Radial Viscous Flow between Two Parallel Annular Plates

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(June 25, 2000; updated July 30, 2008)

1 Problem

Deduce the velocity distribution of steady flow of an incompressible fluid of density ρ and viscosity η between two parallel, coaxial annular plates of inner radii r_1, outer radii r_2 and separation h when pressure difference ΔP is applied between the inner and outer radii.

As an exact solution of the Navier-Stokes equation appears to be difficult, it suffices to give an approximate solution assuming that the velocity is purely radial, $\mathbf{v} = v(r, z)\hat{\mathbf{r}}$, in a cylindrical coordinate system (r, ϕ, z) whose z axis coincides with that of the two annuli. Deduce a condition for validity of the approximation.

This problem arises, for example, in considerations of a rotary joint between two sections of a pipe. Here, we ignore the extra complication of the effect of the rotation of one of the annuli on the fluid flow.

2 Solution

For an incompressible fluid, the velocity distribution obeys the continuity equation

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

in which case the Navier-Stokes equation for steady, viscous flow is

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \eta \nabla^2 \mathbf{v}. \tag{2}$$

There are only three examples in which analytic solutions to this equation have been obtained when the nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is nonvanishing [1].

We first review the simpler case of two-dimensional flow between parallel plates in sec. 2.1, and then take up the case of radial flow in sec. 2.2. We will find an analytic solution to the nonlinear Navier-Stokes equation (2) for radial flow, but this solution cannot satisfy the the boundary conditions,

$$\mathbf{v}(z = 0) = 0 = \mathbf{v}(z = h), \tag{3}$$

that the flow velocity vanish next to the plates. However, in the linear approximation to eq. (2) we obtain an analytic form for the radial flow between two annular plates.

2.1 Two-Dimensional Flow between Parallel Plates

For guidance, we recall that an analytic solution is readily obtained for the related problem of two-dimensional viscous flow between two parallel plates. For example, suppose that the plates are at the planes $z = 0$ and $z = h$, and that the flow is in the x direction, *i.e.*, $\mathbf{v} = v(x,z)\hat{\mathbf{x}}$. The equation of continuity (1) then tells us that $\partial v/\partial x = 0$, so that the velocity is a function of z only,

$$\mathbf{v} = v(z)\hat{\mathbf{x}}. \tag{4}$$

The z component of the Navier-Stokes equation (2) reduces to $\partial P/\partial z = 0$, so that the pressure is a function of x only. The x component of eq. (2) is

$$\frac{\partial P(x)}{\partial x} = \eta \frac{\partial^2 v(z)}{\partial z^2}. \tag{5}$$

Since the lefthand side is a function of x, and the righthand side is a function of z, equation (5) can be satisfied only if both sides are constant. Supposing that the pressure decreases with increasing x, we write

$$-\frac{\partial P}{\partial x} = \text{constant} = \frac{\Delta P}{\Delta x} > 0. \tag{6}$$

Using the boundary conditions (3), we quickly find that

$$v(z) = \frac{\Delta P}{\Delta x}\frac{z(h-z)}{2\eta} = 6\bar{v}\frac{z}{h}\left(1 - \frac{z}{h}\right), \tag{7}$$

where the average velocity $\bar{v}$ is given by

$$\bar{v} = \frac{1}{h}\int_0^h v(z)dz = \frac{\Delta P}{\Delta x}\frac{h^2}{12\eta}. \tag{8}$$

2.2 Radial Flow between Parallel Annular Plates

Returning to the problem of radial flow between two annular plates, we seek a solution in which the velocity is purely radial, $\mathbf{v} = v(r,z)\hat{\mathbf{r}}$. The continuity equation (1) for this hypothesis tells us that

$$\frac{1}{r}\frac{\partial(rv)}{\partial r} = 0, \tag{9}$$

so that

$$\mathbf{v} = \frac{f(z)}{r}\hat{\mathbf{r}}. \tag{10}$$

Following the example of two-dimensional flow between parallel plates, we expect a parabolic profile in z as in eq. (7),

$$f(z) \propto z(h-z), \tag{11}$$

which satisfies the boundary conditions (3).

Using the trial solution (10), the z component of the Navier-Stokes equation (2) again tells us that the pressure must be independent of z: $P = P(r)$. The radial component of eq. (2) yields the nonlinear form

$$\eta r^2 \frac{d^2 f}{dz^2} + \rho f^2 = r^3 \frac{dP}{dr}. \tag{12}$$

The hoped-for separation of this equation can only be achieved if $f(z) = F$ is constant, which requires the pressure profile to be $P(r) = A - \rho F^2/2r^2$. The boundary conditions (3) cannot be satisfied by this solution. Further, this solution exists only for the case that the pressure is increasing with increasing radius. The fluid flow must be then be inward, so the constant F must be negative. The Navier-Stokes equation is not time-reversal invariant due to the dissipation of energy associated with the viscosity, and so reversing the velocity of a solution does not, in general, lead to another solution.

While we have obtained an analytic solution to the nonlinear Navier-Stokes equation (2), it is not a solution to the problem of radial flow between two annuli. It is hard to imagine a physical problem involving steady, radially inward flow of a long tube of fluid, to which the solution could apply.

Instead of an exact solution, we are led to seek an approximate solution in which the nonlinear term f^2 of eq. (12) can be ignored. In this case, the differential equation takes the separable form

$$\frac{d^2 f}{dz^2} = \frac{r}{\eta}\frac{dP}{dr} = \text{constant}. \tag{13}$$

Following eq. (7) we write the solution for f that satisfies the boundary conditions (3) as

$$f(z) = 6\bar{f}\frac{z}{h}\left(1 - \frac{z}{h}\right), \tag{14}$$

where $\bar{f}$ is the average of $f(z)$ over the interval $0 \leq z \leq h$. The part of eq. (13) that describes the pressure leads to the solution

$$P(r) = \frac{P_1 \ln r_2/r + P_2 \ln r/r_1}{\ln r_2/r_1}, \tag{15}$$

where $P_i = P(r_i)$. Plugging the solutions (14) and (15) back into eq. (13), we find that

$$\bar{f} = \frac{h^2 \Delta P}{12\eta \ln r_2/r_1}, \tag{16}$$

where $\Delta P = P_1 - P_2$. Hence, the flow velocity is

$$\mathbf{v}(r, z) = \frac{z(h - z)\Delta P}{2\eta r \ln r_2/r_1}\hat{\mathbf{r}}, \tag{17}$$

whose average with respect to z is $\bar{v}(r) = \bar{f}/r$. As with all solutions to the linearized Navier-Stokes equation, the velocity is independent of the density. Note also that the direction of the flow is from the high pressure region to the low.

For the approximate solution (17) to be valid, the term $f^2 \approx \bar{f}^2$ must be small in eq. (12), which requires

$$\frac{\rho h^4 \Delta P}{144\eta^2 r_1^2 \ln r_2/r_1} \ll 1. \tag{18}$$

When this condition is not satisfied, the solution must include velocity components in the z direction that are significant near the inner and outer radii, while the flow pattern at intermediate radii could be reasonably well described by eq. (17).

Note added July 30, 2008: In the case of low viscosity η we make an approximate analysis of Eq. (12) by taking the factor f to be the constant $\bar{f}^2$ of eq. (14). Then,

$$\frac{dP}{dr} \approx -\frac{12\eta\bar{f}}{h^2 r} + \frac{\rho\bar{f}^2}{r^3}, \tag{19}$$

which integrates to

$$P(r) \approx P_1 - \frac{12\eta\bar{f}}{h^2}\ln\frac{r}{r_1} + \frac{\rho\bar{f}^2}{2}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right), \tag{20}$$

where P_1 is the pressure at radius r_1. Evaluating this at radius r_2 where the pressure is P_2, we can write

$$\Delta P = P_1 - P_2 \approx \frac{12\eta\bar{f}}{h^2}\ln\frac{r_2}{r_1} + \frac{\rho\bar{f}^2}{2}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) = \Delta P_\eta + \Delta P_\rho, \tag{21}$$

where the term

$$\Delta P_\rho = \frac{\rho\bar{v}_2^2}{2} - \frac{\rho\bar{v}_1^2}{2} \tag{22}$$

is the familiar change in pressure associated with the change in velocity described by Bernoulli's law for fluids with zero viscosity. The constant $\bar{f} = r\bar{v}_r$ is determined by eq. (22) to be

$$\bar{f} = -\frac{12\eta r_1^2 r_2^2 \ln(r_2/r_1)}{\rho h^2(r_2^2 - r_1^2)} \pm \sqrt{\left[\frac{12\eta r_1^2 r_2^2 \ln(r_2/r_1)}{\rho h^2(r_2^2 - r_1^2)}\right]^2 + \frac{2r_1^2 r_2^2 \Delta P}{\rho(r_2^2 - r_1^2)}}. \tag{23}$$

The direction of the flow can be either from high pressure to low or vice versa, but with different velocities in these two cases.

An experimental study of a case well approximated by eq. (20) is reported in [2].

If one of the annuli is rotating at angular velocity ω, the radial flow velocity should still be given approximately by eq. (17) so long as $\omega r_2 \lesssim \bar{v}(r_2) = \bar{f}/r_2$.

3 References

[1] L. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, Oxford, 1987), chap. 2.

[2] J. Armengol *et al.*, *Bernoulli correction to viscous losses: Radial flow between two parallel discs*, Am. J. Phys. **76**, 730 (2008), http://puhep1.princeton.edu/~mcdonald/examples/fluids/armengol_ajp_76_730_08.pdf