

THUS IN A GENERAL FRAME

$$U = \gamma \int (T_{00}^{*} + P_{00}^{*}) dvol^{*} \quad c\bar{P} = \gamma \beta \int (T_{00}^{*} + P_{00}^{*}) dvol^{*}$$

WHICH IS A PROPER 4-VECTOR TRANSFORMATION AT LAST.

THE MASS ASSOCIATED WITH THESE FIELDS IS  $Mc^2 = U^{*} = \int (T_{00}^{*} + P_{00}^{*}) dvol^{*}$   
 BUT IT IS NOT PURELY ELECTROMAGNETIC!

THE RADIATION REACTION (SEE SEC 4, VOL II OF BECKER)

SO FAR WE HAVE CONSIDERED THE TROUBLES WITH THE CONCEPT OF A POINT CHARGE AT REST OR IN UNIFORM MOTION. NOW WE LOOK AT THE QUESTION OF AN ACCELERATING POINT CHARGE.

THE ELECTROSTATIC SELF ENERGY OF A "POINT" CHARGE IS DIVERGENT ( $e^2/a \rightarrow \infty$  AS  $a \rightarrow 0$ ), SO WE EXPLORE THE POSSIBILITY THAT A "POINT" CHARGE HAS A SMALL BUT FINITE RADIUS  $a$ .

WHEN SUCH A CHARGE IS ACCELERATING, THE ELECTROMAGNETIC INTERACTION AMONG VARIOUS PORTIONS OF THE CHARGE RESULTS IN A SELF FORCE, FIRST ANALYZED BY LORENTZ, AND IN GREATER DETAIL BY ABRAMAM (1895-1905).

SUCH ANALYSES REMAIN CONTROVERSIAL. WE PRESENT ONLY SIMPLIFIED ARGUMENTS THAT ILLUSTRATE THE MAIN FEATURES.

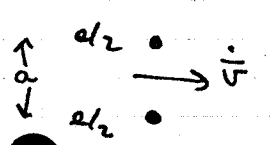
FOR A CHARGE THAT IS ACCELERATING, BUT INSTANTANEOUSLY AT REST, THE SELF FORCE IS

$$\bar{F}_{\text{SELF}} = \int \rho \bar{E}_{\text{SELF}} dvol.$$

FOR A SYMMETRICAL CHARGE DISTRIBUTION, THE ELECTROSTATIC PART OF  $\bar{E}$  DOES NOT CONTRIBUTE. BUT FROM THE VECTOR POTENTIAL,

$$\bar{E} = -\frac{1}{c} \frac{\partial \bar{A}}{\partial t} = -\frac{1}{c^2} \int \frac{[\dot{j}]}{r} dvol \rightarrow -\frac{e}{c^2 r} [\dot{v}] \quad \text{FOR A SINGLE CHARGE } e. \quad (\text{RECALL P. 178.})$$

A USEFUL MODEL OF CHARGE  $e$  IS A PAIR OF CHARGES  $e/2$ , SEPARATED BY DISTANCE  $a$ ,  $\perp$  TO  $\dot{v}$ .



THEN 
$$\bar{F}_{\text{SELF}} = 2 \frac{e}{2} \left( -\frac{e}{2c^2 a} [\dot{v}] \right) = -\frac{e^2}{2c^2 a} \dot{v} \left( t - \frac{a}{c} \right) \approx -\frac{e^2}{2c^2 a} \left( \dot{v} - \frac{a}{c} \ddot{v} \right)$$

$$\approx -\frac{e^2}{2c^2 a} \dot{v} + \frac{e^2}{2c^2} \ddot{v}$$

THE EQUATION OF MOTION IS

$$m \dot{v} = \bar{F}_{\text{EXT}} + \bar{F}_{\text{SELF}} \Rightarrow \left( m + \frac{e^2}{2c^2 a} \right) \dot{v} = \bar{F}_{\text{EXT}} + \frac{e^2}{2c^2} \ddot{v}$$

PH 206 LECTURE 22

THE DIVERGENT TERM,  $\frac{e^2}{2c^2 a} \approx \frac{U_{\text{ELECTROSTATIC}}}{c^2} \equiv \text{ELECTROMAGNETIC MASS}$ , IS CONVENTIONALLY "RENORMALIZED" INTO THE TOTAL MASS.

THE SECOND NEW TERM,  $\frac{e^2 \ddot{v}}{2c^2}$ , WAS INTERPRETED BY LORENTZ AS A KIND OF REACTION TO THE RADIATION OF THE ACCELERATED CHARGE.

IF AN EXTERNAL FORCE IS CAUSING THE CHARGE TO ACCELERATE, THIS FORCE MUST PROVIDE FOR THE RADIATED ENERGY OF THE FIELDS, AS WELL AS THE KINETIC ENERGY OF THE CHARGE. WE MAY ACCOMMODATE THIS BY SUPPOSING, ACCORDING TO NEWTON'S 3RD LAW, THAT THE RADIATION EXERTS A REACTION FORCE BACK ON THE CHARGE.

THIS REACTION FORCE ABSORBS ENERGY AT THE RATE

$$\vec{F}_R \cdot \vec{v} = - \frac{dU_{\text{RADIATION}}}{dt} = - \frac{2}{3} \frac{e^2 \dot{v}^2}{c^3} \quad (v \ll c)$$

ACCORDING TO LARMOR.

LORENTZ NOTED THAT IF THE ACCELERATION LASTS FROM  $t_1$  TO  $t_2$ , WE CAN WRITE

$$\begin{aligned} \int_{t_1}^{t_2} \vec{F}_R \cdot \vec{v} dt &= - \frac{2}{3} \frac{e^2}{c^3} \int_{t_1}^{t_2} \dot{v} \cdot \dot{v} dt \\ &= - \frac{2}{3} \frac{e^2}{c^3} \left[ \vec{v} \cdot \dot{v} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \vec{v} \cdot \ddot{v} dt \right] \end{aligned}$$

IF  $t_2 - t_1$  IS FINITE, OR IF THE MOTION IS OSCILLATORY, WE MAY SAFELY NEGLECT THE FIRST TERM.

$$\int_{t_1}^{t_2} \left( \vec{F}_R - \frac{2e^2 \ddot{v}}{3c^3} \right) \cdot \vec{v} dt = 0$$

HENCE WE IDENTIFY  $\vec{F}_{\text{REACTION}} = \frac{2e^2 \ddot{v}}{3c^3} = \frac{2}{3} \frac{e^2}{c^3} \overset{\text{3RD}}{\text{DERIVATIVE}} \ddot{x}$

THE EQUATION OF MOTION IS THUS

$$\vec{F}_{\text{EXT}} + \vec{F}_R = (M + M_{\text{EL}}) \ddot{x}$$

OR  $M_{\text{TOT}} \ddot{x} = \vec{F}_{\text{EXT}} + \frac{2}{3} \frac{e^2}{c^3} \overset{\text{3RD}}{\text{DERIVATIVE}} \ddot{x}$

THE RADIATION REACTION TERM DOES NOT INVOLVE THE RADIUS OF THE POINT CHARGE, AND SEEMS MUCH MORE SOUNDLY MOTIVATED THAN THE ELECTRO MAGNETIC MASS TERM.

NONETHELESS IT TOO HAS ITS PECULIARITIES!

SUPPOSE  $\vec{F}_{EXT} = 0$ . THEN  $\ddot{\vec{x}} = \frac{3mc^3}{2e^2} \dot{\vec{x}} = \frac{d}{dt} (\dot{\vec{x}})$

THIS HAS THE RUNAWAY SOLUTION  $\ddot{\vec{x}} = \ddot{\vec{x}}_0 e^{\frac{3}{2} \frac{c}{r_0} t}$

WHERE  $r_0 = \frac{e^2}{mc^2} =$  CLASSICAL ELECTRON RADIUS

THE CHARACTERISTIC BLOW-UP TIME IS  $\tau_0 = \frac{2r_0}{3c} \sim 10^{-23}$  SECONDS!

TO GET RID OF THE RUNAWAY SOLUTIONS PEOPLE HAVE MADE CONTOURIONS OF A SPECTACULAR NATURE, WHICH WE WILL NOT DESCRIBE HERE. BUT WE WILL SUGGEST THAT THIS PATHOLOGY INDICATES THAT THERE WILL BE TROUBLE WITH OUR CLASSICAL MODEL OF A POINT CHARGE IN SITUATIONS WHERE

$(F_{EXT}) \lesssim (F_R)$

TO GET AN IDEA OF THIS LIMITATION WE SUPPOSE THE EXTERNAL FORCE IS DUE TO EXTERNAL ELECTROMAGNETIC FIELDS.

THEN  $m \ddot{\vec{x}} = e \vec{E} + e \frac{\vec{v}}{c} \times \vec{B} + \underbrace{\frac{2e^2}{3c^3} \ddot{\vec{x}}}_{\text{HOPEFULLY SMALL}}$

SO  $\ddot{\vec{x}} \approx \frac{e \vec{E}}{m} + \frac{e}{m} \frac{\vec{v}}{c} \times \vec{B} + \frac{e}{m} \frac{\vec{v}}{c} \times \dot{\vec{B}}$

FURTHER SUPPOSING THAT  $\vec{v}/c$  IS SMALL

$\ddot{\vec{x}} = \dot{\vec{v}} \approx \frac{e \vec{E}}{m}$  SO  $\ddot{\vec{x}} \approx \frac{e \vec{E}}{m} + \frac{e^2}{m^2 c} \vec{E} \times \vec{B}$

$\vec{F}_{REACTION} \approx \frac{2}{3} \frac{e^2}{c^3} \left( \frac{e}{m} \dot{\vec{E}} + \frac{e^2}{m^2 c} \vec{E} \times \dot{\vec{B}} \right)$

FOR OSCILLATORY EXTERNAL FIELDS,  $\vec{E} = \vec{E}_0 e^{-i\omega t}$ ,  $\dot{\vec{E}} \sim \omega \vec{E}$

SO  $\vec{F}_{REACTION} \sim \frac{2}{3} \frac{e^2}{mc^2} \frac{\omega}{c} eE + \frac{2}{3} \left( \frac{e^2}{mc^2} \right)^2 \frac{eEB}{\omega}$   
 $\sim F_{EXT} \left( \frac{r_0}{\lambda} + \frac{r_0^2 B}{e} \right)$

HENCE WE EXPECT BREAKDOWNS IN CLASSICAL E & M

IF  $\lambda \lesssim r_0 \sim 10^{-13} \text{ cm} = \frac{e^2}{mc^2} \cdot \frac{e^2}{\hbar c} \frac{\hbar}{mc} = \alpha \lambda_{\text{COMPTON}} \ll \lambda_{\text{COMPTON}}$

OR  $E, B \gtrsim \frac{e}{r_0^2} = \frac{m^2 c^4}{e^3} = \frac{m^2 c^3}{e \hbar} \frac{\hbar c}{e^2} = \frac{E_{\text{CRIT}}}{\alpha} \gg E_{\text{CRIT}}$

NOTE THAT THESE REMARKS DON'T DEPEND ON THE SIZE OF  $\hbar$ , ONLY THAT IT EXISTS!

IN EITHER CASE ONE FINDS THE NEED FOR QUANTUM ELECTRODYNAMICS; I.E. WE ARE OUTSIDE THE VALIDITY OF CLASSICAL PHYSICS.

UNIFORMLY ACCELERATED MOTION

RETURNING TO THE EXPRESSION

$$\vec{F}_{\text{REACTION}} = \frac{2}{3} \frac{e^2}{c^3} \overset{\dots}{\ddot{\vec{x}}}$$

F. ROHRLICH 'CLASSICAL CHARGED PARTICLES'  
 J.D. HAMILTON, AM. J. PHYS 46, 83 (1978)  
 J. COHN AM. J. PHYS 46, 225 (1978)  
 D.C. BOULWARE ANN. PHYS 124, 169 (1980)

ONE OF THE SIMPLEST APPLICATIONS MIGHT BE TO THE CASE OF UNIFORMLY ACCELERATED MOTION:  $\overset{\dots}{\ddot{\vec{x}}} = \text{CONSTANT}$

IF SO,  $\vec{F}_R = 0!$

IN THE EARLY DAYS THIS LED MANY FAMOUS PEOPLE TO CONCLUDE THAT A UNIFORMLY ACCELERATED CHARGE DOES NOT RADIATE! THIS CERTAINLY WOULD BE A BLOW FOR ALL THE EFFORT WE SPENT IN TRYING TO UNDERSTAND RADIATION.

TO OUR RELIEF THE MODERN CONSENSUS IS THAT INDEED A UNIFORMLY ACCELERATED CHARGE RADIATES, AND THAT THE RADIATION REACTION IS ALSO ZERO. THIS APPARENT CONTRADICTION IS RESOLVED BY A DETAILED EXAMINATION OF THE FIELD ENERGY. TYPICALLY WE WOULD SAY THAT

$$U_{\text{FIELD}} = U_{\text{RADIATION}} + U_{\text{NEAR}}$$

WHERE  $U_{\text{NEAR}}$  IS THAT ENERGY WHICH IS TIED TO THE MOVING CHARGE.

WHAT APPEARS TO HAPPEN FOR UNIFORMLY ACCELERATED MOTION IS THAT  $U_{\text{FIELD}} = \text{CONSTANT}$ , BUT  $U_{\text{RADIATION}}$  GROWS WITH TIME WHILE  $U_{\text{NEAR}}$  SHRINKS. THEN NO MECHANICAL ENERGY GOES INTO CHANGING  $U_{\text{FIELD}}$ , AS WOULD NORMALLY BE EXPECTED.

SOME PLAUSIBILITY MAY BE LENT TO THESE STATEMENTS IF WE EXAMINE THE CHARACTER OF UNIFORMLY ACCELERATED MOTION.

FIRST WE MUST DEFINE WHAT IS MEANT BY 'UNIFORM ACCELERATION' IN SPECIAL RELATIVITY. WE REQUIRE THAT THE ACCELERATION AS

VIEWS IN THE INSTANTANEOUS REST FRAME OF THE PARTICLE BE CONSTANT.

FOR MOTION ALONG A STRAIGHT LINE THIS LEADS TO FAIRLY SIMPLE RELATIONS. THEN  $\vec{a}$  IS  $\parallel$  TO  $\vec{v}$  AND SO

$$a^{*2} = -c^4 a_\mu a^\mu = \gamma^6 a^2 \quad (\text{P221 LECTURE 18})$$

$$\text{SO } \gamma^3 a = a^* = \text{CONSTANT}$$

$$\text{NOW } \gamma^3 a = \frac{d\gamma v}{dt} \Rightarrow \gamma v = a^* t$$

$$\Rightarrow v = \frac{dx}{dt} = \frac{ca^* t}{\sqrt{c^2 + (a^* t)^2}} \quad \left[ \text{USING } \gamma = \sqrt{1 + \gamma^2 \beta^2} \right]$$

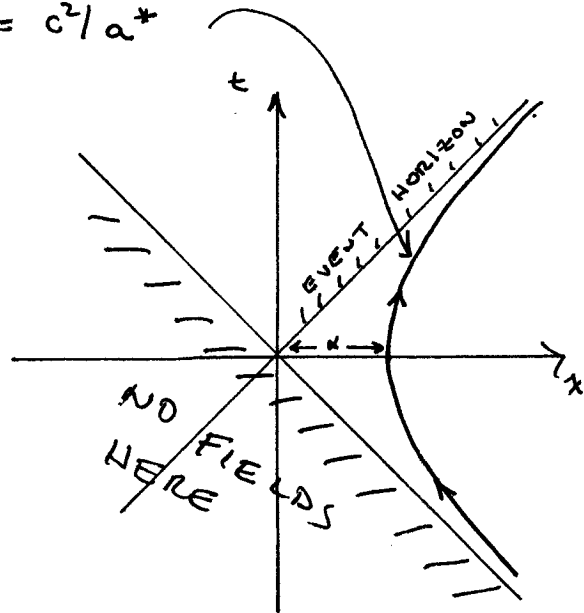
$$\Rightarrow x = \frac{c}{a^*} \sqrt{c^2 + (a^* t)^2} \quad \left( + \text{CONST.} = -x \text{ IF WANT } x(0) = 0 \right)$$

$$\text{OR } x = \sqrt{x^2 + c^2 t^2} \quad \text{WHERE } x = c^2/a^*$$

THIS IS A HYPERBOLA, AND SO UNIFORMLY ACCELERATED MOTION IS SOMETIMES CALLED "HYPERBOLIC MOTION".

AS  $t \rightarrow \pm \infty$   $v \rightarrow \pm c$  WHICH LENDS A SPECIAL CHARACTER TO THIS MOTION.

FOR EXAMPLE, THE RADIATION FIELDS MOVE WITH VELOCITY  $c$ , SO AS  $t \rightarrow \pm \infty$  THE RADIATION CAN HARDLY BE DISTINGUISHED FROM THE NEAR FIELD!



ALSO NOTE THAT THERE CAN BE NO FIELDS AT ALL FROM THE CHARGE IN THE LOWER LEFT HALF SPACE AS SHOWN. ON THE OTHER HAND, ALONG THE BOUNDARY OF THIS HALF SPACE THERE IS A 'SHOCK WAVE' DISCONTINUITY DUE TO THE TREMENDOUS AMOUNT OF RADIATION FROM  $t \sim -\infty$ .

SIMILARLY, NO OBSERVER IN THE UPPER LEFT HALF SPACE CAN COMMUNICATE WITH THE MOVING CHARGE! THE LINE  $x = ct$  IS SAID TO BE AN 'EVENT HORIZON' FOR THE CHARGE. THE CONCEPT OF 'EVENT HORIZON' IS MOST OFTEN HEARD OF IN CONJUNCTION WITH 'BLACK HOLES'....

SOME FINAL TITILLATIONS INVOLVE THE PRINCIPLE OF EQUIVALENCE.

1. WE CERTAINLY AGREE THAT AN OBSERVER AT REST WATCHING ANOTHER CHARGE AT REST WILL SEE NO RADIATION.
2. IF THE CHARGE IS ACCELERATED AND THE OBSERVER IS AT REST THEN RADIATION IS OBSERVED.
3. SUPPOSE THE CHARGE IS AT REST, BUT THE OBSERVER IS ACCELERATED. WHAT DOES SHE SEE? RADIATION!
4. SUPPOSE BOTH THE CHARGE AND OBSERVER ARE ACCELERATED, SO THAT THEIR VELOCITIES ARE ALWAYS EQUAL (IN SOME LAB FRAME) NOW THE OBSERVER SEES NO RADIATION!

EINSTEIN HAS REVEALED HOW AN OBSERVER AT REST IN A UNIFORM GRAVITATIONAL FIELD SHOULD FIND THE SAME PHYSICAL PHENOMENA AS A UNIFORMLY ACCELERATED OBSERVER.

MORE BASICALLY, HE NOTED THAT AN OBSERVER FALLING FREELY IN A GRAVITATIONAL FIELD WILL THINK HE IS IN AN INERTIAL FRAME!

CAN WE RELATE RADIATION FROM ACCELERATED CHARGES TO RADIATION FROM CHARGES IN A GRAVITATIONAL FIELD? CERTAINLY!

(SOME PEOPLE HAVE SAID NO, BECAUSE THE PRINCIPLE OF EQUIVALENCE IS A LOCAL PRINCIPLE, WHILE RADIATION IMPLIES ACTIVITY THAT IS REMOVED FROM THE LOCAL OF THE OBSERVER. THIS OBJECTION IS SPURIOUS.)

BUT WE MUST BE CAREFUL! THE TRICK QUESTION IS WHETHER WE SHOULD EXPECT A CHARGE SITTING IN FRONT OF US TO RADIATE JUST BECAUSE IT IS IN A UNIFORM GRAVITATIONAL FIELD? (IF SO WE HAVE A WAY OF DETECTING GRAVITY...)

THERE IS NO DIFFICULTY IF <sup>WE</sup> SPECIFY THE RELATIVE ROLE OF CHARGE AND OBSERVER.

5. IF BOTH CHARGE AND OBSERVER ARE AT REST IN A GRAVITATIONAL FIELD, THEY ARE EQUIVALENT TO BOTH BEING ACCELERATED. THIS IS CASE 4  $\Rightarrow$  NO RADIATION.
6. LIKEWISE IF BOTH CHARGE AND OBSERVER ARE FALLING FREELY, WE EXPECT NO RADIATION.
- 7.8. BUT IF ONE IS AT REST AND THE OTHER FALLING FREELY, RADIATION IS OBSERVED!

A FOOTNOTE ON ANGULAR MOMENTUM

WE HAVE PREVIOUSLY REMARKED THAT THE RADIATION FIELDS CAN CARRY AWAY ANGULAR MOMENTUM. WE CAN ACCOUNT FOR THE CONSEQUENT LOSS OF ANGULAR MOMENTUM AT THE SOURCE BY MEANS OF THE TORQUE DUE TO THE RADIATION REACTION FORCE.

NAMELY 
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_R = \frac{2e^2}{3c^3} \vec{r} \times \vec{\ddot{r}} = \frac{2}{3c^3} [\dot{e}\vec{r} \times \dot{e}\vec{\ddot{r}}]$$

BUT WE NOTE THAT  $e\vec{r} = \vec{p} =$  DIPOLE MOMENT

[IF SEVERAL CHARGES ARE PRESENT WE HAVE A SUM OF TERMS AS ABOVE.  
WE CAN STILL INTRODUCE  $\vec{p} =$  TOTAL DIPOLE MOMENT.]

Thus 
$$\frac{d\vec{L}}{dt} = \frac{2}{3c^3} \dot{\vec{p}} \times \ddot{\vec{p}} = \frac{2}{3c^3} \left[ \frac{d}{dt} (\vec{p} \times \ddot{\vec{p}}) - \dot{\vec{p}} \times \ddot{\vec{p}} \right]$$

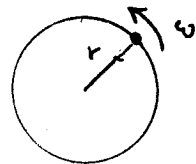
AS FOR THE ENERGY LOSS RELATION, WE OBTAIN A SIMPLER FORM IF WE AVERAGE OVER TIME. IN PARTICULAR, FOR PERIODIC MOTION, THE AVERAGE OF THE FIRST TERM VANISHES. (IF WE IGNORE THE SLOW DECREASE IN  $\ddot{\vec{p}}$  DUE TO THE ENERGY LOSS!)

so 
$$\left\langle \frac{d\vec{L}}{dt} \right\rangle = -\frac{2}{3c^3} \langle \dot{\vec{p}} \times \ddot{\vec{p}} \rangle$$

EXAMPLE POINT CHARGE MOVING IN A CIRCLE (PROB 3, SET 8)

$$\vec{p} = er (\hat{x} + i\hat{y}) e^{-i\omega t}$$

$$\dot{\vec{p}} = -i\omega \vec{p} \quad \ddot{\vec{p}} = -\omega^2 \vec{p}$$



$$\left\langle \frac{d\vec{L}}{dt} \right\rangle = -\frac{2\omega^3}{3c^3} \frac{\text{Re}(i\vec{p} \times \vec{p}^*)}{2}$$

BY AN EXTENSION OF OUR TRICKS FOR THE TIME AVERAGE OF COMPLEX QUANTITIES

$$= -\frac{2\omega^3}{3c^3} P_0^2 \hat{z}$$

$$[P_0 \equiv er]$$

$$= -\frac{1}{\omega} \left( \frac{2}{3} \frac{\omega^4 P_0^2}{c^3} \right) \hat{z}$$

$$= \frac{\langle dU_{\text{lost}}/dt \rangle}{\omega} \hat{z}$$

AS FOUND ON THE PROBLEM SET!