Substituting, we obtain

$$M = -4\pi^2(1-\mu^2)\left\{\frac{b^2}{2c}\frac{dP_1}{d\mu} - \frac{1\,.\,3}{2\,.\,4}\frac{b^4}{3c^3}\frac{dP_3}{d\mu} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{b^6}{5c^5}\frac{dP_5}{d\mu} - \text{etc.}\right\}.$$

The following are the values for the coefficients (Ferrers's *Spherical Harmonics*, p. 23), both in terms of μ and when we substitute $\mu^2 = 1 - \frac{\beta^2}{c^2}$:

$$\frac{dP_1}{d\mu} = 1,$$

$$\frac{dP_3}{d\mu} = \frac{3}{2}(5\mu^2 - 1) = \frac{3}{2}\left(4 - 5\frac{\beta^2}{c^2}\right),$$

$$\frac{dP_5}{d\mu} = \frac{15}{8}(21\mu^4 - 14\mu^2 + 1) = \frac{15}{8}\left(8 - 28\frac{\beta^2}{c^2} + 21\frac{\beta^4}{c^4}\right),$$

$$\frac{dP_7}{d\mu} = \frac{3003\mu^6 - 3465\mu^4 + 945\mu^2 - 35}{16} = \frac{448 - 3024\frac{\beta^2}{c^2} + \text{etc.}}{16},$$

$$\frac{dP_9}{d\mu} = \frac{109395\mu^8 - 180180\mu^6 + 90090\mu^4 - 13860\mu^2 + 315}{128}$$

$$= \frac{5760 + \text{etc.}}{128}.$$

Substituting these values and putting $c^2 = a^2 + \beta^2$, we obtain

$$M = -4\pi\frac{b^2\beta^2}{a^3}\left\{\frac{1}{2} - \frac{3}{4}\frac{b^2 + \beta^2}{a^2} + \frac{15}{16}\frac{b^4 + 3b^2\beta^2 + \beta^4}{a^4} - \frac{35}{32}\frac{b^6 + 6b^4\beta^2 + 6b^2\beta^4 + \beta^6}{a^6} + \text{etc.}\right\}.$$

The more useful form is obtained by retaining c. If we take the two circles of equal radius (i.e. $b = \beta$), we obtain

$$M = -4\pi^2\frac{b^4}{c^3}\left\{\frac{1}{2} - \frac{3}{4}\frac{b^2}{c^2} + \frac{15}{8}\frac{b^4}{c^4} - \frac{35}{8}\frac{b^6}{c^6} + \frac{2835}{256}\frac{b^8}{c^8} - \text{etc.}\right\}.$$

Collected Scientific Papers
J. H. Poynting
(Cambridge U. Press 1920)

10.

ON THE TRANSFER OF ENERGY IN THE ELECTROMAGNETIC FIELD.

[*Phil. Trans.* **175**, 1884, pp. 343–361.]

[*Received* December 17, 1883. *Read* January 10, 1884.]

A space containing electric currents may be regarded as a field where energy is transformed at certain points into the electric and magnetic kinds by means of batteries, dynamos, thermoelectric actions, and so on, while in other parts of the field this energy is again transformed into heat, work done by electromagnetic forces, or any form of energy yielded by currents. Formerly a current was regarded as something travelling along a conductor, attention being chiefly directed to the conductor, and the energy which appeared at any part of the circuit, if considered at all, was supposed to be conveyed thither through the conductor by the current. But the existence of induced currents and of electromagnetic actions at a distance from a primary circuit from which they draw their energy has led us, under the guidance of Faraday and Maxwell, to look upon the medium surrounding the conductor as playing a very important part in the development of the phenomena. If we believe in the continuity of the motion of energy, that is, if we believe that when it disappears at one point and reappears at another it must have passed through the intervening space, we are forced to conclude that the surrounding medium contains at least a part of the energy, and that it is capable of transferring it from point to point.

Upon this basis Maxwell has investigated what energy is contained in the medium, and he has given expressions which assign to each part of the field a quantity of energy depending on the electromotive and magnetic intensities and on the nature of the matter at that part in regard to its specific inductive capacity and magnetic permeability. These expressions account, as far as we know, for the whole energy. According to Maxwell's theory, currents consist essentially in a certain distribution of energy in and around a conductor, accompanied by transformation and consequent movement of energy through the field.

Starting with Maxwell's theory, we are naturally led to consider the problem: How does the energy about an electric current pass from point to point—that is, by what paths and according to what law does it travel from the part of the circuit where it is first recognisable as electric and magnetic to the parts where it is changed into heat or other forms?

The aim of this paper is to prove that there is a general law for the transfer of energy, according to which it moves at any point perpendicularly to the plane containing the lines of electric force and magnetic force, and that the amount crossing unit of area per second of this plane is equal to the product of the intensities of the two forces, multiplied by the sine of the angle between them, divided by 4π; while the direction of flow of energy is that in which a right-handed screw would move if turned round from the positive direction of the electromotive to the positive direction of the magnetic intensity. After the investigation of the general law several applications will be given to show how the energy moves in the neighbourhood of various current-bearing circuits.

The following is a general account of the method by which the law is obtained.

If we denote the electromotive intensity at a point (that is, the force per unit of positive electrification which would act upon a small charged body placed at the point) by $\mathfrak{E}$, and the specific inductive capacity of the medium at that point by K, the magnetic intensity (that is, the force per unit pole which would act on a small north-seeking pole placed at the point) by $\mathfrak{H}$ and the magnetic permeability by μ, Maxwell's expression for the electric and magnetic energies per unit volume of the field is

$$K\mathfrak{E}^2/8\pi + \mu\mathfrak{H}^2/8\pi. \quad \text{..............................(1)}$$

If any change is going on in the supply or distribution of energy the change in this quantity per second will be

$$K\mathfrak{E}\frac{d\mathfrak{E}}{dt}\Big/4\pi + \mu\mathfrak{H}\frac{d\mathfrak{H}}{dt}\Big/4\pi. \quad \text{..........................(2)}$$

According to Maxwell the true electric current is in general made up of two parts, one the conduction-current $\mathfrak{K}$, and the other due to change of electric displacement in the dielectric, this latter being called the displacement-current. Now, the displacement is proportional to the electromotive intensity, and is represented by $K\mathfrak{E}/4\pi$, so that when change of displacement takes place, due to change in the electromotive intensity, the rate of change, that is, the displacement-current, is $K\frac{d\mathfrak{E}}{dt}\Big/4\pi$, and this is equal to the difference between the true current $\mathfrak{C}$ and the conduction-current $\mathfrak{K}$. Multiplying this difference by the electromotive intensity $\mathfrak{E}$ the first term in (2) becomes

$$\frac{K\mathfrak{E}}{4\pi}\frac{d\mathfrak{E}}{dt} = \mathfrak{C}\mathfrak{E} - \mathfrak{K}\mathfrak{E}. \quad \text{..............................(3)}$$

The first term of the right side of (3) may be transformed by substituting for the components of the total current their values in terms of the components of the magnetic intensity, while the second term, the product of the conduction-current and the electromotive intensity, by Ohm's law, which states that $\mathfrak{K} = C\mathfrak{E}$, becomes $\mathfrak{K}^2/C$, where C is the specific conductivity. But this is the energy appearing as heat in the circuit per unit volume according to Joule's law. If we sum up the quantity in (3) thus transformed, *for the whole space within a closed surface*, the integral of the first term can be integrated by parts, and we find that it consists of two terms—one an expression depending on the surface alone to which each part of the surface contributes a share depending on the values of the electromotive and magnetic intensities at that part, the other term being the change per second in the magnetic energy (that is, the second term of (2)) with a negative sign. The integral of the second term of (3) is the total amount of heat developed in the conductors within the surface per second. We have then the following result.

The change per second in the electric energy within a surface = (a quantity depending on the surface) − (the change per second in the magnetic energy) − (the heat developed in the circuit).

Or rearranging:

The change per second in the sum of the electric and magnetic energies within a surface together with the heat developed by currents is equal to a quantity to which each element of the surface contributes a share depending on the values of the electric and magnetic intensities at the element. That is, the total change in the energy is accounted for by supposing that the energy passes in through the surface according to the law given by this expression.

On interpreting the expression it is found that it implies that the energy flows as stated before, that is, perpendicularly to the plane containing the lines of electric and magnetic force, that the amount crossing unit area per second of this plane is equal to the product

$$\frac{\text{electromotive intensity} \times \text{magnetic intensity} \times \text{sine included angle}}{4\pi},$$

while the direction of flow is given by the three quantities, electromotive intensity, magnetic intensity, flow of energy, being in right-handed order.

It follows at once that the energy flows perpendicularly to the lines of electric force, and so along the equipotential surfaces where these exist. It also flows perpendicularly to the lines of magnetic force, and so along the magnetic equipotential surfaces where these exist. If both sets of surfaces exist their lines of intersection are the lines of flow of energy.

The following is the full mathematical proof of the law:

The energy of the field may be expressed in the form (Maxwell's *Electricity*, vol. 2, 2nd ed., p. 253)

$$\tfrac{1}{2}\iiint(Pf + Qg + Rh)\,dxdydz + \frac{1}{8\pi}\iiint(a\alpha + b\beta + c\gamma)\,dxdydz,$$

the first term the electrostatic, the second the electromagnetic energy.

But since $f = \frac{K}{4\pi}P$, with corresponding values for g and h, and $a = \mu\alpha$, $b = \mu\beta$, $c = \mu\gamma$, substituting, the energy becomes

$$\frac{K}{8\pi}\iiint(P^2 + Q^2 + R^2)\,dxdydz + \frac{\mu}{8\pi}\iiint(\alpha^2 + \beta^2 + \gamma^2)\,dxdydz. \quad \ldots(1)$$

Let us consider the space within any fixed closed surface. The energy within this surface will be found by taking the triple integrals throughout the space.

If any changes are taking place the rate of increase of energy of the electric and magnetic kinds per second is

$$\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt} + Q\frac{dQ}{dt} + R\frac{dR}{dt}\right)dxdydz + \frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt} + \beta\frac{d\beta}{dt} + \gamma\frac{d\gamma}{dt}\right)dxdydz. \quad (2)$$

Now Maxwell's equations for the components of the true current are

$$u = p + \frac{df}{dt}, \qquad v = q + \frac{dg}{dt}, \qquad w = r + \frac{dh}{dt},$$

where p, q, r are components of the conduction-current.

But we may substitute for $\frac{df}{dt}$ its value $\frac{K}{4\pi}\frac{dP}{dt}$, and so for the other two, and we obtain

$$\left.\begin{aligned}\frac{K}{4\pi}\frac{dP}{dt} &= u - p\\ \frac{K}{4\pi}\frac{dQ}{dt} &= v - q\\ \frac{K}{4\pi}\frac{dR}{dt} &= w - r\end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots(3)$$

Taking the first term in (2) and substituting from (3) we obtain—

$$\begin{aligned}&\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt} + Q\frac{dQ}{dt} + R\frac{dR}{dt}\right)dxdydz\\ &= \iiint\{P(u - p) + Q(v - q) + R(w - r)\}\,dxdydz\\ &= \iiint(Pu + Qv + Rw)\,dxdydz - \iiint(Pp + Qq + Rr)\,dxdydz. \ldots(4)\end{aligned}$$

Now the equations for the components of electromotive force are (Maxwell, vol. 2, p. 222)

$$\left.\begin{aligned}P &= c\dot{y} - b\dot{z} - \frac{dF}{dt} - \frac{d\psi}{dx} = c\dot{y} - b\dot{z} + P'\\ Q &= a\dot{z} - c\dot{x} - \frac{dG}{dt} - \frac{d\psi}{dy} = a\dot{z} - c\dot{x} + Q'\\ R &= b\dot{x} - a\dot{y} - \frac{dH}{dt} - \frac{d\psi}{dz} = b\dot{x} - a\dot{y} + R'\end{aligned}\right\}, \quad \ldots\ldots\ldots(5)$$

where P', Q', R' are put for the parts of P, Q, R which do not contain the velocities.

Then

$$\begin{aligned}Pu + Qv + Rw &= (c\dot{y} - b\dot{z})\,u + (a\dot{z} - c\dot{x})\,v + (b\dot{x} - a\dot{y})\,w + P'u + Q'v + R'w\\ &= -\{(vc - wb)\,\dot{x} + (wa - uc)\,\dot{y} + (ub - va)\,\dot{z}\} + P'u + Q'v + R'w\\ &= -(X\dot{x} + Y\dot{y} + Z\dot{z}) + P'u + Q'v + R'w,\end{aligned}$$

where X, Y, Z are the components of the electromagnetic force per unit of volume (Maxwell, vol. 2, p. 227).

Now substituting in (4) and putting for u, v, w their values in terms of the magnetic force (Maxwell, vol. 2, p. 233) and transposing we obtain

$$\begin{aligned}&\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt} + Q\frac{dQ}{dt} + R\frac{dR}{dt}\right)dxdydz\\ &\quad + \iiint\{(X\dot{x} + Y\dot{y} + Z\dot{z}) + (Pp + Qq + Rr)\}\,dxdydz\\ &= \iiint(P'u + Q'v + R'w)\,dxdydz\\ &= \frac{1}{4\pi}\iiint\left\{P'\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) + Q'\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + R'\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\}dxdydz\\ &= \frac{1}{4\pi}\iiint\left(R'\frac{d\beta}{dx} - Q'\frac{d\gamma}{dx}\right)dxdydz\\ &\quad + \frac{1}{4\pi}\iiint\left(P'\frac{d\gamma}{dy} - R'\frac{d\alpha}{dy}\right)dxdydz\\ &\quad + \frac{1}{4\pi}\iiint\left(Q'\frac{d\alpha}{dz} - P'\frac{d\beta}{dz}\right)dxdydz\end{aligned}$$

(Integrating each term by parts)

$$\begin{aligned}&= \frac{1}{4\pi}\iint(R'\beta - Q'\gamma)\,dydz + \frac{1}{4\pi}\iint(P'\gamma - R'\alpha)\,dzdx + \frac{1}{4\pi}\iint(Q'\alpha - P'\beta)\,dxdy\\ &\quad - \frac{1}{4\pi}\iiint\left\{\beta\frac{dR'}{dx} - \gamma\frac{dQ'}{dx} + \gamma\frac{dP'}{dy} - \alpha\frac{dR'}{dy} + \alpha\frac{dQ'}{dz} - \beta\frac{dP'}{dz}\right\}dxdydz\end{aligned}$$

(The double integral being taken over the surface)

$$\begin{aligned}&= \frac{1}{4\pi}\iint\{l\,(R'\beta - Q'\gamma) + m\,(P'\gamma - R'\alpha) + n\,(Q'\alpha - P'\beta)\}\,dS\\ &\quad - \frac{1}{4\pi}\iiint\left\{\alpha\left(\frac{dQ'}{dz} - \frac{dR'}{dy}\right) + \beta\left(\frac{dR'}{dx} - \frac{dP'}{dz}\right) + \gamma\left(\frac{dP'}{dy} - \frac{dQ'}{dx}\right)\right\}dxdydz, \ldots(6)\end{aligned}$$

where l, m, n are the direction-cosines of the normal to the surface outwards.

But from the values of P', Q', R' in (5) we see that

$$\frac{dQ'}{dz} - \frac{dR'}{dy} = -\frac{d^2G}{dt\,dz} - \frac{d^2\psi}{dx\,dz} + \frac{d^2H}{dt\,dy} + \frac{d^2\psi}{dz\,dx}$$
$$= \frac{d}{dt}\left(\frac{dH}{dy} - \frac{dG}{dz}\right)$$
$$= \frac{da}{dt} = \mu\frac{d\alpha}{dt} \text{ (Maxwell, vol. 2, p. 216),}$$

similarly

$$\frac{dR'}{dx} - \frac{dP'}{dz} = \frac{db}{dt} = \mu\frac{d\beta}{dt},$$
$$\frac{dP'}{dy} - \frac{dQ'}{dx} = \frac{dc}{dt} = \mu\frac{d\gamma}{dt}.$$

Whence the triple integral in (6) becomes

$$-\frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt} + \beta\frac{d\beta}{dt} + \gamma\frac{d\gamma}{dt}\right)dx\,dy\,dz.$$

Transposing it to the other side we obtain

$$\frac{K}{4\pi}\iiint\left(P\frac{dP}{dt} + Q\frac{dQ}{dt} + R\frac{dR}{dt}\right)dx\,dy\,dz + \frac{\mu}{4\pi}\iiint\left(\alpha\frac{d\alpha}{dt} + \beta\frac{d\beta}{dt} + \gamma\frac{d\gamma}{dt}\right)dx\,dy\,dz$$
$$+ \iiint(X\dot{x} + Y\dot{y} + Z\dot{z})\,dx\,dy\,dz + \iiint(Pp + Qq + Rr)\,dx\,dy\,dz$$
$$= \frac{1}{4\pi}\iint\{l(R'\beta - Q'\gamma) + m(P'\gamma - R'\alpha) + n(Q'\alpha - P'\beta)\}\,dS. \quad \ldots(7)$$

The first two terms of this express the gain per second in electric and magnetic energies as in (2). The third term expresses the work done per second by the electromagnetic forces, that is, the energy transformed by the motion of the matter in which currents exist. The fourth term expresses the energy transformed by the conductor into heat, chemical energy, and so on; for P, Q, R are by definition the components of the force acting at a point per unit of positive electricity, so that $Pp\,dx\,dy\,dz$ or $P\,dx\,p\,dy\,dz$ is the work done per second by the current flowing parallel to the axis of x through the element of volume $dx\,dy\,dz$. So for the other two components. This is in general transformed into other forms of energy, heat due to resistance, thermal effects at thermoelectric surfaces, and so on.

The left side of (7) thus expresses the total gain in energy per second within the closed surface, and the equation asserts that this energy comes through the bounding surface, each element contributing the amount expressed by the right side.

This may be put in another form, for if $\mathfrak{E}'$ be the resultant of P', Q', R', and θ the angle between its direction and that of $\mathfrak{H}$, the magnetic intensity, the direction-cosines L, M, N of the line perpendicular to the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$ are given by

$$L = \frac{R'\beta - Q'\gamma}{\mathfrak{E}'\mathfrak{H}\sin\theta}; \qquad M = \frac{P'\gamma - R'\alpha}{\mathfrak{E}'\mathfrak{H}\sin\theta}; \qquad N = \frac{Q'\alpha - P'\beta}{\mathfrak{E}'\mathfrak{H}\sin\theta};$$

so that the surface-integral becomes

$$\frac{1}{4\pi}\iint\mathfrak{E}'\mathfrak{H}\sin\theta\,(Ll + Mm + Nn)\,dS.$$

If at a given point dS be drawn to coincide with the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$, it then contributes the greatest amount of energy to the space; or in other words the energy flows perpendicularly to the plane containing $\mathfrak{E}'$ and $\mathfrak{H}$, the amount crossing unit area per second being $\mathfrak{E}'\mathfrak{H}\sin\theta/4\pi$. To determine in which way it crosses the plane take $\mathfrak{E}'$ along Oz, $\mathfrak{H}$ along Oy. Then

$$P' = 0, \qquad Q' = 0, \qquad \frac{R'}{\mathfrak{E}'} = 1,$$
$$\alpha = 0, \qquad \frac{\beta}{\mathfrak{H}} = 1, \qquad \gamma = 0,$$

and if $\sin\theta = 1$ $\qquad L = 1, \qquad M = 0, \qquad N = 0.$

If now the axis Ox be the normal to the surface outwards, $l = 1$, $m = 0$, $n = 0$, so that this element of the integral contributes a positive term to the energy within the surface on the negative side of the yz plane; that is, the energy moves along xO, or in the direction in which a screw would move if its head were turned round from the positive direction of the electromotive to the positive direction of the magnetic intensity. If the surface be taken where the matter has no velocity, $\mathfrak{E}'$ becomes equal to $\mathfrak{E}$, and the amount of energy crossing unit area perpendicular to the flow per second is

$$\frac{\text{electromotive intensity} \times \text{magnetic intensity} \times \text{sine included angle}}{4\pi}.$$

Since the surface may be drawn anywhere we please, then wherever there is both magnetic and electromotive intensity there is flow of energy.

Since the energy flows perpendicularly to the plane containing the two intensities, it must flow along the electric and magnetic level surfaces, when these exist, so that the lines of flow are the intersections of the two surfaces.

We shall now consider the applications of this law in several cases.

Applications of the Law of Transfer of Energy.

(1) *A straight wire conveying a current.*

In this case very near the wire, and within it, the lines of magnetic force are circles round the axis of the wire. The lines of electric force are along the wire, if we take it as proved that the flow across equal areas of the crosssection is the same at all parts of the section. If AB, Fig. 1, represents the wire, and the current is from A to B, then a tangent plane to the surface at any point contains the directions of both the electromotive and magnetic intensities (we shall write E.M.I. and M.I. for these respectively in what follows), and energy is therefore flowing in perpendicularly through the surface, that is, along the radius towards the axis. Let us take a portion of the wire bounded by two plane sections perpendicular to the axis. Across the ends no energy is flowing, for they contain no component of the E.M.I. The whole of the energy then enters in through the external surface of the wire, and by the general theorem the amount entering in must just account for the heat developed owing to the resistance, since if the current is steady there is no other alteration of energy. It is, perhaps, worth while to show independently in this case that the energy moving in, in accordance with the general law, will just account for the heat developed.

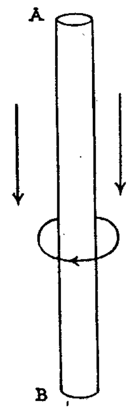

Fig. 1.

Let r be the radius of the wire, i the current along it, a the magnetic intensity at the surface, P the electromotive intensity at any point within the wire, and V the difference of potential between the two ends. Then the area of a length l of the wire is $2\pi rl$, and the energy entering from the outside per second is

$$\frac{\text{area} \times \text{E.M.I.} \times \text{M.I.}}{4\pi} = \frac{2\pi rl \,.\, P \,.\, a}{4\pi}$$
$$= \frac{2\pi ra \,.\, Pl}{4\pi}$$
$$= \frac{4\pi i V}{4\pi}$$
$$= iV,$$

for the line-integral of the magnetic intensity $2\pi ra$ round the wire is $4\pi \times$ current through it, and $Pl = V$.

But by Ohm's law $V = iR$ and $iV = i^2R$, or the heat developed according to Joule's law.

It seems then that none of the energy of a current travels along the wire, but that it comes in from the non-conducting medium surrounding the wire, that as soon as it enters it begins to be transformed into heat, the amount crossing successive layers of the wire decreasing till by the time the centre is reached, where there is no magnetic force, and therefore no energy passing, it has all been transformed into heat. A conduction-current then may be said to consist of this inward flow of energy with its accompanying magnetic

and electromotive forces, and the transformation of the energy into heat within the conductor.

We have now to inquire how the energy travels through the medium on its way to the wire.

(2) *Discharge of a condenser through a wire.*

We shall first consider the case of the slow discharge of a simple condenser consisting of two charged parallel plates when connected by a wire of very great resistance, as in this case we can form an approximate idea of the actual path of the energy.

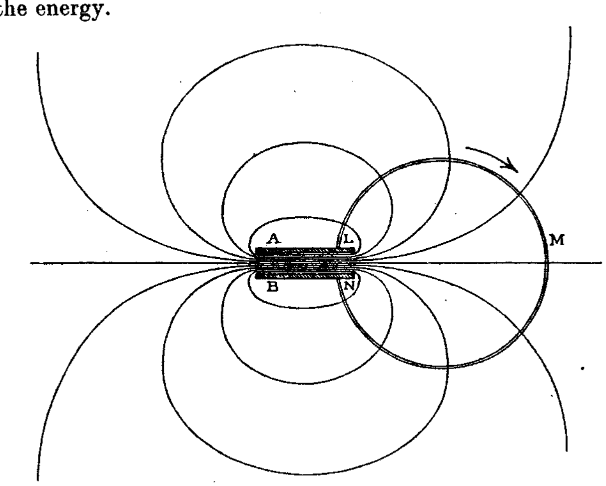

Fig. 2.

Let A and B, Fig. 2, be the two plates of the condenser, A being positively and B negatively electrified. Then before discharge the sections of the equipotential surfaces will be somewhat as sketched. The chief part of the energy resides in the part of the dielectric between the two plates, but there will be some energy wherever there is electromotive intensity. Between A and B the E.M.I. will be from A to B, and everywhere it is perpendicular to the level surfaces. Now connect A and B by a fine wire LMN of very great resistance, following a line of force and with the resistance so adjusted that it is the same for the same fall of potential throughout. We have supposed this arrangement of the resistance so that the level surfaces shall not be disturbed by the flow of the current. The wire is to be supposed so fine that the discharge takes place very slowly.

While the discharge goes on a current flows round LMN in the direction indicated by the arrow, and there is also an equal displacement-current from B to A due to the yielding of the displacement there. The current will be encircled by lines of magnetic force, which will in general form closed curves embracing the circuit. The direction of these round the wire will be from right to left in front, and round the space between A and B from left to right in front. The E.M.I. is always from the higher level surfaces—those nearer A, to the lower—those nearer B, both near the wire and in the space between A and B.

Now, since the energy always moves perpendicularly to the lines of E.M.I. it must travel along the equipotential surfaces. Since it also moves perpendicularly to the lines of M.I. it moves, as we have seen in case No. (1), inwards on all sides to the wire, and is there all converted into heat—if we suppose the discharge so slow that the current is steady during the time considered. But between A and B the E.M.I. is opposed to the current, being downwards, while the M.I. bears the same relation to the current as in the wire. Remembering that E.M.I., M.I., and direction of flow of energy are connected by the right-handed screw relation, we see that the energy moves outwards from the space between A and B. As then the strain of the dielectric between A and B is gradually released by what we call a discharge current along the wire LMN, the energy thus given up travels outwards through the dielectric, following always the equipotential surfaces, and gradually converges once more on the circuit where the surfaces are cut by the wire. There the energy is transformed into heat. It is to be noticed that if the current may be considered steady the energy moves along at the same level throughout.

(3) *A circuit containing a voltaic cell.*

When a circuit contains a voltaic cell we do not know with certainty what is the distribution of potential, but most probably it is somewhat as follows*:—Suppose we have a simple copper, zinc, and acid cell producing a steady current. There is probably a considerable sudden rise in passing from the zinc to the acid, the place where the chemical energy is given up, a fall through the acid depending on the resistance, a sudden fall on passing from the acid to the copper, where some energy is absorbed with evolution of hydrogen,

* It seems probable that the only legitimate mode of measuring the difference of potential between two points in a circuit consisting of dissimilar conductors carrying a steady current, consists in finding the total quantity of energy given out in the part of the circuit between the two points while unit quantity of electricity passes either point. If this is the case, it seems impossible that the surface of contact of dissimilar metals can be the chief seat of the electromotive force, for we have only the very slight evolution or absorption of energy there due to the Peltier effect. I have therefore adopted the theory of the voltaic circuit in which the seat of at least the chief part of the electromotive force is at the contact of the acid and metals. The large differences of potential found by electrometer methods between the air near two different metals in contact are, in this theory, to be accounted for by the supposition that the air acts in a similar manner to an oxidising electrolyte. A short statement of the theory is given in a letter by Professor Maxwell in the *Electrician* for April 26th, 1879, quoted in a note on page 149 of his *Elementary Treatise on Electricity*. (See also § 249, vol. 1, Maxwell's *Electricity and Magnetism*.) June 19, 1884.

and then a gradual fall through the wire of the circuit round to the zinc again. There will be a slight change of potential in passing from copper to zinc, but this we shall neglect for simplicity. The equipotential surfaces will probably then be somewhat as sketched in Fig. 3*, all the surfaces starting from where the acid comes in contact with the zinc, some of the highest potential passing through the acid, others passing between the acid and copper, and crowding in there, the rest lower than these cutting the circuit at right angles in points at intervals representing equal falls of potential.

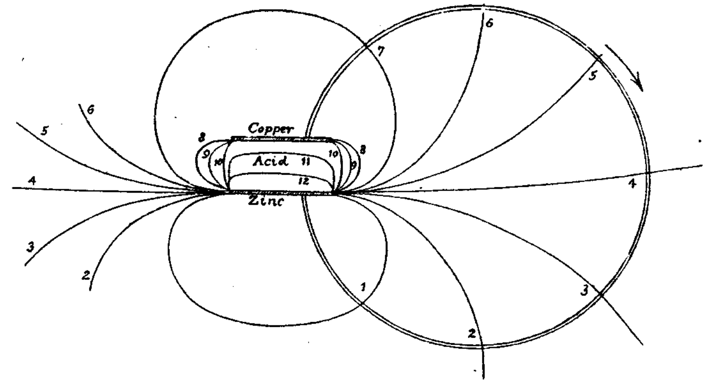

Fig. 3.

If this be the actual arrangement, then it is seen that the current, which travels round the circuit from zinc through acid to copper, is in opposition to the E.M.I. between the zinc and acid, while the M.I. is related to the current in the ordinary way. The energy will therefore pass outwards from there along the level surfaces. In fact, the medium between the zinc and acid behaves like the medium between the plates of the condenser in case No. (2), and it seems possible that the chemical action produces continually fresh 'electric displacement' from acid towards zinc which yields as rapidly as it is formed, the energy of the displacement moving out sideways.

Some of this energy which travels along the highest level surfaces will converge on the acid, and there be, at any rate ultimately, converted into heat. Some of it will move along those surfaces which crowd in between the acid and copper and there converge to supply the energy taken up by the escaping hydrogen. The rest spreads out to converge at last at different parts of the circuit, and there to be transformed into heat according to Joule's law.

It may be noticed that if the level surfaces be drawn with equal differences

* In this and the succeeding cases the circuit is alone supposed to cause the distribution of potential. In actual cases the surfaces would probably be very much deflected from their normal positions in the dielectric through the presence of conductors, electrified matter, and so on.

of potential, equal amounts of energy travel out per second between successive pairs of surfaces. For the amount transformed in the circuit in a length having a given difference of potential V between its ends will be $V \times$ current, and therefore the amount transformed between each pair of surfaces drawn with the same potential difference will be the same. But since the current and the field are steady, the energy transformed will be equal to the energy moving out from the cell between the same surfaces—the energy never crossing level surfaces. This admits of a very easy direct proof, but the above seems quite sufficient.

This result has a consequence which, though already well known, is worth mentioning here. Let V_1 be the difference of potential between the zinc and acid, V_2 that between the acid and copper. If i be the current, $V_1 i$ is the total energy travelling out per second from the zinc surface. Of this $V_2 i$ is absorbed at the copper surface, the rest, viz., $(V_1 - V_2)\, i$, being transformed in the circuit. The fraction, therefore, of the whole energy sent out which is transformed in the circuit is $\frac{V_1 - V_2}{V_1}$, a result analogous to the expression for the amount of heat which can be transformed into work in a reversible heat-engine.

One or two interesting illustrations of this movement of energy may be mentioned here in connection with the voltaic circuit.

Suppose that we are sending a current through a submarine cable by a battery with, say, the zinc to earth, and suppose that the sheath is everywhere at zero potential. Then the wire will everywhere be at higher potential than the sheath, and the level surfaces will pass from the battery through the insulating material to the points where they cut the wire. The energy then which maintains the current, and which works the needle at the further end, travels through the insulating material, the core serving as a means to allow the energy to get in motion.

Again, when the only effect in a circuit is the generation of heat, we have energy moving in upon the wire, there undergoing some sort of transformation, and then moving out again as heat or light. If Maxwell's theory of light be true, it moves out again still as electric and magnetic energy, but with a definite velocity and intermittent in type. We have in the electric light, for instance, the curious result that energy moves in upon the arc or filament from the surrounding medium, there to be converted into a form which is sent out again, and which, though still the same in kind, is now able to affect our senses.

(4) *Thermoelectric circuits.*

Let us first take the case of a circuit composed of two metals, neither of which has any Thomson effect. Let us suppose the current at the hot junction flows from the metal A to the metal B, Fig. 4. According to Professor Tait's theory it would appear that the E.M.I. at the hot junction is to that at the cold as the absolute temperature at the hot is to that at the cold junction. If the current is steady there is probably then a sudden rise in potential from A to B at the hot junction, a gradual fall along B, a sudden fall at the cold junction—less, however, than the sudden rise at the other—and a gradual fall along A. The level surfaces will then all start from the hot junction, the higher ones cutting the circuit at successive points along B, several converging at the cold junction, and the rest cutting the circuit at successive points along A. The heat at the hot junction is converted into electric and magnetic energy, which here moves outwards, since the current is against the E.M.I. Some of this energy converges upon B and A, to be converted

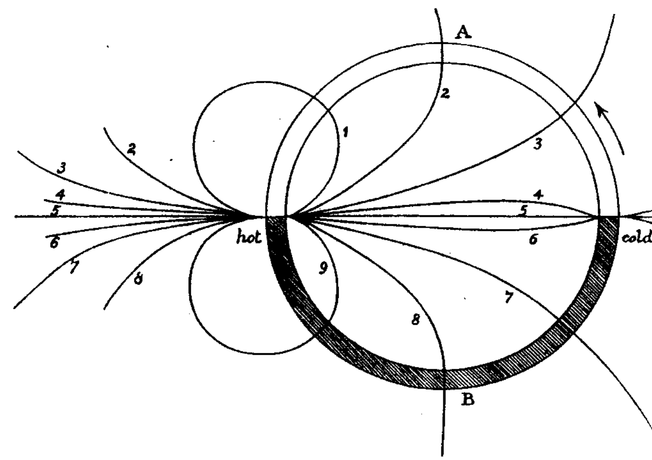

Fig. 4.

into heat, according to Joule's law, and some on the cold junction, there producing the Peltier heating effect.

Let us now suppose that we have a circuit of the same two metals, now all at the same temperature, but with a battery interposed in B, which sends a current in the same direction as before (Fig. 5). Then if C be the junction which was hot, and D that which was cold in the last case, we know that the current will tend to cool C and to heat D. In going from A to B at C there will be a sudden rise of potential, and in going from B to A at D there will be a sudden fall. Then, since the potential falls, as we go with the current along A, there will be a point on A near C which has the same potential as B at the junction. From this point to C, A will have lower potentials, and points with the same potentials will exist on B between C and the battery. Then either the level surfaces passing through C are closed surfaces, cutting A or B, and not passing through the battery at all, or, as seems much more probable, the surfaces from the battery which pass through C cut the circuit in three points in all outside the battery: once somewhere along A, once at C, and once somewhere along B. I have drawn and numbered the surfaces in the figure on this supposition. The heat developed in the parts of the circuit near C will thus be partly supplied from the junction C, where the

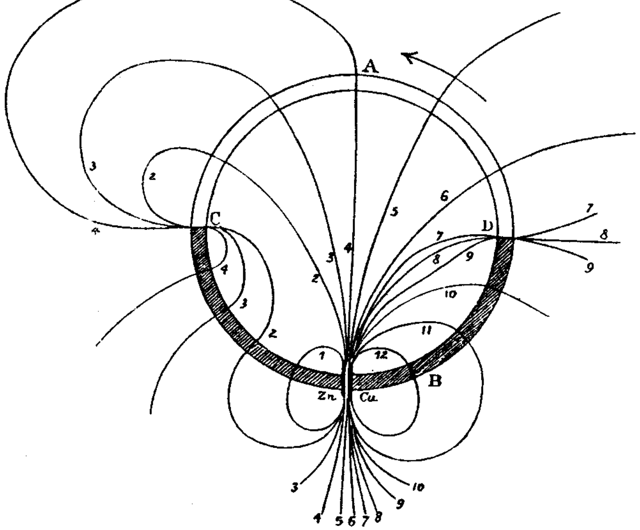

Fig. 5.

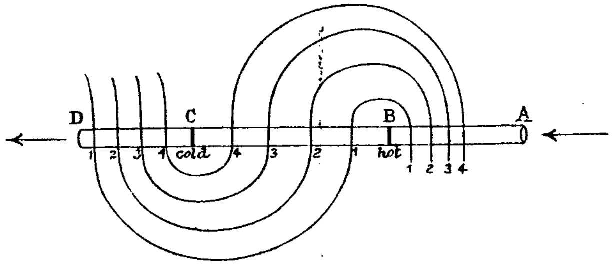

Fig. 6.

current is against the E.M.I. The energy therefore moves out thence, giving a cooling effect.

The Thomson effect may be considered in somewhat the same way. Let us suppose that a metal BC of the iron type, and with temperature falling from B to C, forms part of a circuit between two neutral metals of the lead type AB and CD, Fig. 6, and let us further, for simplicity, suppose that these

metals are each at the neutral temperatures with respect to BC, so that there is no E.M.I. at the junction. If we drive a current from A to D by means of some external E.M.I., say at a junction elsewhere in the circuit, the potential will tend to fall from A to D. But a current in iron from hot to cold cools the metal, that is, the E.M.I. appears to be in opposition to the current, so that the energy moves outwards. The potential, therefore, tends to rise from B to C, and actually will do so if the resistance of BC is negligible compared with that of the rest of the circuit. In this case the level surfaces will probably be somewhat as indicated in Fig. 6, where they are numbered in order, each surface which cuts BC also cutting AB and CD, and the energy moving outwards will come into the circuit again at the parts of AB and CD near the junctions, where it will be transformed once more into heat. If the resistance of BC be gradually increased the fall of potential, according to Ohm's law, will tend to lessen the rise, and fewer surfaces will cut BC. It would seem possible so to adjust matters that the two exactly neutralised each other so that no energy either entered or left BC. In this case we should only have lines of magnetic force round BC, and no other characteristic of a current in that part of the circuit*.

If this is the true account of the Thomson effect it would appear that it should be described not as an absorption of heat or development of heat by the current but rather as a movement of energy outwards or inwards, according as the E.M.I. in the unequally heated metal opposes or agrees with the direction of the current.

(5) *A circuit containing a motor.*

This case closely resembles the third case of a circuit containing a copper-zinc cell, the motor playing a part analogous to that of the surface of contact of the acid with the copper. Let us, for simplicity, suppose that the motor has no internal resistance. When it has no velocity all the level surfaces cut the circuit, and the energy leaving the dynamo or battery is all transformed into heat due to resistance. But if the motor is being worked the current diminishes, the level surfaces begin to converge on the motor and fewer cut the circuit. Some of the energy therefore passes into the motor, and is there transformed into work. As the velocity increases the number cutting the rest of the circuit decreases, for the current diminishes, and, therefore, by Ohm's law, the fall of potential along the circuit is less; and ultimately when the velocity of the motor becomes very great the current becomes very small. In the limit no level surface cuts the circuit, all converging on the motor.

* Perhaps this is only true of the wire as a whole. If we could study the effects in minute portions it is possible that we should find the seat of the E.M.I. due to difference of temperature not the same as that which neutralises it, which is according to Ohm's law. One, for instance, might be between the molecules, the other in their interior, so that there might be an interchange of energy still going on, though no balance remained over to pass out of the wire.

That is, all the energy passes into the motor when it is transformed into work, and the efficiency of the arrangement is perfect, though the rate of doing work is infinitely slow.

(6) *Induced currents.*

It is not so easy to form a mental picture of the movement of energy which takes place when the field is changing and induced currents are created. But we can see in a general way how these currents are accounted for. When there is a steady current in a field there is corresponding to it a definite distribution of energy. If there is a secondary circuit present, so long as the primary current is constant, there is no E.M.I. in the secondary circuit for it is all at the same potential. The energy neither moves into nor out of it, but streams round it somewhat as a current of liquid would stream round a solid obstacle. But if the primary current changes there is a redistribution of the energy in the field. While this takes place there will be a temporary E.M.I. set up in the conducting matter of the secondary circuit, energy will move through it, and some of the energy will there be transformed into heat or work, that is, a current will be induced in the secondary circuit.

(7) *The electromagnetic theory of light.*

The velocity of plane waves of polarised light on the electromagnetic theory may be deduced from the consideration of the flow of energy. If the waves pass on unchanged in form with uniform velocity the energy in any part of the system due to the disturbance also passes on unchanged in amount with the same velocity. If this velocity be v, then the energy contained in unit volume of cubical form with one face in a wave-front will all pass out through that face in $1/v$th of a second. Let us suppose that the direction of propagation is straightforward, while the displacements are up and down; then the magnetic intensity will be right and left. If $\mathfrak{E}$ be the E.M.I. and $\mathfrak{H}$ the M.I. within the volume, supposed so small that the intensities may be taken as uniform through the cube, then the energy within it is $K\mathfrak{E}^2/8\pi + \mu\mathfrak{H}^2/8\pi$. The rate at which energy crosses the face in the wave-front is $\mathfrak{E}\mathfrak{H}/4\pi$ per second, while it takes $1/v$th of a second for the energy in the cube to pass out.

Then $$\frac{\mathfrak{E}\mathfrak{H}}{4\pi v} = \frac{K\mathfrak{E}^2}{8\pi} + \frac{\mu\mathfrak{H}^2}{8\pi}. \qquad \text{(1)}$$

Now, if we take a face of the cube perpendicular to the direction of displacement, and therefore containing the M.I., the line-integral of the M.I. round this face is equal to $4\pi \times$ current through the face. If we denote distance in the direction of propagation from some fixed plane by z, the line-integral of the M.I. is $-\frac{d\mathfrak{H}}{dz}$, while the current, being an alteration of displacement, is $\frac{K}{4\pi}\frac{d\mathfrak{E}}{dt}$.

Therefore $$-\frac{d\mathfrak{H}}{dz} = K\frac{d\mathfrak{E}}{dt}. \qquad \text{(2)}$$

But since the displacement is propagated unchanged with velocity v, the displacement now at a given point will alter in time dt to the displacement now a distance dz behind, where $dz = v\,dt$.

Therefore $$\frac{d\mathfrak{E}}{dt} = -v\frac{d\mathfrak{E}}{dz}. \qquad \text{(3)}$$

Substituting in (2) $$\frac{d\mathfrak{H}}{dz} = Kv\frac{d\mathfrak{E}}{dz},$$

whence $$\mathfrak{H} = Kv\mathfrak{E}, \qquad \text{(4)}$$

the function of the time being zero, since $\mathfrak{H}$ and $\mathfrak{E}$ are zero together in the parts which the wave has not yet reached.

If we take the line-integral of the E.M.I. round a face perpendicular to the M.I. and equate this to the decrease of magnetic induction through the face, we obtain similarly

$$\mathfrak{E} = \mu v\mathfrak{H}. \qquad \text{(5)}$$

It may be noticed that the product of (4) and (5) at once gives the value of v, for dividing out $\mathfrak{E}\mathfrak{H}$ we obtain

$$1 = \mu K v^2$$

or $$v = \frac{1}{\sqrt{\mu K}}.$$

But using one of these equations alone, say (4), and substituting in (1) K for $\mathfrak{H}$ and dividing by $\mathfrak{E}^2$, we have

$$\frac{K}{4\pi} = \frac{K}{8\pi} + \frac{\mu K^2 v^2}{8\pi}$$

or $$1 = \mu K v^2,$$

whence $$v = \frac{1}{\sqrt{\mu K}}.$$

This at once gives us the magnetic energy equal to the electric energy, for

$$\frac{\mu\mathfrak{H}^2}{8\pi} = \frac{\mu K^2 v^2 \mathfrak{E}^2}{8\pi} = \frac{K\mathfrak{E}^2}{8\pi}.$$

It may be noted that the velocity $\frac{1}{\sqrt{\mu K}}$ is the greatest velocity with which the two energies can be propagated together, and that they must be equal when travelling with this velocity. For if v be the velocity of propagation and θ the angle between the two intensities, we have

$$\frac{\mathfrak{E}\mathfrak{H}\sin\theta}{4\pi v} = \frac{K\mathfrak{E}^2}{8\pi} + \frac{\mu\mathfrak{H}^2}{8\pi},$$

or $$v = \frac{2\sin\theta}{\frac{K\mathfrak{E}}{\mathfrak{H}} + \frac{\mu\mathfrak{H}}{\mathfrak{E}}}.$$

The greatest value of the numerator is 2 when θ is a right angle, and the least value of the denominator is $2\sqrt{\mu K}$, when the two terms are equal to each other and to $\sqrt{\mu K}$.

The maximum value of v therefore is $\frac{1}{\sqrt{\mu K}}$, and occurs when $\theta = \frac{\pi}{2}$ and $K\mathfrak{E}^2 = \mu\mathfrak{H}^2$.

The preceding examples will suffice to show that it is easy to arrange some of the known experimental facts in accordance with the general law of the flow of energy. I am not sure that there has hitherto been any distinct theory of the way in which the energy developed in various parts of the circuit has found its way thither, but there is, I believe, a prevailing and somewhat vague opinion that in some way it has been carried along the conductor by the current. Probably Maxwell's use of the term 'displacement' to describe one of the factors of the electric energy of the medium has tended to support this notion. It is very difficult to keep clearly in mind that this 'displacement' is, as far as we are yet warranted in describing it, merely a something with direction which has some of the properties of an actual displacement in incompressible fluids or solids. When we learn that the 'displacement' in a conductor having a current in it increases continually with the time, it is almost impossible to avoid picturing something moving along the conductor, and it then seems only natural to endow this something with energy-carrying power. Of course it may turn out that there is an actual displacement along the lines of electromotive intensity. But it is quite as likely that the electric 'displacement' is only a function of the true displacement, and it is conceivable that many theories may be formed in which this is the case, while they may all account for the observed facts. Mr Glazebrook has already worked out one such theory in which the component of the electric displacement at any point in the direction of x is $\frac{1}{8\pi}\nabla^2\xi$, where ξ is the component of the true displacement (*Phil. Mag.* June 1881). It seems to me then that our use of the term is somewhat unfortunate, as suggesting to our minds so much that is unverified or false, while it is so difficult to bear in mind how little it really means.

I have therefore given several cases in considerable detail of the application of the mode of transfer of energy in current-bearing circuits according to the law given above, as I think it is necessary that we should realise thoroughly that if we accept Maxwell's theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the conductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor and its transformation there into other forms. The current through a seat of so-called electromotive force consists essentially of a divergence of energy from the conductor into the medium. The magnetic lines of force are related to the circuit in the same way throughout, while the lines of electric force are in opposite directions in the two parts of the circuit—with the so-called current in the conductor, against it in the seat of electromotive force. It follows that the total E.M.I. round the circuit with a steady current is zero, or the work done in carrying a unit of positive electricity round the circuit with the current is zero. For work is required to move it against the E.M.I. in the seat of energy, this work sending energy out into the medium, while an equal amount of energy comes in in the rest of the circuit where it is moving with the E.M.I. This mode of regarding the relations of the various parts of the circuit is, I am aware, very different from that usually given, but it seems to me to give us a better account of the known facts.

It may seem at first sight that we ought to have new experimental indications of this sort of movement of energy, if it really takes place. We should look for proofs at points where the energy is transformed into other modifications, that is, in conductors. Now in a conductor, when the field is in a steady state, there is no electromotive intensity, and therefore no motion and no transformation of energy. The energy merely streams round the outside of the conductor, if in motion at all in its neighbourhood. If the field is changing, energy can pass into the conductor, as there may be temporary E.M.I. set up within it, and there will be transformation. But we already know the nature of this transformation, for it constitutes the induced current. Indeed, the fundamental equation describing the motion of energy is only a deduction from Maxwell's equations, which are formed so as to express the experimental facts as far as yet known. Among these are the laws of induction in secondary circuits, and they must therefore agree with the law of transfer. We can hardly hope, then, for any further proof of the law beyond its agreement with the experiments already known until some method is discovered of testing what goes on in the dielectric independently of the secondary circuit.