mathbf{B}_b = \mu \frac{I_a I_b}{c^2} \oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2} \,, \tag{51}$$

where $I_a$ and $I_b$ are the conduction currents in the circuits. We also see that eq. (51) holds even if the wires have permeabilities $\mu_a$ and $\mu_b$ that differ from the permeability $\mu$ of the surrounding medium, since the magnetic field due to wire $b$ at the position of wire $a$ before wire $a$ was introduced is given by $\mathbf{B}_b = \mu \mathbf{H}_b$, which depends on neither $\mu_a$ nor $\mu_b$.

An extensive bibliography on conceptual issues in magnetism is at [17].

# References


[1] See, for example, secs. 4.20-21 of J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[2] L.W. Casperson, *Forces on permeable conductors in magnetic fields*, Am. J. Phys. **70**, 163-168 (2002),
http://puhep1.princeton.edu/~mcdonald/examples/EM/casperson_ajp_70_163_02.pdf

[3] W.F. Brown, Jr. *et al.*, *The teaching of electricity and magnetism at the college level. I. Logical standards and critical issues*, Am. J. Phys. **18**, 1-25 (1950),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brown_ajp_18_1_50.pdf

[4] W.F. Brown, Jr. *et al.*, *The teaching of electricity and magnetism at the college level. II. Two outlines for teachers*, Am. J. Phys. **18**, 69-88 (1950),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brown_ajp_18_69_50.pdf

[5] W.F. Brown, Jr., *Electric and magnetic forces: A direct calculation. I*, Am. J. Phys. **19**, 290-304 (1951),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brown_ajp_19_290_51.pdf

[6] W.F. Brown, Jr., *Electric and magnetic forces: A direct calculation. II*, Am. J. Phys. **19**, 333-350 (1951),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brown_ajp_19_333_51.pdf

[7] K.T. McDonald, *Methods of Calculating Forces on Rigid Magnetic Media* (March 18, 2002), http://puhep1.princeton.edu/~mcdonald/examples/magnetic_force.pdf

[8] F. Rasetti, *Deflection of mesons in magnetized iron*, Phys. Rev. **66**, 1-5 (1944),
http://puhep1.princeton.edu/~mcdonald/examples/EM/rasetti_pr_66_1_44.pdf
This paper mentions the earlier history of erratic results on this topic.

[9] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed., (Butterworth-Heinemann, Oxford, 1984).

[10] F.J. Lowes, *Force on a Wire in a Magnetic Field*, Nature **246**, 208-209 (1973),
http://puhep1.princeton.edu/~mcdonald/examples/EM/lowes_nature_246_208_73.pdf
Related papers by this author include: *The torque on a magnet*, Proc. Roy. Soc. London A **337**, 555-567 (1974),
http://puhep1.princeton.edu/~mcdonald/examples/EM/lowes_prsla_337_555_74.pdf
*Do Magnetometers Measure B or H?*, Geophys. J. Roy. Astro. Soc. **37**, 151 (1974),
http://puhep1.princeton.edu/~mcdonald/examples/EM/lowes_gjras_37_151_74.pdf

[11] W.R. McKinnon, S.P. McAlister and C.M. Hurd, *Origin of the force on a current-carrying wire in a magnetic field*, Am. J. Phys. **49**, 493-494 (1981),
http://puhep1.princeton.edu/~mcdonald/examples/EM/mckinnon_ajp_49_493_81.pdf

[12] Secs. 641-645 of J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd. ed. (Clarendon Press, Oxford, 1891; Dover Publications, New York, 1954).

[13] H. von Helmholtz, *Über die auf das Innere magnetisch order dielectrisch polarisirter Körper wirkenden Kräfte*, Ann. d. Phys. **13**, 385-406 (1882).

[14] H.A. Lorentz, *Versuch einer Theorie der electrischen und optischen Ersheinungen in bewegten Körpern* (E.J. Brill, Leiden, 1895), Art. 12.

[15] S.-D. Poisson, Mem. d. l'Acad. **V**, 247 (1824).

[16] Poisson worked exclusively with the magnetic field $\mathbf{H}$, but realized that the force on a fictitious magnetic pole $p$ is not necessarily $\mathbf{F} = p\mathbf{H}$, since poles are always found inside bulk media, which results in an additional force on the pole depending on the assumed shape of the surrounding cavity. W. Thomson (Lord Kelvin) noted in 1850 that for a pole in a disk-shaped cavity with axis parallel to the magnetization $\mathbf{M}$ of the medium, the force would be $\mathbf{F} = p(\mathbf{H} + 4\pi\mathbf{M})$, and therefore he introduced the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ "according to the electromagnetic definition". See p. 499 of W. Thomson, *Papers on Electrostatics and Magnetism* (Macmillan, London, 1884). In sec. 400 of his *Treatise* [12], Maxwell follows Thomson in stating that the force on a magnetic pole is usefully considered to be $\mathbf{F} = p\mathbf{B}$. This is very reasonable since in linear magnetic media fictitious poles are only found on surfaces, for which an appropriate surrounding cavity is disk-shaped.

Another view of this issue is that the force on the fictitious poles in a magnetic medium depends on that magnetic field which is the macroscopic average of the microscopic field. Whether one labels the microscopic magnetic field $\mathbf{B}$ or $\mathbf{H}$ is a matter of convention since they are equal, the magnetization $\mathbf{M}$ being defined only macroscopically. As emphasized in sec. 29 of [9], the macroscopic field $\mathbf{B}$, and not $\mathbf{H}$, is the average of the microscopic magnetic field field. Hence, we again expect the force on a magnetic pole in a medium to be $\mathbf{F} = p\mathbf{B}$.

[17] http://www.aw-verlag.ch/Ref/Ref_Elektrodynamik.htm