QUESTIONS AND ANSWERS

Contributions to this section, both Questions and Answers, are welcomed. Please submit four copies to the editorial office. Please include a *title* for each submission, include name and address at the end, and put references in the standard format used in the American Journal of Physics. For further suggestions, sample Questions and Answers, and requested form for both Questions and Answers, see Robert H. Romer, "Editorial: 'Questions and Answers,' a new section of the American Journal of Physics," Am. J. Phys. **62** (6) 487–489 (1994).

Questions at any level and on any appropriate AJP topic, including the "quick and curious" question, are encouraged.

Question #24. Can an electron *be* at rest?

In the spirit of this new section, we have asked this question of many people. If the electron is at rest, does this mean that its momentum is zero? But what happens in the indeterminacy principle? Is the position of the electron "everywhere?" Does "rest" mean that the expectation value for P is zero? What about an electron oscillating in an infinite potential with a definite energy state—a stationary state? Does an electron at "rest" imply a particular coordinate system? Perhaps a rest electron is "beable"[1] but not observable? Maybe classical questions can't be answered when referenced to an inherently nonclassical situation?

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[1] J. S. Bell, "Speakable and unspeakable in quantum mechanics" (Cambridge University Press, New York, 1993).

Question #25. Escape velocity from the universe

To derive the escape velocity from the earth, one first integrates the gravitational force on a test mass m over the distance from the surface of the earth to infinity. This gives the work required to escape the earth, GMm/R, where G is the gravitational constant, M is the earth's mass, and R is its radius. This work is then set equal to the kinetic energy of the test mass as it leaves the earth, $(1/2)mv^2$, and the equation is solved for v, the escape velocity. The result is this equation for escape velocity:

$$v=\sqrt{2GM/R}.$$

This equation is general to any planet, the sun, or any collection of mass that is distributed symmetrically about its center.

Assuming the universal matter to be distributed symmetrically about its center, I used this equation to calculate the escape velocity from the universe. This is a simplistic calculation in that it treats space as Euclidean and disregards relativity. Nevertheless, it seems surprising that the result should be the speed of light within the accuracy of the values of the variables. Taking, for example, the density of the universe to be 10^{-29} gm/cm^3 and the radius to be 15 billion light years, the calculated escape velocity is 3.4×10^{10} cm/s.

Is this a coincidence, or does the speed of light actually have some connection with the escape velocity from the universe?

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Question #26. Electromagnetic field momentum

What *is* the correct expression for electromagnetic field momentum ($\mathbf{P}_f$), especially when *static* fields are involved? I was brought up to believe that one should attribute to the field, whether static or dynamic, a linear momentum (in SI units[1]) of

$$\mathbf{P}_f=\epsilon_0\int \mathbf{E}\times\mathbf{B}\,dv \tag{1}$$

and an energy of

$$U_f=\frac{1}{2}\int (\epsilon_0E^2+B^2/\mu_0)dv. \tag{2}$$

This point of view has been forcefully presented over the years by many people (with no apparent qualms), in this journal by Romer,[2] Pugh and Pugh,[3] and many others, in many textbooks that I admire,[4] and very recently, again in this journal, by the present editor.[5]

Yet every now and then I have an uneasy feeling, as I recall some statements labeling these expressions as erroneous, especially those by Rohrlich,[6] who observes that Eqs. (1) and (2) do not have the correct transformation properties to form a four-vector and unambiguously states that Eqs. (1) and (2) are incorrect in the case of "bound" fields, fields that are not tied to their sources, fields other than radiation fields. I want to emphasize that I am not trying to raise the delicate question of the correct expressions to use in the presence of "material media" (dielectrics, paramagnetic materials, etc.). I am likewise not interested here in the "localization" of energy, momentum, or angular momentum; let us just focus on the integrals over all space. And, as I will explain below, I do not wish to challenge the existence of "hidden momentum," a concept which is important in examples like the one to be discussed but is not the focus of this Question.

To be very specific, let me concentrate on the linear field momentum of the arrangement described in Ref. 5. Consider a shell of charge (radius, a), with a surface density of charge[7] proportional to $\cos\theta$, with a resulting electric dipole moment, $\mathbf{p}$; this gives a uniform $\mathbf{E}$ field in its interior and a pure dipole field for $r>a$. Consider also a shell of radius b, with a surface current density proportional to $\sin\theta'$; such a shell has a magnetic dipole moment ($\mathbf{m}$) and produces a uniform $\mathbf{B}$ for $r<b$ and a pure dipole field for $r>b$. Suppose $b>a$ (though the $b<a$ case is also easy to do), and let $\mathbf{m}$ and $\mathbf{p}$ be oriented perpendicular to each other. The necessary integrals[8] are straightforward, the integrands drop off so rapidly with increasing r that no one could question the mathematics, and the resulting linear momentum of this configuration of static fields is simply

$$\mathbf{P}_f=(\mu_0/4\pi b^3)\mathbf{m}\times\mathbf{p}. \tag{3}$$

Although "hidden momentum" is not the point of this Question, we may note that because the center of mass (or energy) of this apparatus is undeniably at rest, the *total* momentum (field plus mechanical) of this apparatus is zero.[9] That is, in addition to the linear momentum of the field, there must exist an equal and opposite amount of mechanical momentum, called "hidden" because it is not obvious to the naked eye that anything is moving.[10]

I believe that Eq. (3) gives the linear three-momentum of this static electromagnetic field. Am I correct in this opinion? If so, how exactly can I reconcile this belief with the apparently contrary view of Rohrlich.[6] Rohrlich (p. 17; see also Chap. 6) states quite unambiguously that Eq. (1) above for $\mathbf{P}_f$ "is valid for free fields (radiation fields) but not for bound fields." But it is precisely nonradiation fields, static fields, for which Romer, Pugh and Pugh, Griffiths, and others make a point of enthusiastically endorsing Eqs. (1) and (2), and the fields of the specific example discussed in this Question are most definitely bound to their sources. (For a brief discussion of Rohrlich's views, see a 1983 paper by Griffiths and Owen.[11]) If I am wrong, what is the flaw in the standard arguments that led me to this conclusion? And what expression should replace Eq. (3) above? Should all of us who teach and write about electromagnetism abandon the standard expressions for field momentum and energy, Eqs. (1) and (2)? Or are *all* of us (Rohrlich, Griffiths, myself, etc.) right in some sense? Do Rohrlich's expressions reduce to the familiar ones under certain conditions? If there really is a conflict, is it in some sense a matter of "taste" as to which expressions one should use? Is there some frame of reference in which my result is correct, and if so, how—in general—should I identify that frame? Are Eqs. (1) and (2) correct in some low-velocity limit, for instance, if all the moving charges that constitute the current shell are moving very slowly? Do these standard expressions give the right answer in this particular case, but not in *other* static field configurations? If so, are there simple and exactly calculable instructive examples of configurations in which Eqs. (1) and (2) give incorrect results? I find myself perplexed about a topic that at one time I thought I understood; enlightenment would be greatly appreciated.

It has not escaped my notice that this Question is substantially longer than most such items. Well, an editor should have a privilege or two, and publishing a longer-than-average Question seems a modest demand to make. Moreover, regular readers of AJP know that more often than not I forswear the use of "my" editorial page, occasionally simply for more regular material, more commonly for a Guest Comment. This month there is neither editorial nor Guest Comment; let this lengthy Question serve the purpose. At any rate, I here exercise my editorial prerogative to reveal my confusion on this topic, in the hopes that a few others may also find the physics nontrivial, and especially in the hope that some of the learned readers of AJP will convincingly satisfy my curiosity on this score.

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[1] The true nature of $\mathbf{P}_f$ is more clearly exhibited, in my view, in Gaussian units, where c appears explicitly ($\mathbf{P}_f=(1/4\pi c)\int\mathbf{E}\times\mathbf{B}\,dv=(1/c^2)\int\mathbf{S}\,dv$, where $\mathbf{S}$ is the Poynting vector, $(c/4\pi)\mathbf{E}\times\mathbf{B}$), reminding us that in textbook examples like the one discussed in this Question, the numerical value of $\mathbf{P}_f$ is likely to be extremely small.

[2] R. H. Romer, "Angular Momentum of Static Electromagnetic Fields," Am. J. Phys. **34** (9), 772–778 (1966); R. H. Romer, "Electromagnetic Angular Momentum," Am. J. Phys. **35** (5), 445–446 (1967).

[3] Emerson M. Pugh and George E. Pugh, "Physical Significance of the Poynting Vector in Static Fields," Am. J. Phys. **35** (2), 153–156 (1967).

[4] See, for example, David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1989), 2nd ed., Chap. 7; Mark A. Heald and Jerry B. Marion, *Classical Electromagnetic Radiation* (Saunders, Philadelphia, 1995), 3rd ed., Chap. 4.

[5] Robert H. Romer, "Editorial: 'Questions and Answers,' a new section of the American Journal of Physics," Am. J. Phys. **62**, (6), 487–489 (1994); see especially the "Answer" to sample Question #1956.

[6] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Massachusetts, 1965). [For a very recent discussion, see Patrick Moylan, "An Elementary Account of the Factor of 4/3 in the Electromagnetic Mass," Am. J. Phys. **63** (8), 818-820 (1995)].

[7] I assume that there are no sources of **E** other than the charge shell of radius a. This requires, for instance, that the current shell should not consist of charges flowing within a *conducting* tube, a tube on which shielding charges would be induced, making the total **E** different from the simple field due to the shell of charge. See also the discussion in Refs. 8 and 9 below.

[8] In the case of static fields with sources restricted to a finite region, an alternative expression can be derived from Eq. (1), $\mathbf{P}_f=(1/c^2)\int\phi\mathbf{J}\,dv$, where $\mathbf{E}=-\text{grad}\,\phi$. See M. G. Calkin, "Linear Momentum of the Source of a Static Electromagnetic Field," Am. J. Phys. **39** (5), 513–516 (1971); see also W. H. Furry, "Examples of Momentum Distributions in the Electromagnetic Field and in Matter," Am. J. Phys. **37** (6), 621–636 (1969) and Lev Vaidman, "Torque and force on a magnetic dipole,' Am. J. Phys. **58** (10), 978–983 (1990).

[9] See Sidney Coleman and J. H. Van Vleck, "Origin of 'Hidden Momentum Forces' on Magnets," Phys. Rev. **171** (5), 1370–1375 (1968) and W. H. Furry (Ref. 8) and references therein, especially to papers by W. Shockley & R. P. James. For some recent discussion, see, for example, Lev Vaidman, Ref. 8.

[10] The configuration discussed in this Question is one of a number of examples that the late Philip C. Peters and I concocted in 1981, in which we could calculate $\mathbf{P}_f$ exactly and in which we could also calculate the impulse delivered to the apparatus as we imagined the **E** or **B** field being slowly turned off or on. I actually embedded a small ceramic magnet and a battery (just to create a static **E**, not to deliver any *current*) in a jar of plastic and labeled it: "WARNING! MOMENTUM! $P\cong10^{-18}$ kg·m/s $\cong$1000 MeV/c," with an arrow indicating the approximate direction of $\mathbf{P}_f$. (I sent one such jar to Peters, who at that time I only knew by correspondence, as a Christmas present; in his thank you note, he wrote that when he opened the package, he found the momentum leaking out of the jar but that he had stuffed it back in by wrapping it with aluminum foil. He also pointed out that 1000 MeV/c is the momentum in about 100 cm^3 of sunlight.) We thought for some time that if the temperature of the room were to increase a bit, decreasing the magnet's magnetization, the jar would (in principle!) begin to slide across the shelf to conserve total momentum. Fortunately, before writing a paper about these examples and submitting it to this journal, we reread the papers by Calkin, Furry, and Coleman and Van Vleck cited above and realized that the total momentum of the stationary jar (field plus mechanical) had to be zero and that therefore there must be hidden mechanical momentum in addition to the $\mathbf{P}_f$ we had calculated. As Calkin[8] writes: "Ones's intuition would probably say that the net linear momentum is zero, and this is indeed correct." I still keep my own jar on my desk, as a reminder of the wisdom of thinking carefully about an astonishing result before sending it off to a journal. Depending on the model chosen for the current shell, the hidden momentum may reside in mechanical stresses in the structure or in the mechanical momentum of the charge carriers. Suppose that the currents are due to charge carriers (of both signs, moving in opposite directions) constrained to move within a nonconducting tube, and thus exposed to the static field, **E**, arising from the charged shell. As the charges coast around their circular paths, they gain and lose speed as their electrical potential energies vary. If mechanical momentum were simply given by $m\mathbf{v}$, this would not lead to any net mechanical momentum, for at the "bottom of the hill," where the carriers travel more quickly, they are less closely bunched, and the two effects

(faster speed and greater spacing), would cancel one another out. But with the relativistic expression for momentum, γmv, these two effects do not quite cancel, and one ends up with a net mechanical momentum from the charge carriers. For details see, for instance, Calkin and Furry (Ref. 8, above).

[11] David J. Griffiths and Russell E. Owen, "Mass renormalization in classical electrodynamics," Am. J. Phys. **51** (12), 1120–1126 (1983).

Answer to Question #1. ["How does a Brownian particle at rest get kicked up to kT?," Frank Munley, Am. J. Phys. 62(6), 871 (1994)]

The apparent paradox of why viscosity helps to speed up stationary Brownian particles to a thermal energy of about kT rests on an extrapolation to microscopic systems of our intuition on damping mechanisms obtained by working with macroscopic objects. Viscosity is the parameter that controls damping in a fluid and, as in the case of a macroscopic damped harmonic oscillator, we are familiar with the fact that the larger the damping parameter the smaller the time a macroscopic moving object takes to come to rest. We therefore tend to think of damping as a process that always robs energy from a system: the larger the damping parameter, the faster the energy is lost. However, this is only true if thermal energies (about kT) are negligible with respect to the energy of the object, a normal situation for the macroscopic systems from which our intuitions are derived. In fact, the damping parameter represents the strength of the coupling between the system and a thermal reservoir and, in particular, controls the speed at which energy is transferred between the two, in either direction. If a microscopic system starts with less energy than the required thermal equilibrium energy, the reservoir will provide that energy. As in the more familiar cases, this transfer of energy will be faster for large damping parameters, although the flow of energy will go this time in the opposite direction, from the reservoir to the system. This situation is very common, for example, in damped quantum harmonic oscillators,[1,2] such as in the vibrational modes of molecules or in the modes of the radiation field inside a lossy cavity. Therefore the result derived from Chandrasekhar's equation is not surprising. Small viscosity means a weak coupling with the thermal reservoir and in turn large relaxation times, including from rest to kT.

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[1] W. H. Louisell, *Quantum Properties of Radiation* (John Wiley & Sons, New York, 1973), pp. 331–404.

[2] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (John Wiley & Sons, New York, 1992), pp. 322–333.

Answer to Question #15. ["What space scales participate in cosmic expansion?," Frank Munley, Am. J. Phys. 63(4), 297 (1995)]

In the April issue of the AJP Frank Munley asked if the cosmological expansion applies to all length scales. If so, it would appear the expansion could not be measured, because the number of meter sticks between Earth and any galaxy would be constant. Although I will offer some caveats below, the short answer that, I believe, gets at the gist of Professor Munley's concern is that there is no scale on which space stops expanding, that the space parameters of atomic physics are conceptually the same as those used by cosmologists and that the number of meter sticks that will fit between Earth and a galaxy is constant. But that does not mean the cosmic expansion cannot be measured; astrophysicists see evidence of it when they observe the cosmological red shift.

Cosmologists describe the geometry of spacetime by specifying the "distance squared" between two closely separated events. In the flat Robertson–Walker cosmology this "distance squared" is

$$ds^2 = -dt^2 + R(t)(dx^2 + dy^2 + dz^2);$$

the quotes reflect the fact that the line element given above may be negative. There are theoretical ways to calculate $R(t)$ given an appropriate description of the matter comprising the universe. For example, if the universe is filled with pressureless dust, $R(t) \propto t^{2/3}$. For a universe filled with radiation, $R(t) \propto t^{1/2}$. Finally, for a universe in the false vacuum state considered by the original inflationary cosmology $R(t)$ increases exponentially. In all these cases the scale factor $R(t)$ increases monotonically with time. When cosmologists say "space expands," they are referring to this increase in the scale factor. The reason is that for a body at fixed coordinate distance D, the geometrically significant proper distance, $d = R(t)D$, increases. When they say "galaxies don't move," they mean that remaining at fixed coordinates is a solution to the dynamical equations of relativity.

Theoretical calculations of $R(t)$ for the current "matter" (i.e., pressureless dust) dominated universe show that the scale factor increases with time. In addition, the measured cosmological red shift empirically demonstrates that $R(t)$ is increasing. A nice discussion is presented in Steven Weinberg's *Gravitation and Cosmology*[1] for example. The key point is that light follows a spacetime trajectory such that $ds^2 = 0$. If a galaxy is at a coordinate distance D from us and sends out a crest of light at time t_s, we will receive it at a time t_r given by

$$\int_{t_s}^{t_r} dt/R(t) = D.$$

If a second crest is sent out at time $t_s + dt_s$ (i.e., just a little later than the first crest was sent) and received at time $t_r + dt_r$ then, as above,

$$\int_{t_s+dt_s}^{t_r+dt_r} dt/R(t) = D.$$

If $R(t)$ does not change much over the period of a light wave (as is the case), one may take the difference of the above two equations to find

$$dt_s/R(t_s) = dt_r/R(t_r).$$

Since the period of light oscillation is proportional to the wavelength,

$$\lambda_r/\lambda_s = R(t_r)/R(t_s).$$

The observed red shift of light received from distant galaxies tells us that the scale factor R is increasing with time.

Now for the caveats. The Robertson–Walker distance squared presented above is derived under the assumption that the universe is homogeneous and isotropic. At terrestrial scales it is clear that this is not so, but cosmologists have traditionally assumed that at sufficiently large scales the as-

Answer to Question #26 ["Electromagnetic field momentum," Robert H. Romer, Am. J. Phys. 63(9), 777–779 (1995)]

(1) The difficulty in interpreting Poynting's vector as proportional to momentum for a system that includes sources as well as fields was first pointed out by Poincaré in 1906. A relativistically consistent formalism can only be achieved by adding terms that include stresses in the sources that arise when the fields are generated.

The usual relativistic argument begins by recasting the Lorentz-force 4-vector,

$$f_\mu = F_{\mu\nu} j^\nu = \left(\frac{\mathbf{j}\cdot\mathbf{E}}{c}, \rho\mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B} \right)$$

(in cgs units and with metric $\eta_{00}=1$, $\eta_{11}=\eta_{22}=\eta_{33}=-1$; Greek indices run over 0,1,2,3, while Latin indices run over 1,2,3) as the derivative of a stress tensor:

$$f_\mu = -\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\partial^\nu T_{\mu\nu}.$$

This leads to the result

$$T_{\mu\nu} = \frac{1}{4\pi} F_{\mu\alpha} F^\alpha_\nu + \frac{1}{16\pi} \eta_{\mu\nu} F^\beta_\alpha F^\beta_\alpha$$

$$= \begin{pmatrix} \dfrac{E^2+B^2}{8\pi} & \dfrac{\mathbf{E}\times\mathbf{B}}{4\pi} = \dfrac{\mathbf{S}}{c} \\ \dfrac{\mathbf{E}\times\mathbf{B}}{4\pi} & -\dfrac{1}{4\pi}(E_iE_j + B_iB_j - \delta_{ij}) \dfrac{E^2+B^2}{2} \end{pmatrix}.$$

Next, one makes a trial definition of an energy-momentum 4-vector for the fields as

$$P_\mu = \int T_{0\mu}\, dvol,$$

so that

$$P_0 = \int T_{00}\, dvol = \frac{1}{8\pi} \int (E^2+B^2) dvol = U_f,$$

$$P_i = \int T_{0i}\, dvol = \frac{1}{4\pi} \int \mathbf{E}\times\mathbf{B}\, dvol = c\mathbf{P}_f,$$

where

$$\mathbf{P}_f = \frac{1}{4\pi c} \int \mathbf{E}\times\mathbf{B}\, dvol$$

is the field 3-momentum that is the subject of Question #26. Then, $P_\mu = (U, c\mathbf{P}_f)$ has the appearance of a familiar 4-vector.

(2) If there are no sources present (free-field case), then the Lorentz-force 4-vector vanishes, the 4-divergence of $T_{\mu\nu}$ vanishes also, and one can verify that P_μ really transforms like a 4-vector.

The argument thus far is seconded in the books of Rohrlich and of Jackson, who do not advocate carrying it further.

(3) Poincaré suggests we proceed to the case where sources of the fields are present. By direct application of a Lorentz transformation to the stress tensor $T^\star_{\mu\nu}$, where the ⋆ indicates the rest frame of the sources, one deduces that P_μ fails to transform like a 4-vector if there are nonzero spatial components to the stress tensor, i.e., if some $\int T^\star_{ij} \neq 0$.

Poincaré noted that if some $\int T^\star_{ij}$ are nonzero, then the system of sources is not in mechanical equilibrium until mechanical stresses $\int P^\star_{ij} = -\int T^\star_{ij}$ are developed to counter the electromagnetic stresses. The $P^\star_{ij}$ can be embedded in a 4-tensor $P_{\mu\nu}$ that includes the mechanical rest energy $m_{\text{mech}}c^2 = \int P^\star_{00}$ and the mechanical momentum $c\mathbf{P}_{\text{mech}} = \int P^\star_{0i} = \int P^\star_{i0}$. Then, when one defines

$$P_\mu = \int (T_{0\mu} + P_{0\mu}) dvol,$$

one has a true 4-vector, with

$$P_0 = U + m_{\text{mech}}c^2, \quad P_i = c(\mathbf{P}_f + \mathbf{P}_{\text{mech}}).$$

This formalism does not quite succeed in providing an independent interpretation of the "field momentum" $\mathbf{P}_f$ when sources are present. That is, only the sum $\mathbf{P}_f + \mathbf{P}_{\text{mech}}$ has a dynamical meaning, where $\mathbf{P}_{\text{mech}}$ includes a contribution associated with the mechanical stresses that arise in response to electromagnetic forces.

(4) There remains the specific topic of Question #26: What interpretation should be given when $\mathbf{P}^\star_f \neq 0$ in the "rest frame" of the sources? In view of the difficulty of giving any independent meaning to $\mathbf{P}_f$ when sources are present, this issue is secondary.

It is not very satisfactory to note that one can always find a frame in which $\mathbf{P}_f$ vanishes, since, in general, the center of mass of the sources will be moving in this frame.

Instead, we advocate a fairly trivial solution to the problem. Simply regard the value $\mathbf{P}^\star_f$ as a constant of the system without an interpretation of anything being in motion. This is a consistent view because the dynamical significance of momentum is in its derivative,

$$f_\mu = \frac{dP_\mu}{d\tau},$$

where τ is the proper time, and in conservation laws, both of which are unaffected by an additive constant. In this sense no dynamical meaning can be assigned to the value of $\mathbf{P}^\star_f$, and one can consistently choose not to give it any further interpretation.

We can amplify this point by recalling the Lorentz transformation of the 4-momentum $P_\mu = (U_f, c\mathbf{P}_f)$ in a boost by $\boldsymbol{\beta} = \mathbf{v}/c$ from the rest (⋆) frame:

$$\mathbf{P}_f = \gamma \left(\mathbf{P}^\star_f + \frac{U^\star_f}{c^2} \mathbf{v} \right),$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$. Thus, in a frame where the system moves with the velocity $\mathbf{v}$, the part of the momentum that is proportional to velocity depends on the effective mass $U^\star_f/c^2$ in the rest frame and not on the momentum $\mathbf{P}^\star_f$ in the rest frame. A nonzero value of $\mathbf{P}^\star_f$ in the rest frame has no dynamical effect on the momentum.

We have gotten used to electrons and photons having spin without being able to identify anything that rotates. So I propose that we not worry too much about a nonzero static value for the "field momentum" that has no dynamical consequence. Foregoing any interpretation of $\mathbf{P}^\star_f$ is even easier than for electron spin since that latter has dynamical significance.

(5) I append a further argument (which perhaps has an error) that shows how the "field momentum" $\mathbf{P}_f$ by itself

does not consistently behave like a nonrelativistic momentum, whether or not its value in the rest frame of the sources is zero.

We consider a system that, when at rest, produces fields $\mathbf{E}_0$ and $\mathbf{B}_0$. The corresponding "field momentum" $\mathbf{P}_0$ may or may not be zero, but, in any case, is a constant vector. Only the velocity-dependent part of the "field momentum" will have relevance to $\mathbf{F}=d\mathbf{P}/dt$.

Next, consider the system when it is moving with center-of-mass velocity $\mathbf{v}$, where $v \ll c$. We suppose that there is no change in the state of the system relative to its center of mass, so fields $\mathbf{E}_0$ and $\mathbf{B}_0$ still hold in the rest frame of the system. Then the nonrelativistic limit of the transformation of the electromagnetic field tells us that

$$\mathbf{E}=\mathbf{E}_0-\frac{\mathbf{v}}{c}\times\mathbf{B}_0, \quad \mathbf{B}=\mathbf{B}_0+\frac{\mathbf{v}}{c}\times\mathbf{E}_0,$$

and so the "field momentum" associated with the moving system is

$$\mathbf{P}_f=\mathbf{P}_0+\frac{1}{4\pi c^2}\int [(E_0^2+B_0^2)\mathbf{v}+(\mathbf{E}_0\cdot\mathbf{v})\mathbf{E}_0 + (\mathbf{B}_0\cdot\mathbf{v})\mathbf{B}_0]dvol,$$

neglecting a term in $(v/c)^2$. The rate of change of this momentum is

$$\frac{d\mathbf{P}_f}{dt}=\frac{2U_0}{c^2}\mathbf{a}+\frac{1}{2\pi c^2}\int (\mathbf{E}_0\cdot\mathbf{a})\mathbf{E}_0+(\mathbf{B}_0\cdot\mathbf{a})\mathbf{B}_0]dvol,$$

where $\mathbf{a}=d\mathbf{v}/dt$ is the acceleration of the system and U_0 is the rest-frame field energy:

$$U_0=\frac{1}{8\pi}\int (E_0^2+B_0^2)dvol.$$

While, as expected, the constant value P_0 does not appear in the expression for the rate of change of "field momentum," this expression does not quite have the desired form, $m_{\text{eff}}\mathbf{a}$. I infer that this is another demonstration of the view of Poincaré that the "field momentum" $\mathbf{P}_f$ cannot be interpreted by itself when sources are present.

(6) Regarding the specific example of nested electric and magnetic dipoles, it is easy to see that the diagonal elements of the electromagnetic stress tensor, T_{ii}, are nonvanishing. The sphere of charge and sphere of current-carrying coils would fly apart without some kind of glue. The resulting mechanical stresses change the rest mass of the system and, when it is in motion, its momentum by an amount comparable to the electromagnetic "mass" and momentum contributions. Trying to interpret the electromagnetic momentum without considering the corresponding stress-induced changes in the mechanical momentum is counterproductive.

But the bottom line is that no meaningful interpretation can be given to the nonzero $\mathbf{P}_f^*$ for that system in its rest frame.

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Answer to Question #26 ["Electromagnetic field momentum," Robert H. Romer, Am. J. Phys. 63(9), 777–779 (1995)]

Here are two well-known facts from special relativity: (1) At the location of sources the electromagnetic energy tensor T_{elm} is not divergence-free; (2) the necessary and sufficient condition for the hyperplane integral of a symmetric tensor of second rank T to be independent of the orientation of that hyperplane is that T be divergence-free; and if that integral is orientation-independent, then it is a four-vector. Fact (2) is sometimes called "von Laue's theorem."

From these two facts it follows that the electromagnetic four-momentum when defined as the hyperplane integral of T_{elm} [see, e.g., CCP[1] (Eq. 6–18)] is not a four-vector if there are sources present. This result is physically obvious because the presence of sources implies that the system is not closed unless other (nonelectromagnetic) forces are included. In the particular system considered in Question #26 this is especially evident: When considered in its *rest* frame, the electromagnetic interactions by themselves yield a nonvanishing momentum which must be held in equilibrium by other interactions. (The realization of the system involves current carrying wires, etc.) But when the system is closed, the total energy tensor will be divergence-free and by (2) the total four-momentum will be a four-vector. A well-known special case of this is the extended charged particle with Poincaré stresses.

It would seem that one must deal with a *closed* system if one wants an energy-momentum *four-vector*. However, there is a way around it: Independence of the plane orientation is a sufficient but not a necessary condition for the integral to be a four-vector. One can define the electromagnetic energy-momentum P_{elm} for a system in uniform motion as the Lorentz boosted rest frame [see CCP (Eq. 6–24)]. That integral is not independent of the orientation of the hyperplane: the plane is specified to be the plane in which the system is at rest. So defined, P_{elm} is a four-vector even though there are sources present! In the rest frame, it is just the conventional definition as given in Question #26. Romer's equation (3) is therefore correct and consistent with (Eq. 6–24).

The advantage of this definition lies in its separating the electromagnetic fields from the rest of the system in a covariant way: $P_{\text{total}}=P_{\text{elm}}+P_{\text{other}}$. These are all four-vectors.

Finally, one should note that radiation fields form a T_{elm} that is always divergence-free, so that all the above is of interest only for "bound" fields.

The answer I have just given is contained in abbreviated form in the paper by Griffiths and Owen.[2] But since the issue has been raised again, I have presented it in a more general and explicit form.

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[1] CCP refers to my book *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965 and 1990).

[2] David J. Griffiths and Russell E. Owen, " Mass renormalization in classical electrodynamics," Am. J. Phys. **51** (12), 1120–1126,(1983).

Answer to Question #26 ["Electromagnetic field momentum," Robert H. Romer, Am. J. Phys. 63(9), 777–779 (1995)]

The answer to this question lies in what is sometimes called "Abraham's theorem," which states roughly that provided one has a symmetric stress-energy tensor $T_{\alpha\beta}$ which is conserved—$\partial^\alpha T_{\alpha\beta}=0$—then one can define quantities

$$E=\int d_x^3 T^{00}, \quad P^i=\int d_x^3 T^{0i} \tag{1}$$

which are time independent

$$\frac{dE}{dt}=0, \quad \frac{d\mathbf{P}}{dt}=0 \tag{2}$$

and which transform as a four-vector under a Lorentz transformation; i.e.,

$$P^\mu \rightarrow \Lambda^\mu{}_\nu P^\nu \text{ when } x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu. \tag{3}$$

The structure of the conserved symmetric energy-momentum tensor is well known and is given by[1]

$$T_{\alpha\beta}=T_{\alpha\beta}^{\text{field}}+T_{\alpha\beta}^{\text{particle}}, \tag{4}$$

where

$$T_{\alpha\beta}^{\text{field}}=\epsilon_0\left(F_{\alpha\gamma}F^{\gamma}{}_{\beta}+\frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right) \tag{5}$$

is the Maxwell stress tensor associated with the electromagnetic field and

$$\begin{aligned} T_{\alpha\beta}^{\text{particle}} &= \sum_n p_{n\alpha}\frac{dx_{n\beta}}{dt}\,\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \\ &= \sum_n \int_{-\infty}^{\infty} d\tau\, p_{n\alpha}\frac{dx_{n\beta}}{d\tau}\,\delta^4(x^\mu - x^\mu(\tau)) \end{aligned} \tag{6}$$

is the stress tensor associated with the charged particles which are present. The field components of the energy and momentum are given by

$$E_{\text{field}}=\frac{\epsilon_0}{2}\int d^3x(E^2+c^2B^2), \quad \mathbf{P}_{\text{field}}=\epsilon_0\int d^3x\,\mathbf{E}\times\mathbf{B}, \tag{7}$$

but it is only the *total* energy-momentum tensor—field *plus* particles—which is conserved and is therefore subject to Abraham's theorem. One can certainly consider the expressions given in Eq. (7) as the energy-momentum carried by the electromagnetic field, but they are *not* constant in time, nor will expressions for these quantities as calculated by observers in different inertial frames be related by a simple Lorentz transformation. When augmented by corresponding contributions from the charged particle sector, the *total* energy and momentum *will* be conserved and corresponding expressions obtained by different inertial observers *will* be related by a Lorentz transformation. However, different observers will differ as to the way in which the total is divided into particle and field content.

In the example cited by Professor Romer, involving static external dipole electric and magnetic fields, the fields are generated by a shell distribution of electric charge and current, respectively, and the energy-momentum associated with these moving and static charges must be added to the corresponding field quantities in order to have an overall energy-momentum which is both conserved and which transforms in the proper fashion under a Lorentz transformation. This is always true in the case of a static field configuration—the charge and current distributions which generate such configurations are of necessity nearby and need to be taken into account since they interact with the fields which they generate. In addition I do not believe that it is possible to maintain a static field configuration without the existence of some sort of external constraint, which then must itself be included into the energy-momentum balance. In the shell distribution example, one must counter the forces on the currents due to electric fields of the charges. Simpler examples are that of a capacitor, which has a constant electric field but whose plates would move toward each other except for some force which maintains them at constant separation, or a current carrying wire, which generates a static magnetic dipole field (at large distance) but which would be required to change its shape by magnetic forces except for stresses within the wire which hold it together. An example more relevant to the question at hand is that of a stationary charge q which is situated at position $\mathbf{d}$ with respect to a stationary magnetic dipole $\mathbf{m}$. Imagine that the dipole is a negative charge in rotation about a positive charge—say a hydrogen atom. It is easy to show that in this case to leading order there exists, in general, a nonzero field momentum $\mathbf{P}_{\text{field}}$. Also, in leading order the dipole magnetic field does not affect the stationary charge nor does the electric field of the charge affect the dipole. However, a more careful look reveals that there are additional effects at work. For example, the presence of the charge q induces an electric dipole moment of the atom proportional to the electric polarizability and this dipole electric field in turn acts back on the charge q. The result is an attractive $1/r^4$ interaction between atom and charge, which means that the system can remain stationary only in the presence of an external constraining force. I believe that this is a general result—without such a constraint there cannot exist a truly static configuration of fields.

On the other hand, when one considers electromagnetic radiation the (accelerating) charges which originally produced the radiation are, in general, no longer in the vicinity and can be considered to be isolated from the radiation pulse. In this case the interaction between the radiation and the charged particles which produced that radiation can be safely neglected and the concept of an *independent* field energy and momentum makes sense, as stated by Rohrlich.[2] But if there are no charges present, what keeps the field configuration going? The answer is, of course, given by the Maxwell equations but is described vividly by Feynman:[3] "Suppose the magnetic field were to disappear. There would be a changing magnetic field which would produce an electric field. If this electric field were to go away, the changing electric field would create a magnetic field back again... They maintain themselves in a kind of a dance—one making the other, the second making the first—propagating through space."

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[1] See J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Sec. 12.10.

[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965).

[3] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II. Chap. 18.4.