

On the Definition of Radiation by a System of Charges

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

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1 What is Radiation?

In a broad sense, radiation has come to mean a flow of energy through some medium, possibly vacuum.¹ In a classical view, the energy can be carried by both particles (α and β particle radiation, *etc.*) and by waves (acoustic radiation, electromagnetic radiation, *etc.*). An early view (see, for example, Newton [1]) of optical radiation was that it consists of “rays” which emanate in straight lines from a source.² Then, the number of rays crossing any surface enclosing the source is the same, and the number of rays crossing a area element normal to the rays falls off as the square of the distance from the source. A simple model is that optical rays are particles that move with some constant velocity along the path of the ray. The energy carried by the ray is the kinetic energy of the particles.

In the early 1800’s Young [2] and Fresnel [3] argued that the optical phenomena of interference and stellar aberration imply that optical rays are actually an aspect of (transverse) waves in an æther. The energy carried by these rays was imagined to be that of the kinetic and potential energy of the undulations of the æther. Maxwell [4] identified optical rays with electromagnetic waves, whose energy is now ascribed to that of the electric and magnetic fields, rather than to a mechanical æther.^{3,4}

The concept of rays for waves is only defined on scales larger than a wavelength. A challenge addressed in the present note is to provide an understanding of what can be meant by radiation of electromagnetic energy close to its source(s).

In the quantum theory of electromagnetic fields, they can also be regarded as consisting of particles (photons), and whether their field or particle character is more prominent depends on details of the experiments devised to ascertain that character. This raises the question as to whether Maxwell’s equations for electromagnetic fields can lead to results of a particle-like character when considering electromagnetic radiation. Also, in the quantum view, photons have an extent at least that of a characteristic wavelength, and the flow of energy of these photons is not as highly localized as is assumed to be possible in a classical description. Hence, a classical description of the flow of energy close to charges and currents is expected to have finer detail than that possible in the quantum view. For example, lines of classical energy flow in Young’s double-slit experiment pass through only one slit or the other, while the quantum view of the resulting interference pattern for a single photon is that the photon has a probability amplitude to pass through both slits. A possible lesson is that one should

¹Radiation sometimes has the meaning of the process of emission of energy. Here we take the term “radiation” to describe the result of a process, and not the process itself.

²Latin: *radiat-* = emitted in rays; past participle of the verb *radiare*.

³In the wave theory the direction of a ray is that of the group velocity (see, for example, sec. 2.1 of [5]).

⁴One view of electromagnetic radiation is that it corresponds to any time-dependent electromagnetic field, without reference to flow of energy in that field. I consider this definition to be too broad, and do not consider it further here.

not worry about details of classical energy flow close to matter. Nonetheless, the present note takes on this challenge.

2 The Poynting Vector

The flow of energy in the electromagnetic fields \mathbf{E} and \mathbf{B} is described by the Poynting vector [6],

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (1)$$

(in Gaussian units) where c is the speed of light in vacuum, so an obvious definition is to identify the Poynting vector with electromagnetic radiation:

- A. Electromagnetic radiation is the flow of electromagnetic energy described by the Poynting vector (1).^{5,6,7,8}

Definition A of radiation encompasses more than electromagnetic waves, since time-independent fields can have a nonzero Poynting vector. For example, in a simple DC circuit energy flows from the battery to the resistor through the intervening space rather than through the wire,⁹ as first noted by Poynting [6], and this energy flow is to be called radiation according to definition A. In the broad sense, this is acceptable usage (and in the quantum view¹⁰ this energy flow involves “virtual” photons¹¹).

The electromagnetic field theory of Faraday and Maxwell is a “unified field theory” in which Faraday advocated combining what might be called the “electrostatic” field $-\nabla V$ with what might be called the “electrokinetic” field $-\partial\mathbf{A}/\partial ct$ (where V and \mathbf{A} are the scalar and vector potentials) into the single electric field

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (2)$$

⁵Definition A was given at least as early as 1927. See p. 5 of [7].

⁶Definition A is more general than Sommerfeld’s “radiation condition” [8] that associates radiation only with flow of electromagnetic energy to “infinity” (meaning that the integral of the normal component of the Poynting vector over the surface of any large sphere is nonzero at some times). The “radiation condition” leaves ambiguous the definition of radiation at a finite distances from charges and currents. Indeed, the “radiation condition” has the awkward implication that “radiation” cannot be detected, since only energy that flows to “infinity” is to be called “radiation.” It is better to think of the “radiation condition” as a boundary condition in mathematical physics than as a definition of radiation [9].

⁷According to definition A, radiation is associated with both a charge q with uniform velocity \mathbf{v} and with uniform acceleration \mathbf{a} , although in neither case does the charge radiate energy (since there is no “radiation reaction force” $q^2\dot{\mathbf{a}}/2c^2$ in either case). According to the “radiation condition,” there is no radiation associated with a uniformly moving charge, while radiation is associated with a uniformly accelerated charge.

⁸A variant is to identify radiation with the Poynting vector (1), but only for time-dependent fields.

⁹See [10] for an instructive model calculation of energy flow in a DC circuit.

¹⁰The Poynting vector can be calculated for quantum fields. See, for example, sec. 6 of [11].

¹¹The spirit of the so-called Weizsäcker-Williams method [12] is to consider the conversion of the “virtual” photons that accompany a uniformly moving charge into “real” ones if that motion is perturbed. In this view, an accelerated electron does not “emit” radiation but rather “sheds” it. *Note, however, that a photon can be “real” (massless) only if it is never detected. Only “virtual” photons are detected, so the popular names “real” and “virtual” have somewhat the inverse of their intended meanings when applied to photons.*

and Maxwell identified the wave fields of optics with electromagnetic fields. Definition A of radiation is consistent with the “unified field theory” in that the entire electric and magnetic fields are used to calculate the radiation/Poynting vector. However, attempts to relate radiation to accelerated charges led to decompositions of the electromagnetic fields into “radiation” and “nonradiation” fields, or into “incident” and “reflected/scattered” fields.¹²

3 Radiation Fields

Considerations of a decomposition of the electric field into “radiation” and “nonradiation” parts likely follows from the calculations by Liénard [16] and by Wiechert [17] of the electromagnetic fields of a single, accelerated charge, which can be summarized as¹³

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_{\text{nonrad}} + \mathbf{E}_{\text{rad}}, \quad (3)$$

$$\mathbf{E}_{\text{nonrad}} = q \left[\frac{\hat{\mathbf{r}} - \mathbf{v}/c}{\gamma^2 r^2 (1 - \hat{\mathbf{r}} \cdot \mathbf{v}/c)^3} \right]_{\text{ret}}, \quad \mathbf{E}_{\text{rad}} = q \left[\frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \mathbf{v}/c) \times \mathbf{a}/c]}{cr(1 - \hat{\mathbf{r}} \cdot \mathbf{v}/c)^3} \right]_{\text{ret}}, \quad (4)$$

$$\mathbf{B}(\mathbf{x}, t) = [\hat{\mathbf{r}}]_{\text{ret}} \times \mathbf{E} = \mathbf{B}_{\text{nonrad}} + \mathbf{B}_{\text{rad}} = [\hat{\mathbf{r}}]_{\text{ret}} \times \mathbf{E}_{\text{nonrad}} + [\hat{\mathbf{r}}]_{\text{ret}} \times \mathbf{E}_{\text{rad}}, \quad (5)$$

where $\mathbf{v} = d\mathbf{x}_q/dt$ is the velocity of the charge, $\mathbf{a} = d^2\mathbf{x}_q/dt^2$ is its acceleration, $\gamma = 1/\sqrt{1 - v^2/c^2}$, the distance from the charge to the observer is $\mathbf{r} = \mathbf{x} - \mathbf{x}_q$, and the retarded time is $t' = t - r/c$.¹⁴ Of course, no measurement can distinguish between \mathbf{E}_{rad} and $\mathbf{E}_{\text{nonrad}}$. The decomposition (3) and (5) is purely conceptual, so one must be cautious in assigning physical significance to it (other than that at large distances from a source that contains accelerated charges, the fields fall off inversely with distance).

This decomposition reinforces the notion that “accelerated charges radiate” and that “radiation is due to the acceleration of charges.” These views are popularly represented by the “kink model” of radiation, which was perhaps first introduced by Heaviside [20], and more graphically by J.J. Thomson [21].

¹²Another decomposition of the fields is due to Helmholtz [13, 14], in which $\mathbf{E} = \mathbf{E}_{\text{irr}} + \mathbf{E}_{\text{rot}}$ (and $\mathbf{B} = \mathbf{B}_{\text{rot}}$) where $\nabla \times \mathbf{E}_{\text{irr}} = 0 = \nabla \cdot \mathbf{E}_{\text{rot}}$. Then, one can write $\mathbf{S} = \mathbf{S}_{\text{irr}} + \mathbf{S}_{\text{rot}} = \mathbf{E}_{\text{irr}} \times \mathbf{B} + \mathbf{E}_{\text{rot}} \times \mathbf{B}$. However, calculation of \mathbf{E}_{irr} and \mathbf{E}_{rot} requires instantaneous knowledge throughout the entire Universe, so the Helmholtz decomposition is of a mathematical rather than physical character. An explanation of radiation that uses the Helmholtz decomposition without awareness of this is [15].

¹³Throughout this note the charges are assumed to be in media with unit relative permittivity and permeability.

¹⁴An alternative form of eq. (4) was given by Heaviside [18] and later popularized by Feynman [19],

$$\mathbf{E}_{\text{nonrad}} = q \left[\frac{\hat{\mathbf{R}}}{R^2} \right] + \frac{q}{c} \left[R \frac{d}{dt} \frac{\hat{\mathbf{R}}}{R^2} \right], \quad \mathbf{E}_{\text{rad}} = \frac{q}{c^2} \left[\frac{d^2 \hat{\mathbf{R}}}{dt^2} \right]. \quad (6)$$

Note that Heaviside’s v is our c , his $\mu v^2/4\pi = 1$ in Gaussian units, and that his \mathbf{R}_1 is our $\hat{\mathbf{R}}$. Then, his eq. (32) can be rewritten as

$$\mathbf{E} = \frac{\mu Q}{4\pi} \left\{ \ddot{\mathbf{R}}_1 + \frac{v}{R^2} (R\dot{\mathbf{R}}_1 - 2\mathbf{R}_1\dot{R} + v\mathbf{R}_1) \right\} \rightarrow Q \left[\frac{\hat{\mathbf{R}}}{R^2} + \frac{R}{c} \frac{d}{dt} \frac{\hat{\mathbf{R}}}{R^2} + \frac{1}{c^2} \frac{d^2 \hat{\mathbf{R}}}{dt^2} \right]_{\text{ret}}, \quad (7)$$

which is Feynman’s expression (6).

A decomposition into radiation and nonradiation fields when the sources are charge and current densities ρ and \mathbf{J} can be made using¹⁵

$$\mathbf{E}(\mathbf{x}, t) = \int \frac{[\rho] \hat{\mathbf{R}}}{R^2} d^3 \mathbf{x}' + \frac{1}{c} \int \frac{([\mathbf{J}] \cdot \hat{\mathbf{R}}) \hat{\mathbf{R}} + ([\mathbf{J}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R^2} d^3 \mathbf{x}' + \frac{1}{c^2} \int \frac{([\dot{\mathbf{J}}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R} d^3 \mathbf{x}', \quad (10)$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{c} \int \frac{[\mathbf{J}] \times \hat{\mathbf{R}}}{R^2} d^3 \mathbf{x}' + \frac{1}{c^2} \int \frac{[\dot{\mathbf{J}}] \times \hat{\mathbf{R}}}{R} d^3 \mathbf{x}', \quad (11)$$

where $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $[\mathbf{J}] = \mathbf{J}(\mathbf{x}', t' = t - R/c)$. The radiation fields are the final terms in eqs. (10)-(11).

The decomposition into radiation and nonradiation fields leads many people to a different definition of radiation:

B. Radiation is the flow of energy described by the part of the Poynting vector due to the radiation fields,

$$\mathbf{S}_B = \frac{1}{4\pi c^3} \int \frac{([\dot{\mathbf{J}}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R} d^3 \mathbf{x}' \times \int \frac{[\dot{\mathbf{J}}] \times \hat{\mathbf{R}}}{R} d^3 \mathbf{x}'. \quad (12)$$

However, neither the vectors \mathbf{S} nor \mathbf{S}_B can be written as a single volume integral over the source charges and currents,

$$\mathbf{S}, \mathbf{S}_B \neq \int \frac{\mathbf{s}([\rho], [\mathbf{J}], [\dot{\mathbf{J}}], \hat{\mathbf{R}})}{R^2} d^3 \mathbf{x}', \quad (13)$$

where \mathbf{s} is some vector function (possibly including spatial derivatives of $[\rho]$ and $[\mathbf{J}]$).¹⁶ Rather, we note that Poynting's theorem [6] can be written in the form

$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t} - \mathbf{J} \cdot \mathbf{E}, \quad (15)$$

¹⁵Equations (10)-(11) first appear in [22], although versions of their Fourier transforms appear in [23], and more explicitly in [24, 25]. An alternative approach is to integrate the wave equations,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 4\pi \nabla \rho + \frac{4\pi}{c} \frac{\partial \mathbf{J}}{\partial t} \quad \text{and} \quad \nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{4\pi}{c} \nabla \times \mathbf{J}, \quad (8)$$

using the method of Lorenz [26] to obtain

$$\mathbf{E} = -\int \frac{[\nabla' \rho]}{R} d^3 \mathbf{x}' - \frac{1}{c^2} \int \frac{[\mathbf{J}]}{R} d^3 \mathbf{x}' \quad \text{and} \quad \mathbf{B} = \frac{1}{c} \int \frac{[\nabla' \times \mathbf{J}]}{R} d^3 \mathbf{x}'. \quad (9)$$

With effort, the derivatives ∇' can be transformed away to yield eqs. (10)-(11).

¹⁶One way to see this is to take the d'Alembertian of the Poynting vector (1),

$$\begin{aligned} \frac{4\pi}{c} \left(\nabla^2 \mathbf{S} - \frac{1}{c^2} \frac{\partial^2 \mathbf{S}}{\partial t^2} \right) &= \left(\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \right) \times \mathbf{B} + \mathbf{E} \times \left(\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \right) \\ &+ 2 \sum_{w=x,y,z} \frac{\partial \mathbf{E}}{\partial w} \times \frac{\partial \mathbf{B}}{\partial w} - \frac{2}{c^2} \frac{\partial \mathbf{E}}{\partial t} \times \frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \quad (14)$$

Since the righthand side of eq. (14) cannot be expressed directly in terms of charges and currents (without becoming an integral equation by use of eqs. (8)-(9)), no solution of form (13) exists.

where $u = (E^2 + B^2)/8\pi$ is the density of energy in the electromagnetic field. This has the implication that both a time-varying field-energy density u and the electric current \mathbf{J} act as sources for the Poynting vector. See [27, 28, 29, 30] for analytic discussions of this in the case of pulsed, point (Hertzian) dipoles.

Thus, although we can identify “radiation fields” which depend only on accelerated charges, we cannot say electromagnetic radiation is the flow of electromagnetic energy corresponding to some or all of the Poynting vector (1) such that this radiation depends only on charges and currents in an integration over (retarded) source terms. It appears to this author that the decomposition of the electromagnetic fields into “radiation” and “nonradiation” parts (which goes against the Maxwellian vision of a unified field theory) does not accomplish its underlying goal of relating “radiation” to accelerated charges alone.¹⁷ As such, it is preferable to use definition A for radiation as being described by the entire Poynting vector.

4 The Issue of Cause and Effect

The equations (3)-(5) and (10)-(11) express the electromagnetic fields in terms of charges and currents. This can give the impression that Maxwell’s equations imply that fields are “caused” by charges and currents. However, most electrical currents are “caused” by electric fields, and many electric charge distributions are “induced” by electric fields. That is, Maxwell’s electrodynamics is a complete logical system only when his four differential equations for the fields are supplemented with the laws,

$$\mathbf{F} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad \mathbf{f} = \rho \mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B}, \quad (16)$$

for the force \mathbf{F} on an individual charge (Lorentz) and for the force density \mathbf{f} on charge and current densities, respectively. Thus, categorical identification of “cause” and “effect” (or “before” and “after”) in electrodynamics is not possible in general, being dependent on an assumption as to what constitutes the initial conditions.

Definition B of radiation tends to be associated with a view that the relevant initial conditions involve knowledge of the charge and current distributions, whereas Definition A of radiation is more neutral as to the assumption as to the initial conditions.

An even more explicit attempt to associate the concept of radiation with “before” and “after” is considered in sec. 6.

5 Time-Harmonic Fields

Many important examples of the flow of electromagnetic energy involve such a narrow range of frequencies that the approximation of a single (angular) frequency ω is sufficient. In this approximation the time average of any quantity (at a given point in space) is constant. In

¹⁷An example of an extended system in which definitions A and B lead to the same notion of “radiation” is a uniform sheet of charge that is given a constant velocity in its plane at $t = 0$. See [33] for computation of the radiation here according to definition B.

particular, the time-average field energy density $\langle u \rangle$ is constant, such that the time-average of Poynting's theorem (15) reads

$$\nabla \cdot \langle \mathbf{S} \rangle = - \langle \mathbf{J} \cdot \mathbf{E} \rangle, \quad (17)$$

which permits the interpretation that the time-average Poynting vector has no sources in current-free regions. This contrasts with the case of the Poynting vector for fields with arbitrary time dependence, for which a changing field energy density $\partial u / \partial t$ acts as a source.

The present note concerns the definition of radiation for a collection of charges, particularly those in conductors, in which case the velocities are extremely low and the lab frame is essentially the instantaneous rest frame of all of the charges. Then, the motion of each charge is well approximated as that of an ideal, oscillating, point (Hertzian) electric dipole [34] with (complex) moment \mathbf{p} , for which the electromagnetic fields are (see, for example, sec. 9.2 of [35])

$$\mathbf{E} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{e^{i(kr - \omega t)}}{r} + p [3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{i(kr - \omega t)}, \quad (18)$$

$$\mathbf{B} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \left(\frac{1}{r} - \frac{1}{ikr^2} \right) e^{i(kr - \omega t)}. \quad (19)$$

The “radiation fields” of a Hertzian dipole are

$$\mathbf{E}_{\text{rad}} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{e^{i(kr - \omega t)}}{r}, \quad \mathbf{B}_{\text{rad}} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \frac{e^{i(kr - \omega t)}}{r}. \quad (20)$$

Hence, definitions A and B for the time-average radiation of a collection of oscillating charges q_j are,

- A. The time-average radiation is defined to be the (time-average) Poynting vector of the total fields of the charges.

$$\langle \mathbf{S}_A \rangle = \langle \mathbf{S} \rangle = \frac{c}{8\pi} \text{Re} \left(\sum_j \mathbf{E}(q_j) \times \sum_k \mathbf{B}^*(q_k) \right). \quad (21)$$

- B. The time-average radiation is defined to be the (time-average) Poynting vector due only to the “radiation fields” of the charges,¹⁸

$$\langle \mathbf{S}_B \rangle = \frac{c}{8\pi} \text{Re} \left(\sum_j \mathbf{E}_{\text{rad}}(q_j) \times \sum_k \mathbf{B}_{\text{rad}}^*(q_k) \right). \quad (22)$$

¹⁸It is always possible to represent a time-varying electromagnetic field as a sum of electromagnetic plane waves, of which some are propagating (homogeneous, of form $\mathbf{E} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$, $\mathbf{B} = \hat{\mathbf{k}} \times \mathbf{E}$, where the constant fields \mathbf{E} and \mathbf{B} obey $\mathbf{E} \cdot \mathbf{k} = 0 = \mathbf{B} \cdot \mathbf{k}$) and some are evanescent (inhomogeneous) [36]. We might then entertain definition C, that “radiation” is $\langle \mathbf{S}_C \rangle = (c/8\pi) \text{Re} \sum \mathbf{E}_m \times \mathbf{B}_n^* e^{i[(\mathbf{k}_m - \mathbf{k}_n) \cdot \mathbf{r} - (\omega_m - \omega_n)t]}$, *i.e.*, the (time-average) Poynting vector formed only from the propagating electromagnetic fields. I believe that definition C is equivalent to definition B, in that only the “radiation fields” propagate far from their sources and are therefore represented by the propagating waves in definition C. Then, the objection to definition B discussed below also applies to definition C.

For a single oscillating charge we find that

$$\langle \mathbf{S} \rangle = \langle \mathbf{S}_A \rangle = \langle \mathbf{S}_B \rangle = k^4 |p|^2 \frac{|\hat{\mathbf{r}} \times \hat{\mathbf{p}}|^2}{r^2} \hat{\mathbf{r}}, \quad (23)$$

where unit vector $\hat{\mathbf{r}}$ points from the average position of the charge to the observer.

To see that the quantities $\langle \mathbf{S}_A \rangle$ and $\langle \mathbf{S}_B \rangle$ are not the same when more than one charge is accelerated it suffices to consider a system of only two charges, q_1 and q_2 , at (average) positions \mathbf{x}_1 and \mathbf{x}_2 , with (complex) oscillating dipole moments \mathbf{p}_1 and \mathbf{p}_2 . Then,

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) = & k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \times \hat{\mathbf{r}}_1 \frac{e^{i(kr_1 - \omega t)}}{r_1} + p_1 [3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_1) \hat{\mathbf{r}}_1 - \hat{\mathbf{p}}_1] \left(\frac{1}{r_1^3} - \frac{ik}{r_1^2} \right) e^{i(kr_1 - \omega t)} \\ & + k^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \times \hat{\mathbf{r}}_2 \frac{e^{i(kr_2 - \omega t)}}{r_2} + p_2 [3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_2) \hat{\mathbf{r}}_2 - \hat{\mathbf{p}}_2] \left(\frac{1}{r_2^3} - \frac{ik}{r_2^2} \right) e^{i(kr_2 - \omega t)}, \end{aligned} \quad (24)$$

$$\mathbf{B}(\mathbf{x}, t) = k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \left(\frac{1}{r_1} - \frac{1}{ikr_1^2} \right) e^{i(kr_1 - \omega t)} + k^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \left(\frac{1}{r_2} - \frac{1}{ikr_2^2} \right) e^{i(kr_2 - \omega t)}, \quad (25)$$

where $\mathbf{r}_j = \mathbf{x} - \mathbf{x}_j$, and

$$\begin{aligned} \frac{8\pi}{c} \langle \mathbf{S}_A \rangle = & k^4 |p_1|^2 \frac{|\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1|^2}{r_1^2} \hat{\mathbf{r}}_1 + k^4 |p_2|^2 \frac{|\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2|^2}{r_2^2} \hat{\mathbf{r}}_2 \\ & + k^4 [(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2)(\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \cdot (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) - (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \\ & - (\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_2)(\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2)] \left\{ \frac{2\text{Re}[p_1 p_2^* e^{ik(r_1 - r_2)}]}{r_1 r_2} \right. \\ & \left. + \text{Re} \left[\frac{p_1 p_2^* e^{ik(r_1 - r_2)}}{ikr_1 r_2^2} + \frac{p_2 p_1^* e^{ik(r_2 - r_1)}}{ikr_2 r_1^2} \right] \right\} \\ & + k^2 [3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_1) \hat{\mathbf{r}}_1 - \hat{\mathbf{p}}_1] \times (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \text{Re} \left[p_1 p_2^* e^{ik(r_1 - r_2)} \left(\frac{1}{r_1^3} - \frac{ik}{r_1^2} \right) \left(\frac{1}{r_2} + \frac{1}{ikr_2^2} \right) \right] \\ & + k^2 [3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_2) \hat{\mathbf{r}}_2 - \hat{\mathbf{p}}_2] \times (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \text{Re} \left[p_2 p_1^* e^{ik(r_2 - r_1)} \left(\frac{1}{r_2^3} - \frac{ik}{r_2^2} \right) \left(\frac{1}{r_1} + \frac{1}{ikr_1^2} \right) \right]. \end{aligned} \quad (26)$$

Then, $(8\pi/c) \langle \mathbf{S}_B \rangle$ equals only the first three lines of eq. (26).

Thus, $\langle \mathbf{S}_A \rangle = \langle \mathbf{S}_B \rangle$ in the “far zone” (where $kr_j \gg 1$ for all charges q_j), but they differ in the “near zone” close to the source charges.

The total time-average energy density $\langle u \rangle$ in the electromagnetic fields of the oscillating charges is constant in time at every point in space outside the charges themselves, so energy conservation (Poynting’s theorem) implies that

$$\nabla \cdot \langle \mathbf{S}_A \rangle = -\frac{\partial \langle u \rangle}{\partial t} = 0. \quad (27)$$

However,

$$\frac{4\pi}{ck^4} \nabla \cdot \langle \mathbf{S}_B \rangle = k \frac{\text{Im}[p_1 p_2^* e^{ik(r_1 - r_2)}]}{r_1 r_2} [(\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2)^2 - (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1)^2]$$

$$\begin{aligned}
& -\frac{\text{Re}[p_1 p_2^* e^{ik(r_1-r_2)}]}{r_1 r_2} \left[\frac{(\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2)^2}{r_1} + \frac{(\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1)^2}{r_2} \right. \\
& + (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 - 1) \left(\frac{1}{r_1} + \frac{1}{r_2} \right) (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \cdot (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \\
& - \left(\frac{1}{r_1} + \frac{1}{r_2} \right) (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2) \\
& - (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2) \cdot \left(\frac{(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1) \hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2}{r_1} + \frac{(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1}{r_2} \right) \\
& + (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_2) \cdot (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \left(\frac{\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2}{r_1} - \frac{1}{r_2} \right) \\
& \left. + (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \cdot (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_1) \left(\frac{\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2}{r_2} - \frac{1}{r_1} \right) \right], \tag{28}
\end{aligned}$$

which is nonzero in the “near zone” (and outside the charges).¹⁹ This means that there are sources (and sinks) of the Poynting flux $\langle \mathbf{S}_B \rangle$ in the “near zone” other than the charges themselves. That is, the other part of the time-average Poynting vector, $\langle \mathbf{S} \rangle - \langle \mathbf{S}_B \rangle$, delivers energy steadily to some current-free regions in the “near zone,” where that energy is (mathematically) “converted” into the flow $\langle \mathbf{S}_B \rangle$ and transported to other regions of the “near zone,” where that energy is “reconverted” into the flow $\langle \mathbf{S} \rangle - \langle \mathbf{S}_B \rangle$.^{20,21}

In contrast, the time-average flow of energy described by $\langle \mathbf{S} \rangle = \langle \mathbf{S}_A \rangle$ moves smoothly from source currents through the “near zone” and on to the “far zone” (or into sink currents in the “near zone”).

6 Incident and Reflected/Scattered Radiation

Another notion about radiation that comes from optics is that it can usefully be decomposed into “incident” and “reflected” (or “scattered”) radiation.²² This decomposition is based on three assumptions that are not generally valid: all charges and currents reside in two widely separated regions, such that the back reaction of the “scattered” waves on the source region can be neglected; the observer is many wavelengths away from and charges and currents; and that the directions of the “incident” and “scattered” waves at the observer are obvious,

¹⁹As expected, eq. (28) vanishes when $\mathbf{r}_1 = \mathbf{r}_2$ even if $\mathbf{p}_1 \neq \mathbf{p}_2$, and when either p_1 or p_2 is zero.

²⁰One of the few discussions that shows awareness of this complicated scenario is given in [28]. See also [29, 30].

²¹Another awkwardness of definition B is that in the “radiation” energy density $\langle u_{\text{rad}} \rangle = \langle u_{B,\text{rad}} \rangle + \langle u_{E,\text{rad}} \rangle$ the magnetic and electric “radiation” energy densities are not, in general, equal to one another (although they are everywhere equal for idealized Hertzian dipole radiators). Lack of awareness of this fact has led to numerous faulty analyses of the relation of circuit reactance to electromagnetic fields, as reviewed in [31, 32].

²²In thermal physics one speaks of energy “radiated” and “absorbed” by a material surface at some temperature. If that surface is in thermal equilibrium with its surroundings, the rates of “radiation” and “absorption” of energy are equal. The “radiated” and “absorbed” energy is in the form of incoherent electromagnetic waves emitted by individual atoms or molecules in the material of the surface. Hence, this form of “radiation” is not subject to the issues addressed in this note, which concern coherent “radiation” by a system of charges and currents.

such that a directional detector can distinguish between the “incident” and the “scattered” radiation.

In the “near zone” of charges and currents, these assumptions are not realistic, and the decomposition does not give very meaningful results there. To see this in more detail, consider the decomposition

$$\mathbf{E} = \mathbf{E}_{\text{in}} + \mathbf{E}_{\text{scat}}, \quad \mathbf{B} = \mathbf{B}_{\text{in}} + \mathbf{B}_{\text{scat}}. \quad (29)$$

Then the Poynting vector can be written

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi}(\mathbf{E}_{\text{in}} + \mathbf{E}_{\text{scat}} \times (\mathbf{B}_{\text{in}} + \mathbf{B}_{\text{scat}})) \\ &= \frac{c}{4\pi}\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{in}} + \frac{c}{4\pi}\mathbf{E}_{\text{scat}} \times \mathbf{B}_{\text{scat}} + \frac{c}{4\pi}\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{scat}} + \frac{c}{4\pi}\mathbf{E}_{\text{scat}} \times \mathbf{B}_{\text{in}} \\ &= \mathbf{S}_{\text{in}} + \mathbf{S}_{\text{scat}} + \mathbf{S}_{\text{other}}, \end{aligned} \quad (30)$$

where

$$\mathbf{S}_{\text{in}} = \frac{c}{4\pi}\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{in}}, \quad (31)$$

$$\mathbf{S}_{\text{scat}} = \frac{c}{4\pi}\mathbf{E}_{\text{scat}} \times \mathbf{B}_{\text{scat}}, \quad (32)$$

$$\mathbf{S}_{\text{other}} = \frac{c}{4\pi}\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{scat}} + \frac{c}{4\pi}\mathbf{E}_{\text{scat}} \times \mathbf{B}_{\text{in}}. \quad (33)$$

The existence of the nontrivial cross term $\mathbf{S}_{\text{other}}$ does not permit a decomposition of the energy flow into only “incident” and “scattered” terms. In general, the energy “scattered” from a point is not only due to the direct effect of the “incident” wave, but also due to energy that arrives at that point as the result of “scattering” of the “incident” energy off other charges or currents in the “near zone.”

Of course, the decomposition (29) goes against the spirit of the “unified field theory” of classical electromagnetism, so it is to be expected that it leads to unsatisfactory results in general.²³

7 Radiation Near Good Conductors

In case of good conductors with simple geometry, such as planes, the decomposition (29) of the fields can often be accomplished by solving a boundary-value problem. An interesting example of this is the decomposition of the fields (whose phase velocity exceeds c) inside a rectangular waveguide into plane waves that propagate with velocity c and “zig-zag” down the guide [38]. It is still best to use the total Poynting vector to describe the flow of energy, which is parallel to the surface of the conductors in the region just outside them. On the scale of a wavelength away from the surface the flow can become complex, as, for example, in a waveguide [39] or in the “whirlpools” of energy flow that occur when a Gaussian laser beam reflects off a good conductor [40].

²³For a discussion of the surprising character of \mathbf{S}_{scat} for a plane wave incident on a small conducting sphere, see [37].

Some people appear reluctant to accept definition A because it implies that accelerated charges in good/perfect conductors do not radiate. As noted at least as early as 1897 in a discussion of radiation by wires [41], the tangential component of the electric field must vanish at the surface of a good/perfect conductor.²⁴ As a consequence the total Poynting vector has no component perpendicular to the surface of a good/perfect conductor at any time [45], and hence there is no net flow of energy into or out of a good conductor.²⁵ If we identify radiation with the total flow of energy, as in definition A, then we arrive at the so-called “radiation paradox” that the good conductors of antennas do not radiate (see sec. 6 of [47]). Rather, the radiation originates in the power source (which must contain some elements in which charges flow in other than good conductors), and is thereafter guided by the good conductors of the transmission line (if any) and of the nominal antenna. In this view, the nominal antenna plays only a somewhat passive role, whereas many antenna enthusiasts prefer a vision in which the nominal antenna plays a more active role, and often favor definition B.

8 Summary

In the view of this author, definition A of electromagnetic radiation as being the flow of electromagnetic energy described by the total Poynting vector is preferred over definition B that radiation is only that part of the Poynting vector due to the “radiation fields” in that

1. Definition B is based on a decomposition of the electric field into terms that cannot separately be measured, and thereby violates the spirit of the unified field theory of Faraday and Maxwell.
2. Definition B tends to become associated with the vision that “currents cause fields/ radiation,” while omitting to acknowledge that “fields/radiation cause currents;” whereas definition A defines radiation in terms of fields with less underlying implication of “cause” and “effect.”
3. Definition B does not lead to a concept of “radiation” as being due only to accelerated charges (although the related concept of “radiation fields” does associate parts of the electric and magnetic fields with accelerated charges).
4. In the case of time-harmonic currents and fields, definition A, but not definition B, is such that the time-average radiation (flow of energy) can be traced only to currents.

²⁴This permits formulation of an integral equation for the currents in a good/perfect conductor in terms of a time-harmonic “source” voltage. After solving numerically for the currents (see, for example, [42]), the fields (and the Poynting vector/radiation) can then be calculated. For an analytic review of this approach, see [43]. In the case of general time dependence, Maxwell’s equations and the equations of motion of charges/currents can be integrated numerically for time steps on a mesh, in which the good/perfect conductor boundary condition is enforced at each step [44].

²⁵In a microscopic model of currents as moving charges, there is a small, time-dependent kinetic energy associated with the currents, which energy is exchanged with the energy of the electromagnetic fields outside the conductor. This is accounted for by consideration of the small imaginary part of the conductivity in good, but not perfect conductors. See sec. 3.1 of [46].

5. Definition A is more compatible than definition B with the semiclassical view of radiation considered in the Weizsäcker-Williams model [12].

The appealing concept of “radiation fields” has clear physical significance only in the “far zone,” where other field components can generally be neglected,²⁶ and where definitions A and B of radiation are effectively the same. While “radiation fields” can also be defined in the “near zone,” use of only these fields to characterize the flow of electromagnetic energy in this region is, to this author, unsatisfactory.²⁷ In contrast, identifying the total Poynting vector with radiation provides a simple, consistent description of the flow of electromagnetic energy in all regions, while clarifying that in general not all accelerated charges emit radiation and that not all radiation is due to accelerated charges.

Similarly, a decomposition of the electromagnetic fields into “incident” and “scattered” terms does not lead to a satisfactory description of the flow of energy close to charges and currents, although the results of this decomposition are very appealing in the “far zone.”

The decompositions of the electromagnetic fields into “radiation” and “nonradiation” terms, or into “incident” and “scattered” terms have the possible merit of implying that all accelerated charges are directly involved in the process of “radiation”, while the total Poynting vector traces “radiation” back to only those accelerated charges not in good/perfect conductors, and relegates charges in the latter to the supporting role of “guiding” rather than generating the “radiation.”²⁸

Although the classical view of radiation as energy flow according to the total Poynting vector (definition A) does not fully capture the subtlety of the quantum view of the behavior of photons in the “near zone” of matter (as discussed briefly at the end of sec. 1), it remains the most consistent classical interpretation of radiation. The total Poynting vector provides a detailed (perhaps overly detailed) description of the flow of energy/radiation with minimal bias as to situational notions of “cause” and “effect”.

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²⁶Characterization of the angular momentum of electromagnetic fields in the “far zone” requires consideration of terms that fall off as $1/r^2$ as well as the dominant terms that fall off as $1/r$.

²⁷Some authors who appear to favor definition B (see, for example [48, 49]) seem to this author to be trying to find a particle-like description of classical radiation by emphasizing “time domain analysis” of narrow pulses. However, a “time-domain analysis” of electromagnetic fields is still a field theory, in which the flow of energy obeys Poynting’s theorem (15) and a changing density of electromagnetic field energy acts as a source of further energy flow (or a sink of incoming flow). See, for example, [27, 28, 29, 30]. The classical field theory of Maxwell does not contain the particle-like quantum notion that a photon cannot “split” into two photons (and that two photons cannot merge into one).

²⁸While no net energy flows across the surface of a good/perfect conductor, momentum does. This momentum flow, which can be described by Maxwell’s stress/momentum tensor, provides in principle a classical view of how charges in good conductors affect/guide Poynting flux/radiation.

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