Princeton University

Ph304 Final Examination
Electrodynamics

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Do all work you wish graded in the exam booklets provided.

You may consult Griffiths' text during the exam, but otherwise the exam is closed book, closed notes.

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http://puhep1.princeton.edu/~mcdonald/examples/

Please do all work in the exam booklets provided.

You may use either Gaussian or SI units on this exam.

1. (20 pts.) All electrostatic fields $\mathbf{E}$ (*i.e.*, ones with no time dependence) can be derived from a scalar potential V ($\mathbf{E} = -\nabla V$) and hence obey $\nabla \times \mathbf{E} = -\nabla \times \nabla V = 0$. The latter condition is sometimes considered to be a requirement for electrostatic fields. However, as you are to show in this problem, it is possible to have time-dependent electric fields (waves) that obey $\nabla \times \mathbf{E} = 0$. Such fields have been given the name "electrostatic waves".

   (a) Show, using Maxwell's equations, that if an electric field $\mathbf{E}$ has no time dependence, then $\nabla \times \mathbf{E} = 0$.

   (b) Show that a plane wave with electric field $\mathbf{E}$ parallel to the wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ (a longitudinal wave) can exist in a medium with no time-dependent magnetic field if the (time dependent part of the) electric displacement $\mathbf{D}$ is zero. This cannot occur in an ordinary dielectric medium, but can happen in a plasma. (Time-independent electric and magnetic fields could, of course, be superimposed on the wave field.)

   (c) Write potentials for the "electrostatic wave" of part (b) in both the Coulomb and Lorentz gauges.

   (d) Discuss energy density and flow for such a wave, accounting for both the energy of the electric field and the kinetic energy of the electrons in the plasma in response to that field. You may assume that the positive ions remain at rest.

2. (20 pts.) The "quality factor" Q of a resonant cavity can be defined as 2π times the ratio of the time-averaged stored energy to the energy lost per cycle. What is the maximum Q that can be attained with a cubical cavity of edge a whose walls are made of a "good" conductor of conductivity σ (and relative dielectric constant $\epsilon_r = 1$ and relative permeability $\mu_r = 1$), when the cavity is operated at angular frequency ω? The cavity itself is under vacuum.

   This problem is "straightforward, but lengthy". It may be convenient to break it up into several steps. In the first part, deduce relations for the electromagnetic fields, for the frequency as a function of wave number, and for the time-averaged stored energy assuming perfectly conducting walls. In the second part, suppose the walls have large but finite conductivity σ to deduce relations between the fields inside the material of the walls and those in the cavity, and from this the power transmitted into the walls. Finally, calculate the cavity Q.

3. (20 pts.) A turnstile antenna consists of a pair of half-wave, center-fed linear dipole antennas oriented at 90° to each other, and driven 90° out of phase, as shown below.

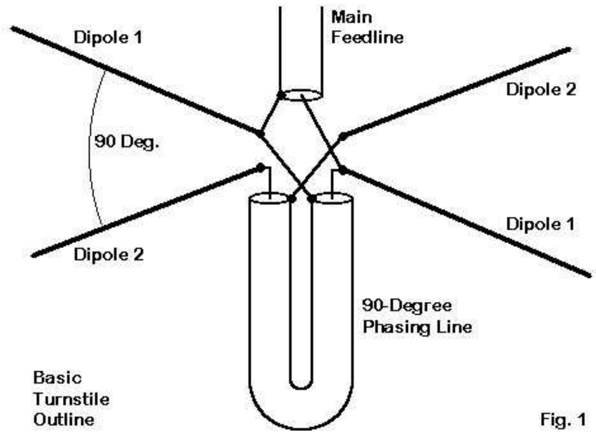


Fig. 1

For simplicity, you may approximate the turnstile radiator as made of a pair of point dipoles of peak strength p_0, oscillating with angular frequency ω, centered on the origin, one oriented along the x and the other along the y axis, with the second dipole driven 90° out of phase with respect to the first.

(a) Deduce the electric and magnetic fields in the far-zone, and the (time-averaged) angular distribution $\langle dP/d\Omega \rangle$ of the radiated power. What is the polarization of radiation in the plane $z = 0$ and along the z axis?

(b) Suppose a second, identical turnstile antenna is mounted with its center at $z = \lambda/4$, and driven in phase with the first antenna. What is the angular distribution of the radiation now?

Solutions

1. (a) $\nabla \times \mathbf{E} = 0$ **if E Has No Time Dependence**

We first verify that Maxwell's equations imply that when an electric field $\mathbf{E}$ has no time dependence, then $\nabla \times \mathbf{E} = 0$.

If $\partial\mathbf{E}/\partial t = 0$, then the magnetic field $\mathbf{B}$ obeys $\partial^2\mathbf{B}/\partial t^2 = 0$, as follows on taking the time derivative of Faraday's law, $c\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ in Gaussian units. In principle, this is consistent with a magnetic field that varies linearly with time, $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r})t$. However, this leads to arbitrarily large magnetic fields at early and late times, and is excluded on physical grounds. Hence, $\partial\mathbf{E}/\partial t = 0$ implies that $\partial\mathbf{B}/\partial t = 0$ also, and $\nabla \times \mathbf{E} = 0$ according to Faraday's law.

(b) **Electrostatic Plane Waves Must have D = 0**

We next consider some general properties of a longitudinal plane electric wave, taken to have the form

$$\mathbf{E} = E_z \hat{\mathbf{z}} e^{i(kz-\omega t)}. \tag{1}$$

This obeys $\nabla \times \mathbf{E} = 0$, and so can be derived from an electric potential, namely

$$\mathbf{E} = -\nabla V \qquad \text{where} \qquad V = i\frac{E_z}{k} e^{i(kz-\omega t)}. \tag{2}$$

The electric wave (1) has no associated magnetic wave, since Faraday's law tells us that

$$0 = \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

and any magnetic field in the problem must be static.

It is well known that electromagnetic waves in vacuum are transverse. A longitudinal electric wave can only exist in a medium that can support a nonzero polarization density $\mathbf{P}$ (volume density of electric dipole moments). The polarization density implies an effective charge density ρ given by

$$\rho = -\nabla \cdot \mathbf{P} \tag{4}$$

which is consistent with the first Maxwell equation,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{5}$$

only if

$$\mathbf{P} = -\frac{\mathbf{E}}{4\pi}, \tag{6}$$

in which case the electric displacement $\mathbf{D}$ of the longitudinal wave vanishes,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = 0. \tag{7}$$

Hence, the (relative) dielectric constant ϵ also vanishes

Strictly speaking, eq. (6) could read $\mathbf{P} = -\mathbf{E}/4\pi + \mathbf{P}'$, for any field $\mathbf{P}'$ that obeys $\nabla \cdot \mathbf{P}' = 0$. However, since any magnetic field in the problem is static, the fourth Maxwell equation tells us that

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\left(\mathbf{J} + \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t}\right) \tag{8}$$

has no time dependence. Recalling that the polarization current is related by

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}\,, \tag{9}$$

we again find relation (6) with the possible addition of a static field $\mathbf{P}'$ that is associated with a truly electrostatic field $\mathbf{E}'$. In sum, a longitudinal electric wave described by eqs. (1), (6) and (7) can coexist with background electrostatic and magnetostatic fields of the usual type.

Maxwell's equations alone provide no relation between the wave number k and the wave frequency ω of the longitudinal wave, and hence the wave phase velocity ω/k is arbitrary. This suggests that purely longitudinal electric waves are best considered as limiting cases of more general waves, for which additional physical relations provide additional information as to the character of the waves.

When $\nabla \times \mathbf{E} = 0$, the magnetic field $\mathbf{B}$ has no time dependence. It is therefore tempting to conclude that the magnetic field $\mathbf{H}$ and the magnetization $\mathbf{M}$ also have no time dependence, since $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$. Then the Maxwell equation $\nabla \times \mathbf{H} = (4\pi/c)\mathbf{J}_{\rm free} + \partial\mathbf{D}/\partial t$ would imply that the displacement $\mathbf{D}$ has no time dependence. However, there exist interesting phenomena called spin waves in which $\mathbf{B} = 0$, but $\mathbf{H} = -4\pi\mathbf{M}$ have nontrivial behavior. See,
http://puhep1.princeton.edu/˜mcdonald/examples/spinwave.pdf

(c) **Gauge Invariance**

Since the electric wave (1) has no associated magnetic field, we can define its vector potential $\mathbf{A}$ to be zero, which is certainly consistent with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. Using eq. (2) for the scalar potential, the potentials in the Coulomb can be written

$$V = i\frac{E_z}{k}e^{i(kz-\omega t)}, \qquad \mathbf{A} = 0 \qquad \text{(Coulomb gauge)}. \tag{10}$$

Suppose, however, we prefer to work in the Lorentz gauge, for which

$$\nabla \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t}. \tag{11}$$

Then, the vector potential will be nonzero, and the electric field is related by

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = E_x \hat{\mathbf{z}} e^{i(kz-\omega t)}. \tag{12}$$

Clearly the potentials have the forms

$$\mathbf{A} = A_z \hat{\mathbf{z}} e^{i(kz-\omega t)}, \qquad V = V_0 e^{i(kz-\omega t)}, \tag{13}$$

which are consistent with $\mathbf{B} = \nabla \times \mathbf{A} = 0$. From the Lorentz gauge condition (11) we have

$$kA_z = \frac{\omega}{c} V_0, \tag{14}$$

and from eq. (12) we find

$$E_z = ikV_0 + i\frac{\omega}{c} A_z. \tag{15}$$

Hence,

$$\mathbf{A} = -i\frac{\omega c}{\omega^2 + k^2c^2} E_x \hat{\mathbf{z}} e^{i(kz-\omega t)}, \quad V = -i\frac{kc^2}{\omega^2 + k^2c^2} E_z e^{i(kz-\omega t)} \quad \text{(Lorentz gauge)}. \tag{16}$$

We could also derive the wave (1) from the potentials

$$\mathbf{A} = -i\frac{c}{\omega} E_z \hat{\mathbf{x}} e^{i(kz-\omega t)}, \qquad V = 0 \qquad \text{(Neumann(?) gauge)}. \tag{17}$$

Thus, an "electrostatic wave" is not necessarily associated with an "electrostatic" scalar potential.

(d) **Energy Considerations**

A common expression for the electric field energy density is $\mathbf{E} \cdot \mathbf{D}/8\pi$. However, this vanishes for longitudinal electric waves, according to eq. (7). Further, since the longitudinal electric wave can exist with zero magnetic field, there is no Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ or momentum density $\mathbf{p}_{\text{field}} = \mathbf{D} \times \mathbf{B}/4\pi c$, according to the usual prescriptions.

Let us recall the origins of the standard lore. Namely, the rate of work done by the field $\mathbf{E}$ on current density $\mathbf{J}$ is

$$\mathbf{J} \cdot \mathbf{E} = \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} = -\frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} = -\frac{\partial E^2/8\pi}{\partial t}, \tag{18}$$

using eqs. (6) and (9). This work is done at the expense of the electric field energy density u_{field}, which we therefore identify as

$$u_{\text{field}} = \frac{E^2}{8\pi} = \frac{E_x^2}{8\pi} \cos^2(kz - \omega t), \tag{19}$$

for the longitudinal wave (1). We readily interpret this energy density as moving in the $+z$ direction at the phase velocity $v_p = \omega/k$, even though the derivation of eq. (18) did not lead to a Poynting vector.

We should also note that energy is stored in the medium in the form of kinetic energy of the electrons (and, in general, ions as well) that contribute to the polarization,

$$\mathbf{P} = Ne(\mathbf{x} - \mathbf{x}_0) = -\frac{\mathbf{E}}{4\pi}. \tag{20}$$

Thus, the velocity of an electron is given by

$$\mathbf{v} = \mathbf{v}_0 - \frac{\dot{\mathbf{E}}}{4\pi Ne} = \mathbf{v}_0 - \frac{\omega E_x \hat{\mathbf{x}}}{4\pi Ne} \sin(kx - \omega t). \tag{21}$$

In squaring this to get the kinetic energy, we neglect the term in $\mathbf{v}_0 \cdot \hat{\mathbf{z}}$, assuming its average to be zero as holds for a medium that is at rest on average (and also holds for a plasma in a tokamak when z is taken as the radial coordinate in a small volume). Then, we find the mechanical energy density to be

$$u_{\rm mech} = \frac{1}{2} N m v^2 = \frac{1}{2} N m v_0^2 + \frac{E_z^2}{8\pi} \frac{\omega^2 m}{4\pi N e^2} \sin^2(kz - \omega t) = u_{\rm mech,0} + \frac{\omega^2}{\omega_p^2} \frac{E_x^2}{8\pi} \sin^2(kz - \omega t), \tag{22}$$

where $\omega_p = \sqrt{4\pi N e^2/m}$ is the plasma frequency. We again can interpret the additional term as an energy density that flows in the $+x$ direction at the phase velocity.

The total, time-averaged energy density associated with the longitudinal wave is

$$\langle u_{\rm wave} \rangle = \frac{\omega^2 + \omega_p^2}{2\omega_p^2} \frac{E_z^2}{8\pi} . \tag{23}$$

If the wave frequency is less than the plasma frequency, as is the case for examples of Bernstein waves discussed in http://puhep1.princeton.edu/~mcdonald/examples/bernstein.pdf, the longitudinal electric field energy density is larger than that of the mechanical energy density of the wave.

2. We can anticipate the answer to this question via dimensional analysis. The stored energy is proportional to the cavity volume times the square of the field strength: $\langle U \rangle \propto a^3 |\mathbf{E}_0|^2$. Energy is lost into the cavity walls, and the fields die out over roughly one skin depth d. The energy lost is therefore proportional to the volume in the walls that contains one skin depth, times the square of the field strength: $\Delta U_{\rm cycle} \propto a^2 d |\mathbf{E}_0|^2$. Hence, the cavity Q varies as $Q \propto \langle U \rangle / \Delta U_{\rm cycle} \propto a/d$.

   (a) **Fields in the cavity, if the walls are perfect conductors**

   The cubical cavity extends over $0 < x < a$, $0 < y < a$, $0 < z < a$. The boundary conditions for perfectly conducting walls are that the tangential component of the electric field, and the normal component of the magnetic field, vanish at the walls.

   In the cavity, the fields obey the wave equation

$$\nabla^2 \mathbf{E}, \mathbf{B} + \frac{1}{c^2} \frac{\partial \mathbf{E}, \mathbf{B}}{\partial t^2} = 0. \tag{24}$$

   We seek standing-wave solutions with angular frequency ω,

$$\mathbf{E}, \mathbf{B} = \mathbf{E}, \mathbf{B}(x, y, z) e^{-i\omega t}, \tag{25}$$

   for which the wave equation (24) becomes the Helmhotz equation

$$\nabla^2 \mathbf{E}, \mathbf{B} + k^2 \mathbf{E}, \mathbf{B} = 0, \tag{26}$$

where the wave number k is defined to be the ratio ω/c. For each of the six scalar components (such as $\psi = E_x$) of eq. (26), we seek a separation-of-variables solution, *i.e.*,

$$\psi = X(x)Y(y)Z(z). \tag{27}$$

Inserting this in the Helmholtz equation (26) and dividing by ψ, we are led to the three separated equations

$$X'' = -k_x^2 X, \qquad Y'' = -k_y^2 Z, \qquad Z'' = -k_z^2 Z, \tag{28}$$

where the separation constants k_x, k_y and k_z obey

$$k^2 = k_x^2 + k_y^2 + k_z^2. \tag{29}$$

Clearly, the functions X, Y and Z are sines and cosines (such as $X = \cos k_x x$, *etc.*).

Consider, for example, $\psi = E_x$. This component is tangential to the walls at $y = 0$, a and at $z = 0$, a, and so must vanish there. Hence, the functions Y and Z for E_x must be sines, not cosines, and the separation constants must be of the form $k_y = m\pi/a$, $k_z = n\pi/a$. When we consider $\psi = E_y$ or E_z we learn that $k_x = l\pi/a$ (where $l,\ m,\ n$ are integers). That is the wave vector $\mathbf{k}$ can be written

$$\mathbf{k} = \frac{\pi}{z}(l, m, n). \tag{30}$$

We also learn that the functions X and Z for E_y, and the functions X and Y for E_z must be sines. So far, we haven't learned of the character of function X of E_x, Y of E_y or Z of E_z.

If we now consider ψ to be a component of the magnetic field, whose normal component vanishes as the wall, we learn that the function X of B_x must have the form $\sin l\pi x/a$, and likewise that function Y of B_y and function Z of B_z are sines also.

So far, we have that

$$E_x = E_{x0} X_{E_x} \sin\frac{m\pi y}{a} \sin\frac{n\pi z}{a} e^{-i\omega t}, \tag{31}$$

$$E_y = E_{y0} \sin\frac{l\pi x}{a} Y_{E_y} \sin\frac{n\pi z}{a} e^{-i\omega t}, \tag{32}$$

$$E_z = E_{z0} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{a} Z_{E_z} e^{-i\omega t}, \tag{33}$$

$$B_x = B_{x0} \sin\frac{l\pi x}{a} Y_{B_x} Z_{B_x} e^{-i\omega t}, \tag{34}$$

$$B_y = B_{y0} X_{B_y} \sin\frac{m\pi y}{a} Z_{B_y} e^{-i\omega t}, \tag{35}$$

$$B_z = B_{z0} X_{E_z} Y_{B_z} \sin\frac{n\pi z}{a} e^{-i\omega t}. \tag{36}$$

To go further, we can use Faraday's law to relate $\mathbf{E}$ and $\mathbf{B}$. For example, the x component of this tells us that

$$-\frac{1}{c}\frac{\partial B_x}{\partial t} = ikB_x = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\,. \tag{37}$$

Comparing with the forms (32), (33) and (34), we see that Y_{E_y} must be $\cos m\pi y/a$, Z_{E_z} must be $\cos n\pi z/a$ and X_{B_x} must be $\sin l\pi x/a$. By similar consideration of the y and z components of Faraday's law, we see that the fields have the form

$$E_x = E_{x0} \cos \frac{l\pi x}{a} \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{a} e^{-i\omega t}, \tag{38}$$

$$E_y = E_{y0} \sin \frac{l\pi x}{a} \cos \frac{m\pi y}{a} \sin \frac{n\pi z}{a} e^{-i\omega t}, \tag{39}$$

$$E_z = E_{z0} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{a} \cos \frac{n\pi z}{a} e^{-i\omega t}, \tag{40}$$

$$B_x = B_{x0} \sin \frac{l\pi x}{a} \cos \frac{m\pi y}{a} \cos \frac{n\pi z}{a} e^{-i\omega t}, \tag{41}$$

$$B_y = B_{y0} \cos \frac{l\pi x}{a} \sin \frac{m\pi y}{a} \cos \frac{n\pi z}{a} e^{-i\omega t}, \tag{42}$$

$$B_z = B_{z0} \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{a} \sin \frac{n\pi z}{a} e^{-i\omega t}. \tag{43}$$

The normalizations

$$\mathbf{E}_0 = (E_{x0}, E_{y0}, E_{z0}) \qquad \text{and} \qquad \mathbf{B}_0 = (B_{x0}, B_{y0}, B_{z0}) \tag{44}$$

are not yet determined. The Maxwell equations $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ require that

$$\mathbf{E}_0 \cdot \mathbf{k} = 0 = \mathbf{B}_0 \cdot \mathbf{k}, \tag{45}$$

while the Maxwell equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial ct$ and $\nabla \times \mathbf{B} = \partial \mathbf{E}/\partial ct$ require that

$$ik\mathbf{B}_0 = \mathbf{k} \times \mathbf{E}_0, \qquad -ik\mathbf{E}_0 = \mathbf{k} \times \mathbf{B}_0. \tag{46}$$

The vectors $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ form an orthogonal triad where $|\mathbf{E}_0| = |\mathbf{B}_0|$. For a given set of indices (l, m, n) there are, in general, two solutions to eq. (45), *i.e.*, two polarizations of the fields. The 5 distinct equations of (45) and (46) imply that of the six components of $\mathbf{E}_0$ and $\mathbf{B}_0$ is independent. That is, for a given set of indices (l, m, n) and of polarization, the strength of the fields $\mathbf{E}$ and $\mathbf{B}$ can be characterized by a single scalar parameter.

(b) Inserting any of the field components (38)-(43) into the Helmholtz equation (26), we find that

$$k = \frac{\omega}{c} = \frac{\pi}{a}\sqrt{l^2 + m^2 + n^2}. \tag{47}$$

One can think of the standing-wave pattern inside a resonant cavity as due to "trapping" of a travelling wave inside a waveguide that is terminated by a pair of "mirrors". Then, the travelling wave factor $e^{i(k_g z - \omega t)}$ *is reduced to the standing-wave factor* $e^{-i\omega t}$*. That is, the "guide wave number"* k_g *goes to zero. Hence, Griffiths' eq. (9.187), in which he denotes by* k *what I have just called* k_g*, now tells us that* $\omega/c = (\pi/a)\sqrt{m^2 + n^2}$*. If we realize that the "trapped" travelling waves can also be thought of as moving in the* x *or* y *directions, the generalization to eq. (47) is to be expected.*

(c) The time-average stored energy is

$$\langle U\rangle = \frac{1}{2}\int \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} d\mathrm{Vol} = \int \frac{|\mathbf{E}|^2}{8\pi} d\mathrm{Vol} = \frac{|\mathbf{E}_0|^2 a^3}{64\pi}, \tag{48}$$

since the average of each of the three spatial factors of any component of $\mathbf{E}$ is 1/2 over the range of its argument.

(d) We now consider the fields inside the material of the walls, whose conductivity is σ. These fields are small, but nonzero. As a consequence, there are currents in the walls, which then dissipate energy due to Joule heating. We desire a relation for the total energy lost to this Joule heating per cycle. It is possible to calculate the current density $\mathbf{J}$ in the walls, and integrate J^2/σ to find the energy dissipation. However, is suffices to calculate the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{49}$$

just inside the walls, since we know that $\mathbf{S}$ points into the material, and that all the energy that enters the walls is eventually dissipated.

We recall that inside a good conductor the magnetic field is much larger than the electric field. Indeed, Faraday's law tells us that

$$i\frac{\omega}{c}\mathbf{B} = i\mathbf{k}\times\mathbf{E}, \tag{50}$$

where inside a good conductor the wave vector $\mathbf{k}$ is given by

$$\mathbf{k} = \frac{\hat{\mathbf{k}}}{d}(1+i), \tag{51}$$

where the skin depth d is

$$d = \frac{c}{\sqrt{2\pi\sigma\omega}}, \tag{52}$$

and the unit vector $\hat{\mathbf{k}}$ points normally into the conductor. Thus,

$$\mathbf{E} = \frac{\omega}{kc}\hat{\mathbf{k}}\times\mathbf{B} = \frac{\omega d}{2c}(1-i)\hat{\mathbf{k}}\times\mathbf{B}, \tag{53}$$

and the time-averaged Poynting vector can be written

$$\langle\mathbf{S}\rangle = \frac{1}{2}\frac{c}{4\pi}Re(\mathbf{E}\times\mathbf{B}^\star) = \frac{\omega d}{16\pi}\hat{\mathbf{k}}\,|\mathbf{B}|^2, \tag{54}$$

Now, the tangential component of the magnetic field is continuous across a boundary, so we can use $\mathbf{B}$ from eqs. (41)-(43) evaluated at the cavity wall (where $\mathbf{B}$ is tangential by construction).

The time-averaged energy flow into the cavity walls is therefore

$$\left\langle\frac{dU}{dt}\right\rangle = \int \langle\mathbf{S}_{\mathrm{wall}}\rangle\cdot d\mathbf{Area} = \frac{\omega a^2 d}{16\pi}\,|\mathbf{B_0}|^2, \tag{55}$$

The energy dissipated per cycle is $2\pi/\omega$ times this:

$$\Delta U_{\rm cycle} = \frac{a^2 d}{8}\left|\mathbf{B_0}\right|^2 = \frac{a^2 d}{8}\left|\mathbf{E_0}\right|^2, \tag{56}$$

The cavity Q is (2π times) the ratio of eqs. (48) and (56),

$$Q = 2\pi\frac{\langle U\rangle}{\Delta U_{\rm cycle}} = \frac{a}{4d} = \frac{a\sqrt{2\pi\sigma\omega}}{4c}. \tag{57}$$

The cavity Q is independent of the indices (l, m, n), and increases as $\sqrt{\sigma\omega}$

3. (a) If we approximate the half-wave dipoles by point dipoles each of strength p_0, then the dipole moment of the system can be written

$$\mathbf{p} = \mathbf{p}_0 e^{-i\omega t} = p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega t}, \tag{58}$$

taking the antenna to be aligned along the x and y axes. The electromagnetic fields in the far zone are then

$$\mathbf{B} = k^2 p_0\frac{e^{i(kr-\omega t)}}{r}\hat{\mathbf{k}}\times(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \qquad \mathbf{E} = \mathbf{B}\times\hat{\mathbf{k}}. \tag{59}$$

We desire the components of $\mathbf{E}$ and $\mathbf{B}$ in spherical coordinates, but it is more convenient to calculate first in rectangular coordinates, where

$$\hat{\mathbf{k}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}. \tag{60}$$

Then,

$$\begin{aligned}
\hat{\mathbf{k}}\times(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) &= -i\cos\theta\hat{\mathbf{x}} + \cos\hat{\mathbf{y}} + (i\sin\theta\cos\phi - \sin\theta\sin\phi)\hat{\mathbf{z}} \\
&= -i\cos\theta(\sin\theta\cos\phi\ \hat{\mathbf{r}} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}) \\
&\quad +\cos\theta(\sin\theta\sin\phi\ \hat{\mathbf{r}} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}) \\
&\quad +(i\sin\theta\cos\phi - \sin\theta\sin\phi)(\cos\theta\hat{\mathbf{r}} - \sin\theta\ \hat{\theta}) \\
&= (\sin\phi - i\cos\phi)\ \hat{\theta} - (\cos\phi + i\sin\phi)\ \hat{\phi}.
\end{aligned} \tag{61}$$

The components of the fields in spherical coordinates are therefore

$$E_r = B_r = \hat{\mathbf{k}}\cdot\mathbf{B} = 0, \tag{62}$$

$$E_\theta = B_\phi = -p_0 k^2\frac{e^{i(kr-\omega t)}}{r}\cos\theta(\cos\phi + i\sin\phi), \tag{63}$$

$$E_\phi = -B_\theta = -p_0 k^2\frac{e^{i(kr-\omega t)}}{r}(\sin\phi - i\cos\phi). \tag{64}$$

In the plane of the antenna, $\theta = 90°$, the electric field has no θ component, and hence no z component; the turnstile radiation in the horizontal plane is horizontally polarized. In the vertical direction, $\theta = 0°$ or $180°$, the radiation is circularly polarized. For intermediate angles θ the radiation is elliptically polarized.

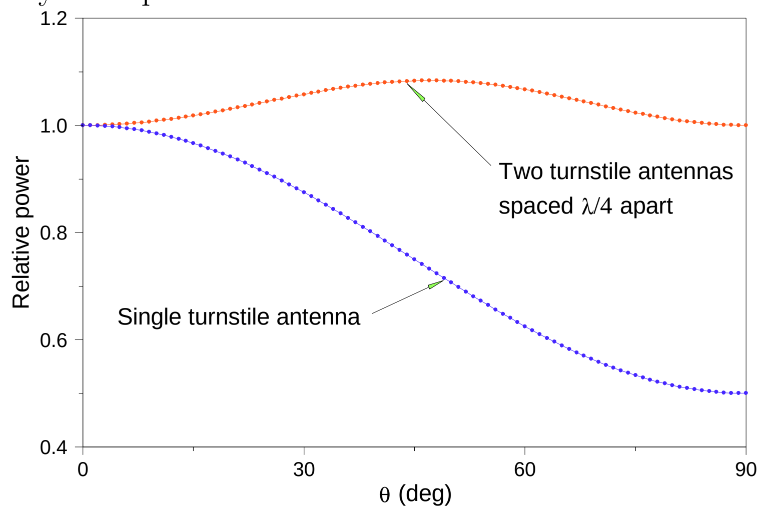
The magnitudes of the fields are

$$E_0 = B_0 = \frac{p_0 k^2}{r}\sqrt{1+\cos^2\theta}, \tag{65}$$

so the time-averaged radiation pattern is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{8\pi}B_0^2 = \frac{p_0^2\omega^4}{8\pi c^3}(1+\cos^2\theta). \tag{66}$$

The intensity of the radiation varies by a factor of 2 over the sphere, that is, by 3 dB, as shown below. Compared to other simple antennas, this pattern is remarkably isotropic.



(b) But we can make the pattern even more isotropic by considering a vertical stack of turnstile antennas.

If the center of the turnstile antenna had been at height z along the z-axis, the only difference in the resulting electric and magnetic fields would be a phase change by $kz\cos\theta$ because the path length to the distant observer differs by $z\cos\theta$. That is, the fields (62)-(64) would simply be multiplied by the phase factor $e^{-ikz\cos\theta}$.

Thus, if we have two turnstile antennas, one whose center is at the origin, and the other whose center is at height z, and we operated them in phase, the fields (62)-(64) would be multiplied by

$$1 + e^{-ikz\cos\theta}. \tag{67}$$

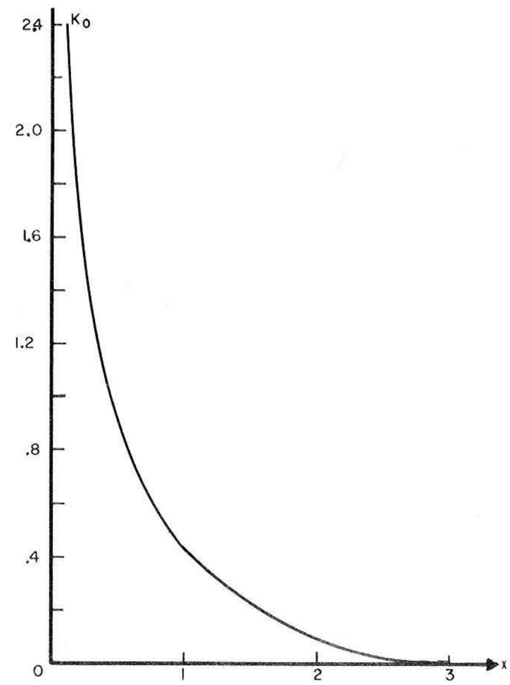
The time-averaged radiated power would therefore by eq. (66) multiplied by the absolute square of eq. (67):

$$\left\langle \frac{dP}{d\Omega} \right\rangle = 2\frac{p_0^2\omega^4}{8\pi c^3}(1+\cos^2\theta)[1+\cos(kz\cos\theta)]. \tag{68}$$

For example, suppose $kz = \pi/2$, *i.e.*, the vertical separation of the two antennas is 1/4 of a wavelength. Then, the peak of the radiation pattern is only 1.08 times

(0.35 db) greater than the minimum, above. For most practical purposes, this double turnstile antenna could be considered to be isotropic.

Saunders[1] has further shown that an infinite array of turnstile antennas yields strictly isotropic radiation provided the number $N(z)$ of such antennas in an interval dz along the vertical axis is proportional to $K_0(kz)$, the so-called modified Bessel function of order zero, whose behavior is sketched below. The antennas are all driven in phase. Since the function $K_0(kz)$ is sharply peaked at $z = 0$, we see that a properly spaced collection of turnstile antennas that extends over only ± 1 wavelength in z could produce an extremely isotropic radiation pattern.



Additional discussion of isotropic antenna is given in
http://puhep1.princeton.edu/˜mcdonald/examples/isorad.pdf

[1]W.K. Saunders, *On the Unity Gain Antenna*, in *Electromagnetic Theory and Antennas*, ed. by E.C. Jordan (Pergamon Press, New York, 1963), Vol. 2, p. 1125.