Wire Polarizers

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1 Problem

In 1861 Fizeau studied transmission of polarized light through a screen with subwavelength scratches [1], and in 1888 Hertz performed similar experiments with radio waves and a grid of wires,[1] which transmited polarized electromagnetic waves, of wavelength λ large compared to the wire spacing d, only if the electric field of the wave was perpendicular to the direction of the wires. Explain these observations.

Discuss also the case of plane conducting sheets with slots, and the applications to this of Babinet's principle.

2 Solution

This solution follows methods perhaps first given by Lord Rayleigh [3, 4] for "wires" being formed by cutting slots in a plane conducting sheet. The case of circular wires was first considered by J.J. Thomson in secs. 359-360 of [5]. A unified discussion of both kinds of "wire" polarizers was given by H. Lamb [6]. A more general theory of the interaction of electromagnetic waves with gratings was given by Rayleigh [9].

Because the effect of the wire polarizer on waves with polarization perpendicular to the wires is small, it has become common so suppose that there is no effect at all in this case based on the approximation (perhaps first made following eq. (1) of [7]) that the incident wave cannot excite currents perpendicular to small wires (see also p. 327 of [8]). We do not make that approximation here, but rather consider this example as a boundary-value problem, following [3, 5, 6].

We restrict our discussion to the case of wire spacing $d < \lambda/2$ and waves of normal incidence, such that the reflected and transmitted waves can only be along the direction of incidence.[2] We will consider only perfectly conducting "wires," either circular or planar strips of radius/half width a (and hence aperture $b = d - a$ between adjacent wires). The centers of the wires lie in the plane $x = 0$ with their axes parallel to the z-axis at positions $y = nd$ for any integer n.[3]

The incident wave propagates in the $+x$-direction with electric and magnetic fields (in Gaussian units)

$$\mathbf{E} = (E_{0y}\,\hat{\mathbf{y}} + E_{0z}\,\hat{\mathbf{z}})\,e^{i(kx-\omega t)}, \qquad \mathbf{B} = (-E_{0z}\,\hat{\mathbf{y}} + E_{0y}\,\hat{\mathbf{z}})\,e^{i(kx-\omega t)}, \tag{1}$$

[1] The waves in Hertz' experiment [2] had $\lambda \approx 50$ cm, the wire spacing was $d = 3$ cm and the wire diameter was $2a = 1$ mm.

[2] We later make the further restriction that $d \ll \lambda$.

[3] This choice of axes anticipates later use of complex functions of the variable $x + iy$.

where $\omega = kc$ supposing that the medium outside the wires is vacuum, in which c is the speed of light. The total fields cannot depend on z, so each scalar component ψ thereof obeys that scalar Helmholtz equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 \psi = 0. \tag{2}$$

The solutions are sums of terms of the form $\psi(x, y, t) = X(x)Y(z)\, e^{-i\omega t}$ where the periodicity of the wires and their symmetry about the plane $y = 0$ indicates that the Y functions have the form $\cos 2n\pi y/d$ for integer $n \geq 0$. Then, the X functions are exponential, $e^{ik_n x}$, where to satisfy the Helmholtz equation (2), $k_0 = \pm k$ and $k_n^2 = k^2 - (2n\pi/d)^2 < 0$ since $d < \lambda/2 = \pi/k$. Writing $k_n = \pm i l_n$ for $n > 0$, we have that

$$l_n^2 = \frac{4n^2\pi^2}{d^2} - k^2 \qquad (n > 0). \tag{3}$$

That is, the functions ψ_n correspond to evanescent waves for $n > 0$, which die out at large $|x|$.

For $x < 0$ the (incident plus reflected) wave has the form

$$\psi(x < 0) = \left[\psi_0\, e^{ikx} + A_0\, e^{-ikx} + \sum_{n=1}^{\infty} A_n\, e^{l_n x} \cos\frac{2n\pi y}{d}\right] e^{-i\omega t}, \tag{4}$$

and for $x > 0$ the (transmitted) wave has the form

$$\psi(x > 0) = \left[B_0\, e^{ikz} + \sum_{n=1}^{\infty} B_n\, e^{-l_n x} \cos\frac{2n\pi y}{d}\right] e^{-i\omega t}. \tag{5}$$

The Fourier coefficients are to be determined by the requirements that ψ and $\partial\psi/\partial x$ be continuous at $x = 0$, and by conditions at the surface of the wires, namely that the tangential component of the electric field and the normal component of the magnetic field vanish at the surface of a perfect conductor.

We now make the further restriction that the wire separation d is small compared to the wavelength λ. In this case,

$$l_n \approx \frac{2n\pi}{d} \qquad (n > 0,\ d \ll \lambda). \tag{6}$$

Even when $d = \lambda/2$ the approximation (6) is not bad. The surface of the wires has $|x| \ll \lambda$, so the component ψ near the wires has the approximate form

$$\psi(x < 0)\, e^{i\omega t} \approx \psi_0 + A_0 + ik(\psi_0 - A_0)x + \sum_{n=1}^{\infty} A_n\, e^{2n\pi x/d} \cos\frac{2n\pi y}{d}, \tag{7}$$

$$\psi(x > 0)\, e^{i\omega t} \approx B_0(1 + ikx) + \sum_{n=1}^{\infty} B_n\, e^{-2n\pi x/d} \cos\frac{2n\pi y}{d}. \tag{8}$$

Close to the wire plane we see that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \psi\, e^{i\omega t} \approx 0, \tag{9}$$

so that $\psi(|x| \ll \lambda)\, e^{i\omega t}$ can be a solution to a two-dimensional electrostatics problem, as noted by J.J. Thomson [5].

2.1 Circular Wires

Following sec. 103 of [10], we consider the 2-dimensional function $u(x, y)$ to be the real part of a function $w(\mathrm{z})$ of the complex variable $\mathrm{z} = x + iy$,

$$\begin{aligned} w &= u + iv = \ln\left(e^{\pi \mathrm{z}/d} - e^{-\pi \mathrm{z}/d}\right) = \ln\left[e^{\pi \mathrm{z}/d}\left(1 - e^{-2\pi \mathrm{z}/d}\right)\right] = \frac{\pi z}{d} + \ln\left(1 - e^{-2\pi \mathrm{z}/d}\right) \\ &= \frac{\pi z}{d} - \sum_{n=1}^{\infty} \frac{e^{-2n\pi \mathrm{z}/d}}{n}, \end{aligned} \tag{10}$$

where the series expansion holds for $0 \leq x < 1$. The real part of w is

$$u(0 \leq x < 1) = \frac{\pi x}{d} - \sum_{n=1}^{\infty} \frac{e^{-2n\pi x/d}}{n} \cos \frac{2n\pi y}{d}, \tag{11}$$

which has the form of eq. (8). To verify that u of eq. (11) is (approximately) constant on small circles about $(x, y) = (0, nd)$ it is helpful to write

$$\begin{aligned} w &= \ln\left(r\, e^{i\theta}\right) = \ln r + i\theta = \frac{1}{2}\ln r^2 + i\theta = \frac{1}{2}\ln[(e^{\pi \mathrm{z}/d} - e^{-\pi \mathrm{z}/d})(e^{\pi \mathrm{z}^\star/d} - e^{-\pi \mathrm{z}^\star/d})] + i\theta \\ &= \frac{1}{2}\ln(e^{\pi(\mathrm{z}+\mathrm{z}^\star)/d} + e^{-\pi(\mathrm{z}+\mathrm{z}^\star)/d} - e^{\pi(\mathrm{z}-\mathrm{z}^\star)/d} - e^{-\pi(\mathrm{z}-\mathrm{z}^\star)/d}) + i\theta \\ &= \frac{1}{2}\ln\left[2\left(\cosh\frac{2\pi x}{d} - \cos\frac{2\pi y}{d}\right)\right] + i\theta \end{aligned} \tag{12}$$

such that

$$u = \frac{1}{2}\ln\left[2\left(\cosh\frac{2\pi x}{d} - \cos\frac{2\pi y}{d}\right)\right] \approx \frac{1}{2}\ln\left[\frac{4\pi^2}{d^2}\left(x^2 + (y - nd)^2\right)\right], \tag{13}$$

where the approximation holds for small x and $y - nd$. *As the thickness of the wires in y approaches d the contours of constant u become elongated in x, so the solution applies to appropriately shaped noncircular wires.*

Equation (13) shows that u is an even function of x (and y), and hence for negative x eq. (11) becomes

$$u(-1 < x \leq 0) = -\frac{\pi x}{d} - \sum_{n=1}^{\infty} \frac{e^{2n\pi x/d}}{n} \cos\frac{2n\pi y}{d}, \tag{14}$$

which has the form of eq. (7).

2.1.1 Electric Field Polarized Parallel to the Wires

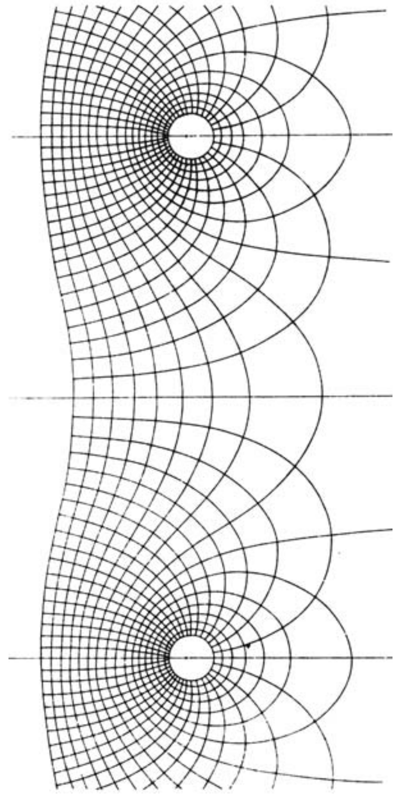
In this case we take $\psi_\parallel = E_z$, and the condition is that $\psi_\parallel = 0$ on the surface of each (circular) wire of radius a.

The forms (11) and (14) suggest that we set $A_n = B_n = A_1/n$ to write eqs. (7)-(8) as

$$\psi_\parallel(x < 0)\, e^{i\omega t} \approx \psi_0 + A_0 + ik(\psi_0 - A_0)x - A_1\left(u + \frac{\pi x}{d}\right), \tag{15}$$

$$\psi_\parallel(x > 0)\, e^{i\omega t} \approx B_0(1 + ikx) - A_1\left(u - \frac{\pi x}{d}\right). \tag{16}$$

The function $\psi_\|$ is of the form shown in plate XII of [10].



Two condition on the remaining unknowns A_0, A_1 and B_0 are that $\psi_\|$ and $\partial\psi_\|/\partial x$ be continuous at $x = 0$, which tells us that

$$B_0 = \psi_0 + A_0, \qquad A_1 = -\frac{ikd}{\pi} A_0. \tag{17}$$

Finally, we require that $\psi_\| = E_z$ vanish at the surface of the wires, and in particular at $(x, y) = (0, a)$, which implies that, recalling eq. (13),

$$\frac{A_0}{\psi_0} = -\frac{1}{1 + \frac{ikd}{2\pi} \ln\left[2\left(1 - \cos\frac{2\pi a}{d}\right)\right]} \equiv -\frac{1}{1 + iC_\| kd}, \qquad \frac{B_0}{\psi_0} = \frac{iC_\| kd}{1 + iC_\| kd} \tag{18}$$

where

$$C_\| = \frac{1}{2\pi} \ln\left[2\left(1 - \cos\frac{2\pi a}{d}\right)\right] \approx \frac{1}{\pi} \ln\frac{2\pi a}{d}, \tag{19}$$

and the approximation holds for wire radius a small compared to the spacing d.

The coefficients of reflected and transmitted wave intensities are

$$R_\| = \left|\frac{A_0}{\psi_0}\right|^2 = \frac{1}{1 + C_\|^2 k^2 d^2}, \qquad T_\| = \left|\frac{B_0}{\psi_0}\right|^2 = \frac{C_\|^2 k^2 d^2}{1 + C_\|^2 k^2 d^2}. \tag{20}$$

For reasonable values of the ratio a/d the quantity $C_\| kd$ is very small when $d \ll \lambda$, and the wave with electric field polarized parallel to the wires is almost completely reflected.

However, there are some oddities in the result (20). First, if a/d is extremely small, $\left|C_\|\right|$ becomes large and the wave is almost completely transmitted. To be in this regime we need

$\ln d/2\pi a \gtrsim d/\lambda$, *i.e.*, $a \lesssim e^{-d/\lambda} d/2\pi$. For $d = \lambda/15 = 3$ cm as in the original experiments of Hertz [2] this implies $a \lesssim 10^{-8} d \approx 10$ Å. *If Hertz had used carbon nanotube rather than copper wires his experiment would not have worked well.*

Also, if $a = d/6$ then $C_\parallel$ is infinite and the transmission coefficient is 1. This corresponds to the case of the wire surfaces following the curves in the figure on the previous page that extend to $x = \infty$, where the (infinitely elongated) "wires" touch. All the wave energy propagates beyond the plane $x = 0$, although we cannot say that the energy is transmitted beyond the "wires."

For use in sec. 2.1.2, we note that the surfaces of the infinitely elongated wires have $\psi_\parallel = \infty$ and minimum $x < 0$ such that $\cosh(2\pi x/d) = 2$, *i.e.*, $x_{\rm min} = -0.21d$.

The case of circular wires with radius $a \lesssim d/2$ is not well treated by the methods used here.

2.1.2 Electric Field Polarized Perpendicular to the Wires

In this case the incident electric field is in the y direction (and the incident magnetic field is in the z direction). The electric field E_y in the plane $x = 0$ can be nonzero, so if we took $\psi_\perp = E_y$ the analysis of sec. 2.1.1 would fail at eq. (18). Instead, we take $\psi_\perp = B_z$.

The condition that the tangential component of the electric field vanish at (and just outside) the surface of the wire also means that according to the fourth Maxwell equation, the tangential component of $\mathbf{\nabla} \times \mathbf{B}$ also vanishes just outside the surface. Thus for the wire centered on $y = 0$, the condition at the surface of the wire $(x, y) = (0, a)$ is that

$$\frac{\partial B_z(0,a)}{\partial y} = \frac{1}{c}\frac{\partial E_x(0,a)}{\partial t} = 0. \tag{21}$$

However, this condition is trivially satisfied by the forms (15)-(16). Instead, we consider the point $(x, y) = (-a, 0)$ on the surface of the wire (if the wire is elongated in x we take $-a$ to be its minimum x coordinate), where $E_y = 0$ and the condition becomes

$$0 = \frac{1}{c}\frac{\partial E_y(-a,0)}{\partial t} = \frac{\partial B_z(-a,0)}{\partial x}\,. \tag{22}$$

If we follow the procedure of sec. 2.1.1 then eq. (15), together with eqs. (13) and (17), only satisfy the condition (22) for $\psi_0 = 0$. To work around this difficulty, Lamb [6] noted that the derivatives of eqs. (11) and (14) with respect to x are

$$\frac{\partial u(0 \le x < 1)}{\partial x} = \frac{\pi}{d} + \frac{2\pi}{d}\sum_{n=1}^{\infty} e^{-2n\pi x/d} \cos\frac{2n\pi y}{d}\,, \tag{23}$$

$$\frac{\partial u(-1 < x \le 0)}{\partial x} = -\frac{\pi}{d} - \frac{2\pi}{d}\sum_{n=1}^{\infty} e^{2n\pi x/d} \cos\frac{2n\pi y}{d}\,. \tag{24}$$

A second trick of Lamb is to take $A_n = A_1 = -B_n$, such that eqs. (7)-(8) can be cast in the forms

$$\psi_\perp(x<0)\, e^{i\omega t} \approx \psi_0 + A_0 + ik(\psi_0 - A_0)x - A_1\left(\frac{d}{2\pi}\frac{\partial u}{\partial x} + \frac{1}{2}\right), \tag{25}$$

$$\psi_\perp(x>0)\, e^{i\omega t} \approx B_0(1 + ikx) - A_1\left(\frac{d}{2\pi}\frac{\partial u}{\partial x} - \frac{1}{2}\right). \tag{26}$$

Continuity of $\psi_\perp$ and $\partial_\perp\psi/\partial x$ at $x = 0$ tell us that

$$B_0 = \psi_0 + A_0 - A_1 = \psi_0 - A_0, \qquad \text{and so} \qquad A_1 = 2A_0. \tag{27}$$

Finally, noting that

$$\frac{\partial^2\psi_\perp(-a,0)}{\partial x^2} = -\frac{2\pi^2}{d^2(\cosh\frac{2\pi a}{d} - 1)} \equiv -\frac{1}{D^2}\,, \tag{28}$$

the condition (22) yields

$$\frac{A_0}{\psi_0} = \frac{iC_\perp kd}{1 + iC_\perp kd}\,, \qquad \frac{B_0}{\psi_0} = \frac{1}{1 + iC_\perp kd}\,, \tag{29}$$

where

$$C_\perp = \frac{\pi D^2}{d^2} = \frac{\cosh\frac{2\pi a}{d} - 1}{2\pi} \approx \frac{\pi a^2}{d^2}\,, \tag{30}$$

and the approximation holds when $a \ll d$.

The coefficients of reflected and transmitted wave intensities are

$$R_\perp = \left|\frac{A_0}{\psi_0}\right|^2 = \frac{C_\perp^2 k^2 d^2}{1 + C_\perp^2 k^2 d^2}\,, \qquad T_\perp = \left|\frac{B_0}{\psi_0}\right|^2 = \frac{1}{1 + C_\perp^2 k^2 d^2}\,. \tag{31}$$

When the wire radius a is small compared to the spacing d, the reflected wave is very small and essentially all wave is transmitted, for the case of electric field polarized perpendicular to the wires.

The maximum value of a is $0.21d$, corresponding to infinitely elongated "wires," as considered at the end of sec. 2.1.1. In this case $\cosh(2\pi a/d) = 2$ and $C_\perp = 1/2\pi$. Even here the reflected intensity is very small.

2.1.3 Surface Charges and Currents

The analysis above involved matching fields at boundaries, with no mention of the charges or currents on the surface of the wires. It has become popular to suppose it is obvious what these distributions are, and that the reflected and transmitted waves can readily be deduced therefrom. However, it seems to this author that only after the fields are known can the charge and current distributions be deduced, as done below when $a \ll d$ such that the above forms hold for circular wires.

We use cylindrical coordinates (r, θ, z). The surface charge and current densities σ and $\mathbf{K}$ on the wires of radius a are given by

$$\sigma(\theta) = \frac{E_r(a,\theta)}{4\pi} = \frac{i}{4\pi ak}\frac{\partial B_z(a,\theta)}{\partial\theta}\,, \quad \mathbf{K}(\theta) = \frac{c}{4\pi}\hat{\mathbf{r}}\times\mathbf{B}(a,\theta) = -\frac{ic}{4\pi k}\hat{\mathbf{r}}\times(\boldsymbol{\nabla}\times\mathbf{E}(a,\theta)). \tag{32}$$

When the electric field is polarized parallel to the wires, $\mathbf{E} = E_z\,\hat{\mathbf{z}}$ and

$$\sigma_\parallel = 0, \qquad K_{\parallel,z} = \frac{ic}{4\pi k}\frac{\partial E_z(a,\theta)}{\partial r}\,, \tag{33}$$

where E_z near the wires follows from eqs. (13)-(19),

$$\frac{E_z}{E_{0z}} e^{i\omega t} \approx 1 + ikr\cos\theta - \frac{1}{1 + iC_\parallel kd}\left(1 + \frac{ikd}{\pi}\ln\frac{2\pi r}{d}\right), \tag{34}$$

for $a \ll d \ll \lambda$, with $C_\parallel \approx (1/\pi)\ln 2\pi a/d$. Hence, the surface current is

$$K_{\parallel,z} \approx \frac{c}{4\pi}E_{0z}\left(\frac{d}{\pi a}\frac{1}{1 + iC_\parallel kd} - \cos\theta\right) e^{-i\omega t}. \tag{35}$$

So long as a/d is not extraordinarily small, the factor $\left|C_\parallel kd\right|$ is small and

$$K_{\parallel,z} \approx \frac{c}{4\pi}E_{0z}\frac{d}{\pi a} e^{-i\omega t}. \tag{36}$$

This current exists only on the wires, of circumference $2\pi a$, and which occupy the fraction $2a/d$ of the plane $x = 0$. Hence, the current (36) is equivalent to a uniform sheet of current $2cE_{0z}/4\pi$ over that entire plane, which is exactly the current that would exist on a conducting mirror at $x = 0$.

When the electric field is polarized perpendicular to the wires, the magnetic field is parallel to them, and

$$\sigma_\perp = \frac{i}{4\pi ak}\frac{\partial B_z(a,\theta)}{\partial\theta}, \qquad K_{\perp,\theta} = -\frac{c}{4\pi}B_z(a,\theta), \tag{37}$$

where B_z at $r = a$ follows from eqs. (25)-(30),

$$\begin{aligned}\frac{B_z}{E_{0y}} e^{i\omega t} &\approx 1 + ika\cos\theta - \frac{iC_\perp kd}{1 + iC_\perp kd}\left(\frac{d^2}{\pi^2 a^2} + ika\cos\theta\right) \\ &\approx 1 - \frac{kd}{\pi} + ika\cos\theta,\end{aligned} \tag{38}$$

for $a \ll d \ll \lambda$, with $C_\perp \approx \pi a^2/d^2$. Hence, the surface charge and current densities are

$$\sigma_\perp \approx \frac{1}{4\pi}E_{0y}\sin\theta\, e^{-i\omega t}, \qquad K_{\perp,\theta} \approx -\frac{c}{4\pi}E_{0y}\left(1 - \frac{kd}{\pi} + ika\cos\theta\right) e^{-i\omega t}, \tag{39}$$

which obey the continuity equation

$$\boldsymbol{\nabla}\cdot\mathbf{K}_\perp = \frac{1}{a}\frac{\partial K_{\perp,\theta}}{\partial\theta} = -\frac{\partial\sigma_\perp}{\partial t} = ikc\,\sigma_\perp. \tag{40}$$

In sum, when the electric field is polarized perpendicular to the wires the currents are roughly those expected on a mirror ($2cE/4\pi$), and the reflection is small because these currents occupy only fraction $2a/d$ of the plane $x = 0$. In contrast, when the electric field is polarized parallel to the wires the currents are enhanced by a factor $d/2\pi a$ over that on a mirror, such that the average current density on the plane $x = 0$ is the same as that for a mirror and the reflection is essentially total.

The case of polarization perpendicular to the wires obeys the approximation of optical diffraction in that the wires, in effect, remove from the transmitted wave the (small) portion of the incident wave that hits them. But for polarization parallel to the wires the latter are "super-reflectors" with enhanced electrical width of d rather than their physical width $2a$, and the near-zero transmission is unexpected from the viewpoint of optical diffraction.

This behavior is somewhat contrary to popular claims [7, 8], which imply that the case of polarization parallel to the wires is easy to understand, while the currents are unusually small when the polarization if perpendicular to the wires.

2.2 Conducting Plane with Slots

Lamb [6] realized that the function $\psi(x, y)$ of eqs. (7)-(8) appropriate for "wires" made by cutting slots in a conducting sheet is related to the real part of the complex function $w(\mathrm{z} = x + iy) = u + iv$ defined by[4]

$$\cosh w = \mu \cosh \frac{\pi \mathrm{z}}{d}, \tag{41}$$

for (real) $\mu > 1$. Then,

$$\cosh u \cos v = \mu \cosh \frac{\pi x}{d} \cos \frac{\pi y}{d}, \qquad \sinh u \sin v = \mu \sinh \frac{\pi x}{d} \sin \frac{\pi y}{d}. \tag{42}$$

The surfaces with $u = 0$ lie only the plane $x = 0$, and have y-values consistent with $|\cos v| = \mu\,|\cos \pi y/d| \le 1$. These are strips centered on $y = (n + 1/2)d$ for any integer n. Denoting the "radius" (half width) of a strip by a, one edge of the strip centered on $(x, y) = (0, d/2)$ is at $(0, d/2 - a)$ where

$$\frac{1}{\mu} = \cos\left[\frac{\pi}{d}\left(\frac{d}{2} - a\right)\right] = \sin \frac{\pi a}{d}, \qquad a = \frac{d}{\pi} \sin^{-1} \frac{1}{\mu}. \tag{43}$$

The functions u and v have contours as sketched in the figure [6] on the next page for $\mu = 1.2$, which implies that $a/d = 0.62$. The origin is where three contours meet at a point near the bottom of the figure. Away from the strips, lines of constant u are nearly parallel to the vertical (y) axis.

The function u is even in x and in y (such that $\partial u(0, y)/\partial x = 0$).

For large x, $v = \pi y/d$ for $|y| < d/2$, and $u = \pi x/d + \ln \mu = \pi x/d - \ln 1/\mu$, as inferred from eq. (42).

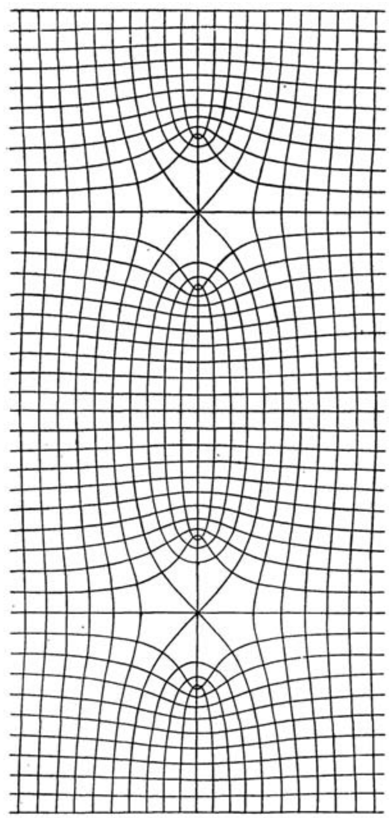
We do not have a closed-form expression for u, but it satisfies Laplace's equation in two dimensions, eq. (9), and has period d in y, which permits a Fourier expansion of the form

$$u(x > 0) = \frac{\pi x}{d} - \ln \frac{1}{\mu} + \sum_{n=1}^{\infty} C_n\, e^{-2n\pi x/d} \cos \frac{2n\pi y}{d}, \tag{44}$$

[4]Lamb indicated that he came to eq. (41) via considerations of the Schwarz transformation, but note also that a strip is the limit of an elliptical cylinder with major axis of length equal to the separation of the foci, and that the relations $x = \cosh u \cos v$ and $y = \sinh u \sin v$ correspond to a transformation to elliptic cylindrical coordinates (u, v). For use of these coordinates in the problem of scattering by small elliptical cylinders and strips, see, for example, sec. 3 of [11].

$$u(x<0) = -\frac{\pi x}{d} - \ln\frac{1}{\mu} + \sum_{n=1}^{\infty} C_n\, e^{2n\pi x/d} \cos\frac{2n\pi y}{d}, \tag{45}$$

since u is even in both x and y. The Fourier coefficients C_n are displayed in [6], but will not be needed here.



2.2.1 Electric Field Polarized Parallel to the Strips

As in sec. 2.1.1 we take $\psi_\parallel = E_z$, and require that $\psi_\parallel = 0$ on the strips.

The forms of eqs. (7)-(8) and (44)-(45) suggest that we set $A_n = B_n = A_1 C_n$ to write

$$\psi_\parallel(x<0)\, e^{i\omega t} \approx \psi_0 + A_0 + ik(\psi_0 - A_0)x + A_1\left(u + \frac{\pi x}{d} + \ln\frac{1}{\mu}\right), \tag{46}$$

$$\psi_\parallel(x>0)\, e^{i\omega t} \approx B_0(1+ikx) + A_1\left(u - \frac{\pi x}{d} + \ln\frac{1}{\mu}\right). \tag{47}$$

Continuity of $\psi_\parallel$ and $\partial\psi_\parallel/\partial x$ across the apertures in the plane $x = 0$ implies that

$$B_0 = \psi_0 + A_0, \qquad A_1 = \frac{ikd}{\pi} A_0. \tag{48}$$

Finally, $\psi_\parallel$ must vanish on the strips (where $u = 0$), which implies that

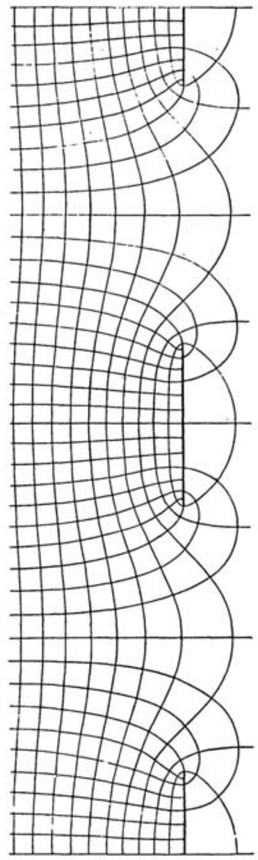
$$\frac{A_0}{\psi_0} = -\frac{1}{1 + iC_\parallel kd}, \qquad \frac{B_0}{\psi_0} = \frac{iC_\parallel kd}{1 + iC_\parallel kd} \tag{49}$$

where

$$C_{\|} = \frac{1}{\pi} \ln \frac{1}{\mu} = \frac{1}{\pi} \ln \sin \frac{\pi a}{d} \approx \frac{1}{\pi} \ln \frac{\pi a}{d}\,, \tag{50}$$

and the approximation holds for strip "radius" a small compared to the spacing d.

The function ψ is of the form shown below [6].



The coefficients of reflected and transmitted wave intensities are

$$R_{\|} = \left|\frac{A_0}{\psi_0}\right|^2 = \frac{1}{1 + k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}\,, \quad T_{\|} = \left|\frac{B_0}{\psi_0}\right|^2 = \frac{k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}{1 + k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}\,. \tag{51}$$

When the strips touch, $a = d/2$, $C_{\|} = 0$, and the wave is totally reflected. *This limit is better treated by the present techniques for the case of wire strips than for circular wires.*

When a/d is extraordinarily small, $\left|C_{\|}\right|$ becomes very large and the wave is totally transmitted.

For a/d small such that $\left|C_{\|}\right|$ is small compared to kd, the wave is largely reflected.

As is to be expected, the behavior for an array of conducting strips is qualitatively similar to that for circular wires.

2.2.2 Electric Field Polarized Perpendicular to the Strips

For perpendicular electric polarization we again associate the scalar function $\psi_\perp$ with the magnetic field B_z. The condition that E_x vanish as the surface of the strips tells us that

$$\frac{\partial B_z(0, y_{\rm strip})}{\partial y} = \frac{1}{c}\frac{\partial E_x(0,a)}{\partial t} = 0. \tag{52}$$

We can again relate $\psi_\perp$ to u of eqs. (42) and (44)-(45), but here the strips are taken to be located where the apertures were for the case of parallel polarization, centered on $y = nd$. This choice is possible because $\partial u/\partial x = 0$ everywhere on the plane $x = 0$. Now the "radius" (half width) a of the strips is given by

$$\frac{1}{\mu} = \cos\frac{\pi a}{d}\,, \qquad a = \frac{d}{\pi}\cos^{-1}\frac{1}{\mu}\,. \tag{53}$$

The forms of eqs. (7)-(8) and (44)-(45) permit us to set $A_n = -B_n = A_1 C_n$, leading to

$$\psi_\perp(x<0)\,e^{i\omega t} \approx \psi_0 + A_0 + ik(\psi_0 - A_0)x + A_1\left(u + \frac{\pi x}{d} + \ln\frac{1}{\mu}\right), \tag{54}$$

$$\psi_\perp(x>0)\,e^{i\omega t} \approx B_0(1+ikx) - A_1\left(u - \frac{\pi x}{d} + \ln\frac{1}{\mu}\right). \tag{55}$$

Continuity of $\psi_\perp$ and $\partial\psi_\perp/\partial x$ across the apertures in the plane $x = 0$ is possible for the choice $A_n = -B_n$ in that both u and $\partial u/\partial x$ are zero there, having identified the strips in this case with the apertures of the previous one. Hence,

$$B_0 = \psi_0 - A_0, \qquad A_1 = -\frac{A_0}{\ln\frac{1}{\mu}}\,. \tag{56}$$

Finally, $\partial\psi_\perp/\partial x$ must vanish on the strips (where $\partial u/\partial x = 0$), which implies that

$$\frac{A_0}{\psi_0} = \frac{iC_\perp kd}{1 + iC_\perp kd}\,, \qquad \frac{B_0}{\psi_0} = \frac{1}{1 + iC_\perp kd}\,, \tag{57}$$

where

$$C_\perp = \frac{1}{\pi}\ln\frac{1}{\mu} = \frac{1}{\pi}\ln\cos\frac{\pi a}{d} \approx -\frac{\pi a^2}{2d^2}\,, \tag{58}$$

and the approximation holds for strip "radius" a small compared to the spacing d.

The coefficients of reflected and transmitted wave intensities are

$$R_\perp = \left|\frac{A_0}{\psi_0}\right|^2 = \frac{k^2d^2\ln^2[\cos(\pi a/d)]/\pi^2}{1 + k^2d^2\ln^2[\cos(\pi a/d)]/\pi^2}\,, \quad T_\perp = \left|\frac{B_0}{\psi_0}\right|^2 = \frac{1}{1 + k^2d^2\ln^2[\cos(\pi a/d)]/\pi^2}\,. \tag{59}$$

When the strips touch, $a = d/2$, $C_\perp \to -\infty$, and the wave is totally reflected. *Again, this limit is better treated by the present techniques for the case of wire strips than for circular wires.*

For small a/d, $C_\perp$ is very small and the wave is largely transmitted.

2.2.3 Babinet's Principle

In optics, Babinet's principle [12] is that the sum of the wave transmitted through a screen and that through its "complement" is just the incident wave (as if no screen were present).

For the case of a conducting plane with (nonconducting) slots of "radius"/half width a and spacing d, the complementary screen has slots of "radius" $d/2 - a$. The transmission coefficients for electric polarization parallel and perpendicular to the slots are given by eqs. (50)-(51) and (58)-(59),

$$T_{\|}(a) = \frac{C_{\|}^2(a) k^2 d^2}{1 + C_{\|}^2(a) k^2 d^2}, \qquad C_{\|}(a) = \frac{1}{\pi} \ln \sin \frac{\pi a}{d}, \qquad C_{\|}(d/2 - a) = \frac{1}{\pi} \ln \cos \frac{\pi a}{d}, \tag{60}$$

$$T_{\perp}(a) = \frac{1}{1 + C_{\perp}^2(a) k^2 d^2}, \qquad C_{\perp}(a) = \frac{1}{\pi} \ln \cos \frac{\pi a}{d}, \qquad C_{\perp}(d/2 - a) = \frac{1}{\pi} \ln \sin \frac{\pi a}{d}. \tag{61}$$

We see that $T_{\|}(a) + T_{\|}(d/2 - a) \neq 1 \neq T_{\perp}(a) + T_{\perp}(d/2 - a)$, contrary to the suggestion of Babinet. However,

$$T_{\|}(a) + T_{\perp}(d/2 - a) = 1 = T_{\perp}(a) + T_{\|}(d/2 - a). \tag{62}$$

Thus, we obtain the electromagnetic version of Babinet's principle, that the sum of the transmission coefficients of waves of orthogonal polarizations through complementary screens is unity. This was first stated by Booker [13], who went on to develop a theory of complementary planar antennas in which the source voltage is applied parallel to a thin strip or perpendicular to a thin slot.[5] *In Russia, the electromagnetic version of Babinet's principle is attributed to Leontovich [14].*

From eqs. (47)-(49) and (55)-(57) we see that

$$\left[\psi_{\|}(x > 0, a) + \psi_{\perp}(x > 0, d/2 - a)\right] e^{i\omega t} \approx \psi_0 (1 + ikx), \tag{63}$$

so that

$$\psi_{\|}(x > 0, a) + \psi_{\perp}(x > 0, d/2 - a) = \psi_0 \, e^{i(kx - \omega t)}, \tag{64}$$

and hence Babinet's principle also implies that the sum of the transmitted wave for parallel polarization of a screen and for perpendicular polarization of the complementary screen equals the incident wave.

The methods used in sec. 2.1 do not apply well to circular wires as a approaches $d/2$, and the results (19)-(20) and (30)-(31) do not illustrate Babinet's principle.

2.2.4 Screens with Slits of Half Width a and Spacing d

For completeness, I record the reflection and transmission coefficients for conducting screens with slits of half width a and spacing d. These are, of course, the coefficients of eqs. (51) and (59) with $a \to d/2 - a$,

$$R_{\|}^{\rm screen} = \frac{1}{1 + k^2 d^2 \ln^2[\cos(\pi a/d)]/\pi^2}, \qquad T_{\|}^{\rm screen} = \frac{k^2 d^2 \ln^2[\cos(\pi a/d)]/\pi^2}{1 + k^2 d^2 \ln^2[\cos(\pi a/d)]/\pi^2}, \tag{65}$$

[5] Towards the end of [3] Rayleigh considered the relation between transmission and reflection of complementary screens, but in a scalar theory corresponding only to the electric field polarized perpendicular to the directions of the slits/strips.

$$R_\perp^{\rm screen} = \frac{k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}{1 + k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}\,, \qquad T_\perp^{\rm screen} = \frac{1}{1 + k^2 d^2 \ln^2[\sin(\pi a/d)]/\pi^2}\,. \tag{66}$$

For $a \ll d$, $T_\parallel^{\rm screen} \approx k^2a^2/(4 + k^2a^2)$, which is near unity for $ka \gg 1$ even though the slits are very narrow, while $T_\parallel^{\rm screen}$ is very small for $ka \ll 1$. For $a \ll d$, $T_\perp^{\rm screen} \approx 1/[1 + k^2d^2 \ln^2(d/\pi a)/\pi^2])$, which is small for $kd \gg 1$, but is large for $kd \ll 1$ and a/d not too small; while for a/d extremely small the transmission again becomes large.

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