

Power Received by a Small Antenna

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1 Problem

Deduce an approximate expression for the power received by a small antenna with a load that include a resistance R (as well as a possible reactance) when the antenna is in a linearly polarized incident plane wave of wavelength λ and (time-average) power per unit area P_{in} . Show that the maximum power received is approximately

$$P_{\text{max}} \approx P_{\text{in}} \frac{\lambda^2}{4\pi}, \quad (1)$$

independent of the physical size of the small antenna.

2 Solution

We first note that the (time-average) incident power per unit area, assuming that the medium surrounding the antenna has unit relative permittivity and unit relative permeability, can be written as

$$P_{\text{in}} = \langle S \rangle = \frac{1}{2} E_0 H_0 = \frac{E_0^2}{2Z_0}, \quad (2)$$

where c is the speed of light in vacuum, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector, E_0 and H_0 are the amplitudes of the incident electric and magnetic fields, and $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377 \, \Omega$ ($= 4\pi/c$ in Gaussian units).

We also note that the small antennas are generally well described by their electric and magnetic dipole moments.¹ So, we consider here the cases of a small linear dipole antenna, for which only its electric dipole moment is significant, and a small loop antenna, for which only its magnetic dipole moment is significant.

2.1 Effective Height and Effective Area

The present analysis builds on that presented in [2]. So far as the receiver circuit is concerned, the antenna can be considered as a two-terminal device which can be characterized, according to Thévenin, by a voltage source V_{oc} and a series impedance Z_A .

The strength of the voltage source is the (open-circuit) voltage across the terminals of the antenna when it is not connected to anything. We can write the open-circuit voltage in terms of an effective height h_{eff} as

$$V_{\text{oc}} = E_0 h_{\text{eff}}, \quad (3)$$

where E_0 is the amplitude of the incident electric field.

¹An interesting exception is a small, counterwound, helical, toroidal antenna [1].

The impedance Z_A equals $V_{\text{on}}/I_{\text{sc}}$ where I_{sc} is the current between the terminals of the antenna when they are short-circuited. The short-circuit current is difficult to estimate accurately by analytic techniques, but for a short antenna it is almost purely out of phase with respect to the incident field. Then, the impedance Z_A is almost purely imaginary, corresponding to the large capacitive reactance of the gap between the antenna terminals. The impedance Z_A relevant to a receiving antenna is the same as the terminal impedance of the antenna when used for transmission, according to an antenna reciprocity theorem. For transmitting antennas the real part of the impedance can be written $R_{\text{Ohmic}} + R_{\text{rad}}$, where R_{Ohmic} is the effective resistance of the antenna due to the finite conductivity of its conductors, and R_{rad} is the so-called radiation resistance of the antenna.

If the load on the receiving antenna is described by an impedance $Z_L = R_L + iX_L$, then the (complex) current I through the load has amplitude

$$I = \frac{V_{\text{oc}}}{Z_A + Z_L}, \quad (4)$$

and the power delivered into the load resistor R_L is

$$\begin{aligned} P &= \frac{1}{2} |I|^2 R = \frac{V_{\text{oc}}^2}{2} \frac{R_L}{|Z_A + Z_L|^2} = \frac{E_0^2 h_{\text{eff}}^2}{2} \frac{R}{|Z_A + Z_L|^2} \\ &= P_{\text{in}} \frac{R_L Z_0}{|Z_A + Z_L|^2} h_{\text{eff}}^2 \equiv P_{\text{in}} A_{\text{eff}}, \end{aligned} \quad (5)$$

where

$$A_{\text{eff}} = \frac{R_L Z_0}{|Z_A + Z_L|^2} h_{\text{eff}}^2. \quad (6)$$

A_{eff} is the **effective area** of the antenna system.

The process of maximizing the effective area of the receiving antenna is called **matching**. A first step is to make the reactance X_L of the load equal and opposite to the reactance X_A of the antenna. When this is done, the effective area becomes

$$A_{\text{eff}} = \frac{R_L Z_0}{(R_{\text{Ohmic}} + R_{\text{rad}} + R_L)^2} h_{\text{eff}}^2 \quad (\text{matching reactance}). \quad (7)$$

It is now clear (if it wasn't before) that the antenna will perform better if the effective Ohmic resistance of its conductors is negligible, which we will assume to be the case from now on. Then,

$$A_{\text{eff}} = \frac{R_L Z_0}{(R_{\text{rad}} + R_L)^2} h_{\text{eff}}^2 \quad (\text{matching reactance, } R_{\text{Ohmic}} \ll R_{\text{rad}}). \quad (8)$$

Finally, we maximize the effective area by choosing a load resistor R_L equal in value to the radiation resistance R_{rad} . This may not be practical, but in principle we obtain

$$A_{\text{eff,max}} = \frac{Z_0}{4R_{\text{rad}}} h_{\text{eff}}^2 \quad (\text{matching reactance, } R_{\text{Ohmic}} \ll R_L = R_{\text{rad}}). \quad (9)$$

2.2 Small Linear Dipole Antenna

As shown, for example, in sec. 2 of [2] the effective height of a small linear antenna is

$$h_{\text{eff}} = h \quad (\text{small linear antenna}), \quad (10)$$

where the length of each of its two arms is $h \ll \lambda$. The radiation resistance of a small linear dipole antenna is (see, for example, p. 192 of [3])

$$R_{\text{rad}} = \frac{2\pi}{3} Z_0 \left(\frac{h}{\lambda} \right)^2 \quad (\text{small linear antenna}). \quad (11)$$

Hence, according to eq. (9) the maximum effective area of a small linear dipole antenna is

$$A_{\text{eff,max}} = \frac{Z_0}{4R_{\text{rad}}} h_{\text{eff}}^2 = \frac{3\lambda^2}{8\pi} \quad (\text{matching inductor, } R_{\text{Ohmic}} \ll R_L = R_{\text{rad}}), \quad (12)$$

noting that the reactance of a small linear antenna is capacitive, so the matching element must be an inductor.

The result (12) is often written to include a factor of D , the **directivity** of the antenna, defined to be the maximum of the angular function $f(\theta, \phi)$ such that the angular distribution of the power radiated by the antenna (when used as a transmitter) is

$$P(\theta, \phi) = \frac{P_{\text{total}}}{4\pi} f(\theta, \phi). \quad (13)$$

For a small dipole antenna (either linear or loop), the function f is $(3/2) \sin^2 \theta$ for polar angle θ to the relevant axis. Hence, $D = 3/2$ is the directivity for a small dipole antenna, and we can write

$$A_{\text{eff,max}} = D \frac{\lambda^2}{4\pi} = \pi \tilde{\lambda}^2 D \quad (\text{matching inductor, } R_{\text{Ohmic}} \ll R_L = R_{\text{rad}}), \quad (14)$$

where $\tilde{\lambda} = \lambda/2\pi$ is the reduced wavelength.

2.3 Small Dipole Loop Antenna

The radiation resistance of a small loop antenna of radius h is (see, for example, p. 192 of [3])

$$R_{\text{rad}} = \frac{8\pi^5}{3} Z_0 \left(\frac{h}{\lambda} \right)^4 \quad (\text{small loop antenna}). \quad (15)$$

According to Faraday's law, the open-circuit voltage for a small loop antenna equals the time rate of change of the magnetic flux through the loop,

$$V_{\text{oc}} = \pi h^2 \omega B_0 = \frac{2\pi^2 h^2}{\lambda} E_0, \quad (16)$$

so the effective height of a small loop antenna is

$$h_{\text{eff}} = \frac{2\pi^2 h^2}{\lambda} \quad (\text{small loop}). \quad (17)$$

Thus, the maximum effective area of the small loop antenna follows from eq. (14) as

$$A_{\text{eff,max}} = \frac{3\lambda^2}{2\pi} = \frac{D\lambda^2}{4\pi} \quad (\text{matching capacitor, } R_{\text{Ohmic}} \ll R = R_{\text{rad}}), \quad (18)$$

recalling that the directivity of a small loop (dipole) antenna is $3/2$, and that since the reactance of a loop is inductive the matching element must be a capacitor. This result is the same as (14) for a small linear dipole antenna.

3 An Effective-Area Theorem

The results (14) and (18) are examples of the fact that for any antenna,

$$A_{\text{eff,max}} = \frac{D\lambda^2}{4\pi} \quad (\text{matching element, } R_{\text{Ohmic}} \ll R = R_{\text{rad}}), \quad (19)$$

such that the maximum power received by the load resistor is $P_{\text{max}} = A_{\text{eff,max}}P_{\text{in}}$.

If it is desired that the effective area be large compared to λ^2 , then the directivity D must be large. This is consistent with the law of diffraction that if a beam is to have a characteristic angular spread of θ (directivity $D \approx 1/\theta^2$), then its narrowest cross section has radius $\approx \lambda/\theta$, and hence area $\approx D\lambda^2$.

We show that eq. (19) describes the maximum effective area for any antenna, arguing from an antenna reciprocity theorem (Appendix A.6).

Consider any two antennas, labeled i and j , that are separated by a distance r such that each antenna is in the far zone of the other. Each antenna is oriented such that the other is at the maximum of the angular distributed of power radiated when the antenna is used as a transmitter. Each antenna includes a matching network such that the terminal impedance of the antenna plus network is R_{rad} , the (real) radiation resistance of the antenna.

If antenna j is used as a receiver, the open-circuit (no load) voltage $V_{\text{oc},j}$ induced across its terminals by the radiation from antenna i with drive current I_i is related to the open-circuit voltage $V_{\text{oc},i}$ induced across the terminals of antenna i (when used as a receiver) by the radiation from antenna j (when used as a transmitter with drive current I_j) according to the reciprocity relation (Appendix A)

$$\frac{V_{\text{oc},j}}{I_i} = \frac{V_{\text{oc},i}}{I_j}. \quad (20)$$

The maximum time-average power received by a load resistor R_j connected to antenna j when it is used as a receiver occurs for $R_j = R_{\text{rad},j}$ (and the receiver includes a matching network whose reactance cancels that of the antenna itself), as discussed in sec. 2.1,

$$P_{j,\text{receive,max}} = \frac{V_{\text{oc},j}^2}{8R_{\text{rad},j}}. \quad (21)$$

The time-average power per unit area incident on antenna j due to current $I_i e^{-i\omega t}$ delivered to the terminals of antenna i is

$$\frac{dP_{ij}}{dA} = \frac{D_i}{4\pi r^2} P_{i,\text{transmit}} = \frac{D_i}{8\pi r^2} R_{\text{rad},i} I_i^2, \quad (22)$$

where D_i is the directivity of antenna i . The (maximum) effective area of antenna j is the ratio of the (maximum) power received to the incident power per unit area,

$$A_{\text{eff,max},j} = \frac{P_{j,\text{receive,max}}}{dP_{ij}/dA} = \frac{V_{\text{oc},j}^2}{8R_{\text{rad},j}} \frac{8\pi r^2}{D_i R_{\text{rad},i} I_i^2} \quad (23)$$

Thus,

$$A_{\text{eff,max},j} D_i = \frac{V_{\text{oc},j}^2}{I_i^2} \frac{\pi r^2}{R_{\text{rad},i} R_{\text{rad},j}} = \frac{V_{\text{oc},i}^2}{I_j^2} \frac{\pi r^2}{R_{\text{rad},i} R_{\text{rad},j}} = A_{\text{eff,max},i} D_j, \quad (24)$$

using the reciprocity relation (20). Hence, the ratio $A_{\text{eff,max}}/D$ is the same for any two antennas. From secs. 2.1 and 2.2 we see that this ratio is $\lambda^2/4\pi$, which confirms the general validity of eq. (19).

This section follows sec. 2.14 of [4], which is the earliest discussion of the effective-area theorem that I have found. However, the theorem is likely much older. Rayleigh discusses the effective area of an acoustic “resonator” in [5] and reports a value close to $\lambda^2/4\pi$.

A Appendix: Reciprocity Theorems

A.1 Green’s Reciprocation Theorem for Electrostatics

The first reciprocity theorem is due to Green (1828, p. 39 of [6]), which states that if a set $\{i\}$ of fixed conductors is at potentials V_i when carrying charges Q_i , and at potentials V'_i when carrying charges Q'_i , then

$$\sum_i V_i Q'_i = \sum_i V'_i Q_i. \quad (25)$$

To see this, we label the 3-dimensional potential distribution associated with charges Q_i by $\Phi(\mathbf{r})$, and that associated with charges Q'_i by Φ' . The space outside the conductors is charge free and with relative dielectric constant $\epsilon = 1$.

We invoke Green’s theorem (p. 23 of [6])

$$\int (\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi) d\text{Vol} = \oint (\Phi \nabla \Phi' - \Phi' \nabla \Phi) \cdot d\mathbf{S}, \quad (26)$$

where we take the bounding surface S to be that of the set of conductors. In the charge-free space outside the conductors we have $\nabla^2 \Phi = 0 = \nabla^2 \Phi'$, and the conductors are equipotentials with $\Phi = V_i$ and $\Phi' = V'_i$ on conductor i , so that

$$0 = \sum_i V_i \oint \nabla \Phi'_i \cdot d\mathbf{S}_i - V'_i \sum_i \oint \nabla \Phi_i \cdot d\mathbf{S}_i = -4\pi \left(\sum_i V_i Q'_i - \sum_i V'_i Q_i \right), \quad (27)$$

using Gauss’ Law (in Gaussian units) that

$$4\pi Q_i = \oint \mathbf{E}_i \cdot d\mathbf{S}_i = - \oint \nabla \Phi_i \cdot d\mathbf{S}_i. \quad (28)$$

A.2 Helmholtz Reciprocity

The next step in enlarging the scope of reciprocity relations appears to have been taken by Helmholtz (1859) [7], who stated that “Wenn in einem mit Luft gefüllten Raume, der teils von endlich ausgedehnten festen Körpern begrenzt, teils unbegrenzt ist, im Punkte A Schallwellen erregt werden, so ist das Geschwindigkeitspotential derselben in einem zweiten Punkte B ebenso groß, als es in A sein würde, wenn nicht in A, sondern in B Wellen von derselben Intensität erregt würden. Auch ist der Unterschied der Phasen des erregenden und erregten Punktes in beiden Fällen gleich.”²

Apparently Helmholtz considered this theorem to be “obvious”, as he offered no proof. Helmholtz’ theorem is for scalar waves generated by point sources, and detected by point observers. In this case we readily write that for a wave of angular frequency ω that is generated at point A with strength S_A , the wave observed at point B a distance r from A is

$$O_B = S_A \frac{e^{i(kr-\omega t)}}{r}. \quad (29)$$

Likewise a source of strength S'_B at point B leads to a wave that is observed at point A to be

$$O'_A = S'_B \frac{e^{i(kr-\omega t)}}{r}. \quad (30)$$

Thus, we obtain the reciprocity relation

$$O'_A S_A = O_B S'_B = S_A S'_B \frac{e^{2i(kr-\omega t)}}{r}. \quad (31)$$

While Helmholtz reciprocity may be “obvious” for point sources and point observers it does not always hold for sources and observers of finite extent [8].

In 1886 Helmholtz returned to the theme of reciprocity, and gave a general argument based on Hamiltonian dynamics [9]. For a commentary in English, see [10].

A.3 Maxwell and Reciprocity

Variants of reciprocity theorems for static mechanical systems were considered by several authors in the 1860’s [11], including a version by Maxwell [12].

Maxwell later discussed Green’s reciprocation theorem in sec. 86 of his *Treatise* [13]. In secs. 280-281 he noted that in a linear circuit that contains only resistors, but which may contain linkages of arbitrary complexity, if a (constant) voltage V_{ij} applied between points i and j leads to a (steady) current I_{kl} between points k and l , then (constant) voltage V'_{kl} applied between point k and l leads to (steady) current I'_{ij} between points i and j (which now has no applied voltage) that obeys the reciprocity relation

$$V_{ij} I'_{ij} = V'_{kl} I_{kl}. \quad (32)$$

²If in a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point A, the resulting velocity potential at a second point B is the same both in magnitude and phase, as it would have been at A, had B been the source of the sound.

Maxwell's reciprocity relation (32) is perhaps "obvious" for a circuit that contains only a single loop, and seems to be little referenced (whereas his name is commonly attached to a reciprocity theorem in mechanics).

We sketch Maxwell's argument, noting that it is readily extended to include capacitors and inductors, and to include time-harmonic voltage sources, so long as radiation is neglected, and with the important restriction that there is no mutual capacitance or inductance between circuit elements.

The system consists of a set $\{i\}$ of nodes, $i = 1, \dots, n$, connected by "wires".

Any or all of the $n(n-1)/2$ pairs of nodes, ij , can be directly connected by a "wires" along which there exists one or more circuit elements (resistor, capacitor, inductor, or external voltage source).

The impedance³ and the external voltage (if any, but which will then contribute to the impedance) along the "wire" that directly connects nodes i and j are written Z_{ij} and V_{ij} , respectively, where V_{ij} is positive when the external voltage is lower near node i .

A key feature of Maxwell's analysis is the assumption that the current I_{ij} that flows from node i to node j can be written

$$I_{ij} = \frac{V_i - V_j + V_{ij}}{Z_{ij}} = -I_{ji}, \quad (33)$$

where V_i is the voltage at node i . This assumption excludes the possibility of mutual capacitance or inductance between "wire" ij and any other "wires" in the circuit.

The sum of the currents at each node is zero,

$$0 = \sum_{j \neq i} I_{ij} = \sum_{j \neq i} \frac{V_i - V_j + V_{ij}}{Z_{ij}} = V_i \sum_{j \neq i} \frac{1}{Z_{ij}} - \sum_{j \neq i} \frac{V_j}{Z_{ij}} + \sum_{j \neq i} \frac{V_{ij}}{Z_{ij}}. \quad (34)$$

A clever trick of Maxwell was to define

$$V_{ii} = 0, \quad \text{and} \quad \frac{1}{Z_{ii}} = - \sum_{j \neq i} \frac{1}{Z_{ij}}, \quad (35)$$

which permits us to rewrite eq. (34) as the set of n equations

$$\sum_j \frac{V_j}{Z_{ij}} = \sum_j \frac{V_{ij}}{Z_{ij}}. \quad (36)$$

Since $V_{ji} = -V_{ij}$ while $Z_{ji} = Z_{ij}$, we have that $\sum_i \sum_j V_j/Z_{ij} = \sum_i \sum_j V_{ij}/Z_{ij} = 0$, and only $n - 1$ of these equations are independent. This is to be expected since the voltages are defined only up to an overall constant. Suppose, for example, that we define $V_n = 0$. Then we consider eqs. (36) only for $i = 1, \dots, n - 1$, and it suffices to sum over index j only for $j = 1, \dots, n - 1$. Thus, we obtain

$$V_i = \frac{\sum_k \Delta_{ik} \sum_l V_{kl}/Z_{kl}}{\Delta}, \quad (37)$$

³The term impedance appears to have been first used for electrical circuits by Lodge in 1888 [14].

where Δ is the determinant of the admittance matrix $A_{ij} = 1/Z_{ij}$, and Δ_{ij} is the minor determinant associated with element A_{ij} . Since impedance is symmetric, $Z_{ji} = Z_{ij}$, matrix A_{ij} is symmetric, and it follows that $\Delta_{ji} = \Delta_{ij}$.

Now suppose that all external voltages are zero except for $V_{kl} = -V_{lk}$. Then, the current in the “wire” from node i to node j is

$$I_{ij} = \frac{V_i - V_j}{Z_{ij}} = V_{kl} \frac{\Delta_{ik} - \Delta_{il} - \Delta_{jk} + \Delta_{jl}}{Z_{ij} Z_{kl} \Delta}. \quad (38)$$

Similarly, if all external voltages are zero except for $V'_{ij} = -V'_{ji}$, then the current in the “wire” from node k to node l is

$$I'_{kl} = \frac{V'_k - V'_l}{Z_{kl}} = V'_{ij} \frac{\Delta_{ki} - \Delta_{kj} - \Delta_{li} + \Delta_{lj}}{Z_{ij} Z_{kl} \Delta} = V'_{ij} \frac{\Delta_{ik} - \Delta_{il} - \Delta_{jk} + \Delta_{jl}}{Z_{ij} Z_{kl} \Delta}. \quad (39)$$

Comparing eqs. (38) and (39) we arrive at the reciprocity relation (32).

A.4 Rayleigh Reciprocity

In 1873 Rayleigh extended the static reciprocity theorems of Green, Maxwell and others to include that case that the “force” (derivable from a scalar potential) was periodic rather than static [15]. Although he acknowledged Helmholtz’ earlier theorem for waves [7], Rayleigh’s discussion did not explicitly include the possibility of energy in the form of radiation. His subsequent exposition in *The Theory of Sound* [16] included various examples to show the generality of reciprocity in quasistatic systems, with the effect that his name is now often associated with reciprocity theorems. The statement of his theorem for electrical circuits is the same as eq. (32), but Rayleigh’s argument holds even when there is mutual capacitance and inductance between circuit elements.

Rayleigh’s original argument, like Maxwell’s, involved minors of a relevant determinant. A shorter derivation was given in 1952 by Tellegen [17], as discussed in sec. A.7.

A.5 Lorentz Reciprocity

The generalization of reciprocity theorems to the vector fields of electromagnetism, including waves, is due to Lorentz (1896 [18]).

Lorentz showed that if a (time-dependent) current distribution \mathbf{J}_1 leads to electric and magnetic fields \mathbf{E}_1 and \mathbf{B}_1 in a linear medium, and that current distribution \mathbf{J}_2 leads to electric and magnetic fields \mathbf{E}_2 and \mathbf{B}_2 , then

$$\int \mathbf{J}_1 \cdot \mathbf{E}_2 d\text{Vol} = \int \mathbf{J}_2 \cdot \mathbf{E}_1 d\text{Vol}. \quad (40)$$

A demonstration of eq. (40) invokes the vector calculus identity that

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}). \quad (41)$$

Thus, for fields with time dependence $e^{-i\omega t}$,

$$\begin{aligned}
\nabla \cdot (\mathbf{E}_1 \times \mathbf{B}_2) &= \mathbf{B}_2 \cdot (\nabla \times \mathbf{E}_1) - \mathbf{E}_1 \cdot (\nabla \times \mathbf{B}_2) \\
&= -\frac{1}{c} \mathbf{B}_2 \cdot \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{E}_1 \cdot \left(\frac{4\pi}{c} \mathbf{J}_2 + \frac{1}{c} \frac{\partial \mathbf{E}_2}{\partial t} \right) \\
&= -\frac{4\pi}{c} \mathbf{E}_1 \cdot \mathbf{J}_2 + \frac{i\omega}{c} (\mathbf{B}_2 \cdot \mathbf{B}_1 + \mathbf{E}_1 \cdot \mathbf{E}_2).
\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}
\nabla \cdot (\mathbf{B}_1 \times \mathbf{E}_2) &= \mathbf{E}_2 \cdot (\nabla \times \mathbf{B}_1) - \mathbf{B}_1 \cdot (\nabla \times \mathbf{E}_2) \\
&= \mathbf{E}_2 \cdot \left(\frac{4\pi}{c} \mathbf{J}_1 + \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t} \right) - \frac{1}{c} \mathbf{B}_1 \cdot \frac{\partial \mathbf{B}_2}{\partial t} \\
&= -\frac{4\pi}{c} \mathbf{E}_2 \cdot \mathbf{J}_1 + \frac{i\omega}{c} (\mathbf{E}_2 \cdot \mathbf{E}_1 + \mathbf{B}_1 \cdot \mathbf{B}_2).
\end{aligned} \tag{43}$$

Hence,

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{B}_1 \times \mathbf{E}_2) = \frac{4\pi}{c} (\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1), \tag{44}$$

and so,

$$\int (\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1) d\text{Vol} = \frac{c}{4\pi} \oint (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{E}_1 \times \mathbf{B}_2) \cdot d\text{Area}. \tag{45}$$

A small delicacy in the argument is that for sources \mathbf{J}_1 and \mathbf{J}_2 contained in a bounded region, the asymptotic radiation fields vary as $1/r$ and have the form $\mathbf{B}_{1(2)} = \hat{\mathbf{r}} \times \mathbf{E}_{1(2)}$, so the surface integral vanishes, and we obtain the reciprocity relation (40), independent of the angular frequency ω .

The assumption of linear media seems necessary only to insure that sources at frequency ω lead to fields only at this frequency. See [19] for a review of Lorentz reciprocity in the time domain.

A.6 An Antenna Reciprocity Theorem

The earliest application of Lorentz' reciprocity relation (40) to antennas that I have found is in the papers of Carson [20, 21, 22], but he appears not to have explicitly deduced CEQ. (49). A result close to this was given by Ballantine (1929, [23]). I first find CEQ. (49) in sec. 11.10 of [24].

Consider two antennas, A and B , which contain the only currents in our system. Then CEQ. (40) can be written

$$\int_A \mathbf{J}_{1A} \cdot \mathbf{E}_{2A} d\text{Vol} + \int_B \mathbf{J}_{1B} \cdot \mathbf{E}_{2B} d\text{Vol} = \int_A \mathbf{J}_{2A} \cdot \mathbf{E}_{1A} d\text{Vol} + \int_B \mathbf{J}_{2B} \cdot \mathbf{E}_{1B} d\text{Vol}. \tag{46}$$

In situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in situation 2. Then the currents \mathbf{J}_{1B}

and \mathbf{J}_{2A} exist only in the conductors of the receiving antennas, and not in the gap between the terminals of these antennas. Of course, these currents flow along the conductors. In the approximation of perfect conductors, the electric fields in/on these conductors have no component parallel to the conductors. Hence,

$$\mathbf{J}_{1B} \cdot \mathbf{E}_{2B} = 0 = \mathbf{J}_{2A} \cdot \mathbf{E}_{1A}, \quad (47)$$

while the remaining integrals in CEQ. (46) have contributions only from the gap between the terminals (where the idealized power sources for the transmitting antennas are located):

$$\int_{A,\text{gap}} \mathbf{J}_{1A} \cdot \mathbf{E}_{2A} d\text{Vol} = \int_{B,\text{gap}} \mathbf{J}_{2B} \cdot \mathbf{E}_{1B} d\text{Vol}. \quad (48)$$

We can write the currents in the gap as $\mathbf{J} = I d\mathbf{l}/d\text{Vol}$, so that CEQ. (48) becomes

$$I_{1A} \int_{A,\text{gap}} \mathbf{E}_{2A} \cdot d\mathbf{l} = I_{1A} V_{2A}^{\text{oc}} = I_{2B} V_{1B}^{\text{oc}} = I_{2B} \int_{B,\text{gap}} \mathbf{E}_{1B} \cdot d\mathbf{l}, \quad (49)$$

where V^{oc} is measured between the terminals of the receiving antennas. This is the form of the reciprocity theorem used in sec. 2.3 above, which holds for both “linear” and “loop” antennas.

A.7 Tellegen’s Theorem

Tellegen [17] has given a network theorem that then leads to a kind of reciprocity theorem. See also [25].

Consider a network with **nodes**, and **links** between some or all pairs of nodes. The network can consist of parts with no links between different parts.

“Currents” flow along links between pairs with nodes, with the same scalar value for the “current” at both nodes. A “current I_{jk} is defined to be positive if it flows from node j to node k . Then, $I_{ji} = -I_{ij}$. The only “physical” assumption underlying Tellegen’s theorem is that

$$\sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0 \quad (50)$$

for all nodes j .

We also suppose that every node can be assigned a scalar “voltage” V_j . However, there is not necessarily any “physical” relation between “current” and “voltage”.

It follows immediately from eq. (50) that

$$\sum_j V_j \sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0. \quad (51)$$

The “current” in a link appears exactly twice in the sum (51), in the form

$$V_j I_{jk} = V_k I_{kj} = (V_j - V_k) I_{jk} \equiv V_{jk} I_{jk}. \quad (52)$$

Summing over all links, we obtain Tellegen’s theorem,

$$\sum_{\text{links}} V_{jk} I_{jk} = 0. \quad (53)$$

Another consequence of the definition of the “voltage drop” $V_{jk} = V_j - V_k$ is that the directed sum of “voltage drops” around any closed loop of links is zero.

We can obtain a kind of reciprocity theorem from eq. (53) by considering a second set of “voltages” V'_j and the corresponding “voltage drops” V'_{jk} . Since eq. (53) does not require there to be any “physical” relation between “current” and “voltage”, we also have that $\sum_{\text{links}} V'_{jk} I_{jk} = 0$. Likewise, we can consider another set of “currents” I'_{jk} that are not necessarily related to either the V_{jk} or the V'_{jk} (other than applying to the same network topology), for which we can write $\sum_{\text{links}} V_{jk} I'_{jk} = 0$. Hence, we obtain Tellegen’s reciprocity relation

$$\sum_{\text{links}} V'_{jk} I_{jk} = \sum_{\text{links}} V_{jk} I'_{jk} = 0. \quad (54)$$

However, we cannot deduce from this that $V'_{jk} I_{jk} = V_{mn} I'_{mn}$ for a pair of links jk and mn .

The “nonphysical” character of Tellegen’s theorem clarifies how the reciprocity theorems are somewhat abstract “bookkeeping” constructs, rather than a manifestation of “cause and effect”.

We can, however, obtain the antenna reciprocity theorem of Appendix 6 from Tellegen’s Theorem, but only for “linear” antennas. For this, we consider a network of two disconnected parts, antennas A and B , with three links, $A1$, $A2$, $A3$, *etc.* in each part in the case of a “linear” antenna. Link 2 is the gap between the physical conductors of the antenna (between its terminals).

In the unprimed situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in the primed situation. Then the “currents” in this example are the physical electrical currents, so that I_{B2} and I'_{A2} are zero, since the receiving antennas are operated “open circuit”.⁴ We define the “voltage difference” between the ends of a link to be $\int \mathbf{E} \cdot d\mathbf{l}$. Hence, $V_{A1} = V_{A3} = V_{B1} = V_{B3} = V'_{A1} = V'_{A3} = V'_{B1} = V'_{B3} = 0$, in the approximation that the conductors of the antenna are perfect. The voltages V_{jA} at the four nodes of antenna A are then $V_{1A} = V_{2A}$ and $V_{3A} = V_{4A}$.

Thus, of the 12 terms in eq. (54) for the pair of “linear” antennas only two are nonzero, and we have

$$V'_{A2} I_{A2} = V_{B2} I'_{B2}, \quad (55)$$

as in eq. (20), where again the “voltages” V'_{A2} and V_{B2} are “open circuit”.

However, a similar argument for a “loop” antenna, representing it by two links connected to two nodes, fails. If link 2 again represent the gap between the terminals, then the “voltage” $V_{1A} = \int_{\text{link } 1A} \mathbf{E} \cdot d\mathbf{l} = 0$, but $V_{2A} = \int_{\text{link } 2A} \mathbf{E} \cdot d\mathbf{l}$ is nonzero in general, so a unique scalar “voltage” cannot be assigned to the two nodes of antenna A .

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⁴The currents along the conductors of the antenna are not constant but they do satisfy the node condition (50). The currents at the tips of a “linear” antenna are zero, where only one link is connected to the tip node. For the transmitting antennas the current I_{12} does not equal $-I_{21}$ and I_{34} does not equal $-I_{43}$. However, Tellegen’s relation (54) still holds because $V_{12} = 0 = V_{34}$, as discussed below.

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