

# Reactance of a Sinusoidally Driven Antenna

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## 1 Problem

Use Poynting's theorem for complex, time-harmonic fields to deduce general expressions for the reactance of an antenna that is operated at angular frequency  $\omega$ . Discuss whether this reactance can be separated into capacitive and inductive reactances. Consider also possible meanings of the concept of "reactive field energy."

## 2 Solution

*This problem is based on the overoptimistic claim in [1] that the reactance of an antenna can be decomposed into capacitive and inductive reactances. Deduction of antenna reactance via so-called complex Poynting theorem (sec. 13.14 of [2], sec. 2.20 of [3]) goes back at least to sec. 5 of [4]. See also chap. 8 of [5]. We consider only media with unit relative permittivity and permeability.*

For any system in which the charges and currents have time dependence  $e^{j\omega t}$  it is convenient to consider fields as complex vectors, of which only their real parts have physical meaning. For example, the real part of the complex Poynting vector,

$$\tilde{\mathbf{S}} = \frac{\mathbf{E} \times \mathbf{B}^*}{2\mu_0}, \quad (1)$$

is the time-average flow of energy in the electromagnetic fields.

Poynting's theorem [6] can be expressed in terms of complex fields as follows,

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{S}} &= \nabla \cdot \frac{\mathbf{E} \times \mathbf{B}^*}{2\mu_0} = \frac{\mathbf{B}^*}{2\mu_0} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \frac{\mathbf{B}^*}{2\mu_0} \\ &= -j\omega \frac{|B|^2}{2\mu_0} + j\omega \frac{\epsilon_0 |E|^2}{2} - \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} = -2j\omega(\langle u_B \rangle - \langle u_E \rangle) - \frac{\mathbf{E} \cdot \mathbf{J}^*}{2}, \end{aligned} \quad (2)$$

where the time-average densities of energy in the electromagnetic fields are

$$\langle u_B \rangle = \frac{|B|^2}{4\mu_0}, \quad \text{and} \quad \langle u_E \rangle = \frac{\epsilon_0 |E|^2}{4}. \quad (3)$$

When we integrate eq. (2) over some volume we obtain

$$\begin{aligned} \int \nabla \cdot \tilde{\mathbf{S}} d\text{Vol} &= \oint \tilde{\mathbf{S}} \cdot d\text{Area} = -2j\omega \int (\langle u_B \rangle - \langle u_E \rangle) d\text{Vol} - \int \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} d\text{Vol} \\ &= -2j\omega(\langle U_B \rangle - \langle U_E \rangle) - \int \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} d\text{Vol}, \end{aligned} \quad (4)$$

where  $\langle U_B \rangle = \int \langle u_B \rangle d\text{Vol}$  and  $\langle U_E \rangle = \int \langle u_E \rangle d\text{Vol}$  are the total, time-average energies of the magnetic and electric fields in that volume.

In the case of an antenna, we take the volume to be all the space outside the antenna and outside its power source, where we suppose that the power source for the antenna fits in the (small) space between its two terminals. Then,  $\mathbf{J} = 0$  everywhere in this volume, so  $\int \mathbf{E} \cdot \mathbf{J}^* d\text{Vol} = 0$ .

The surface of this volume has three regions: a sphere at “infinity,” the surface of the conductors of the antenna (excluding the small surfaces of the terminals that face the power supply), and the (small) surface of the power supply that does not face the terminals. Then,

$$\begin{aligned} \oint \tilde{\mathbf{S}} \cdot d\mathbf{Area} &= \oint_{\infty} \tilde{\mathbf{S}} \cdot d\mathbf{Area} + \oint_{\text{antenna}} \tilde{\mathbf{S}} \cdot d\mathbf{Area} + \oint_{\text{power supply}} \tilde{\mathbf{S}} \cdot d\mathbf{Area} \\ &= \langle P_{\text{rad}} \rangle + \langle P_{\text{Ohmic}} \rangle - \langle P_{\text{power supply}} \rangle. \end{aligned} \quad (5)$$

The integral of the Poynting vector over a sphere at “infinity” is the (time-average) power  $\langle P_{\text{rad}} \rangle$  that is “radiated to infinity.” The integral of the Poynting vector into the surface of the antenna is  $\langle P_{\text{Ohmic}} \rangle$ , taking note of the fact that the flow of energy across the surface of the antenna is zero in the limit of a perfect conductor (since then  $\mathbf{E}_{\text{tangential}} = 0$  and therefore  $\mathbf{S}_{\perp} = 0$ ), and that energy must flow into a resistive conductor to replace the Ohmic losses. Similarly, the power supply does not deliver energy into the antenna, but rather energy flows directly from the power supply into the volume outside it (and the antenna).<sup>1</sup>

Defining  $I$  to be the (complex) current at the terminals of the antenna, we write the (time-average) power as  $Z |I|^2 / 2$ , where for the surface at “infinity” we define  $Z = R_{\text{rad}} = 2 \langle P_{\text{rad}} \rangle / |I|^2$  = radiation resistance, for the antenna conductors we define  $Z = R_{\text{Ohmic}} = 2 \langle P_{\text{Ohmic}} \rangle / |I|^2$  = terminal resistance,<sup>2</sup> and for the power supply we define  $Z = Z_{\text{antenna}} =$  total terminal impedance of the antenna as seen by the power supply.

Combining eqs. (4) and (5) we have,

$$Z_{\text{antenna}} = \frac{2 \langle P_{\text{power supply}} \rangle}{|I|^2} = R_{\text{rad}} + R_{\text{Ohmic}} + \frac{4j\omega(\langle U_B \rangle - \langle U_E \rangle)}{|I|^2} \equiv R_{\text{antenna}} + jX_{\text{antenna}}, \quad (6)$$

where

$$R_{\text{antenna}} = R_{\text{rad}} + R_{\text{Ohmic}}, \quad \text{and} \quad X_{\text{antenna}} = \frac{4\omega(\langle U_B \rangle - \langle U_E \rangle)}{|I|^2} \equiv \omega L - \frac{1}{\omega C}, \quad (7)$$

introducing the antenna reactance  $X_{\text{antenna}}$ , inductance  $L$  and capacitance  $C$ . The total energies  $\langle U_B \rangle$  and  $\langle U_E \rangle$  are infinite, so we cannot immediately identify  $L = 4\omega \langle U_B \rangle / |I|^2$  and  $C = |I|^2 / 4\omega^2 \langle U_E \rangle$ .

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<sup>1</sup>The conductors of the antenna guide the energy from the power supply into the space around the antenna, but they do not generate this power. The antenna can be thought of as a waveguide, or an inside-out resonant cavity, which suggests that the concepts of capacitance and inductance may be relevant here.

<sup>2</sup>The terminal resistance depends on details of the current distribution in the antenna, and is not directly measurable by an “Ohm-meter.”

We note that the electric and magnetic fields can be related to the charge and current densities  $\rho$  and  $\mathbf{J}$  in the antenna and the power supply according to [7]

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) = & \frac{1}{4\pi\epsilon_0} \int \frac{\rho \hat{\mathbf{R}}}{R^2} e^{j(\omega t - kR)} d^3 \mathbf{x}' + \frac{\mu_0 c}{4\pi} \int \frac{(\mathbf{J} \cdot \hat{\mathbf{R}}) \hat{\mathbf{R}} + (\mathbf{J} \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R^2} e^{j(\omega t - kR)} d^3 \mathbf{x}' \\ & + \frac{j\mu_0 \omega}{4\pi} \int \frac{(\mathbf{J} \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R} e^{j(\omega t - kR)} d^3 \mathbf{x}', \end{aligned} \quad (8)$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2} e^{j(\omega t - kR)} d^3 \mathbf{x}' + \frac{j\mu_0 \omega}{4\pi c} \int \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R} e^{j(\omega t - kR)} d^3 \mathbf{x}', \quad (9)$$

where  $\mathbf{R} = \mathbf{x} - \mathbf{x}'$  and  $k = \omega/c$ , and  $c$  is the speed of light in vacuum. At large distances from the sources, the magnitudes of the electric and magnetic fields are related by  $E = cB$ . Hence, the difference  $U_B - U_E$  in the field energies is finite, and the reactance  $X_{\text{antenna}}$  is meaningfully calculated according to eq. (7).

It is common to identify the ‘‘radiation fields’’ as the terms in eqs. (8)-(9) whose integrand varies as  $1/R$ ,

$$\mathbf{E}_{\text{rad}} = \frac{j\mu_0 \omega}{4\pi} \int \frac{(\mathbf{J} \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{R} e^{j(\omega t - kR)} d^3 \mathbf{x}', \quad \mathbf{B}_{\text{rad}} = \frac{j\mu_0 \omega}{4\pi c} \int \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R} e^{j(\omega t - kR)} d^3 \mathbf{x}'. \quad (10)$$

*We note that while the total fields are asymptotically equal to the ‘‘radiation fields,’’ no measurement can distinguish between them at any finite distance from the sources. This warns us that results based on use of the ‘‘radiation fields’’ at finite distances from the source may not be physically meaningful.*

The energy densities  $\langle u_{B,\text{rad}} \rangle$  and  $\langle u_{E,\text{rad}} \rangle$  are asymptotically equal, so the difference  $\langle U_{B,\text{rad}} \rangle - \langle U_{E,\text{rad}} \rangle$  between the total energies in the ‘‘radiation fields’’ is finite (but nonzero in general<sup>3</sup>). If we define the ‘‘nonradiative’’ field energies as

$$\langle U_{B_{\text{nonrad}}} \rangle = \langle U_B \rangle - \langle U_{B_{\text{rad}}} \rangle, \quad \text{and} \quad \langle U_{E_{\text{nonrad}}} \rangle = \langle U_E \rangle - \langle U_{E_{\text{rad}}} \rangle, \quad (11)$$

then

$$\langle U_{B_{\text{nonrad}}} \rangle - \langle U_{E_{\text{nonrad}}} \rangle = \langle U_B \rangle - \langle U_E \rangle - \langle U_{B_{\text{rad}}} \rangle + \langle U_{E_{\text{rad}}} \rangle. \quad (12)$$

In [1] it is proposed that the antenna inductance and capacitance be identified as

$$L = \frac{4 \langle U_{B_{\text{nonrad}}} \rangle}{|I|^2}, \quad \text{and} \quad C = \frac{|I|^2}{4\omega^2 \langle U_{E_{\text{nonrad}}} \rangle} \quad (\text{as proposed in [1]}). \quad (13)$$

However, because  $\langle U_{B_{\text{rad}}} \rangle \neq \langle U_{E_{\text{rad}}} \rangle$ ,

$$\omega L - \frac{1}{\omega C} = \frac{4\omega}{|I|^2} (\langle U_{B_{\text{nonrad}}} \rangle - \langle U_{E_{\text{nonrad}}} \rangle) = \frac{4\omega}{|I|^2} (\langle U_B \rangle - \langle U_E \rangle - \langle U_{B_{\text{rad}}} \rangle + \langle U_{E_{\text{rad}}} \rangle) \neq X_{\text{antenna}}, \quad (14)$$

for the inductance and capacitance given in eq. (13).

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<sup>3</sup>Only for an idealized point antenna, such as a Hertzian dipole, does  $\langle U_{B,\text{rad}} \rangle = \langle U_{E,\text{rad}} \rangle$ . See also the Appendix.

While the concept of antenna impedance is well defined, the relation of that impedance to an inductance and a capacitance is ambiguous.<sup>4</sup> The latter are well defined only for antennas that are small compared to a wavelength, such that  $|E| \approx c|B|$  everywhere, and hence  $\langle U_{B_{\text{rad}}} \rangle - \langle U_{E_{\text{rad}}} \rangle \approx 0$ . In this case the capacitance and inductance of the antenna can be evaluated by quasistatic methods [10].

### 3 Can We Identify Reactive Field Energy?

The form of the antenna reactance found in eq. (7) suggests that we identify  $\langle U_B \rangle - \langle U_E \rangle$  as the time average of the reactive field energy [4, 5], in which case  $\langle u_B \rangle - \langle u_E \rangle$  is the (time-average) density of reactive field energy.<sup>5</sup> Can we also identify

$$u_{\text{reactive}} = u_B - u_E = \frac{(Re\mathbf{B})^2}{2\mu_0} - \frac{\epsilon_0(Re\mathbf{E})^2}{2} \quad (15)$$

as the instantaneous density of reactive field energy? Because the energy densities  $u_{B,\text{rad}}$  and  $u_{E,\text{rad}}$  of the radiation fields are asymptotically equal the density of reactive field energy falls off faster than  $1/r^2$  from source charges and currents, and we infer that any flow of reactive field energy does not extend to “infinity.”<sup>6</sup>

Writing the electric and magnetic fields as

$$\mathbf{E}(\mathbf{x}, t) = \tilde{\mathbf{E}}(\mathbf{x}) e^{j\omega t} = Re\tilde{\mathbf{E}} \cos \omega t - Im\tilde{\mathbf{E}} \sin \omega t + j(Im\tilde{\mathbf{E}} \cos \omega t + Re\tilde{\mathbf{E}} \sin \omega t), \quad (16)$$

$$\mathbf{B}(\mathbf{x}, t) = \tilde{\mathbf{B}}(\mathbf{x}) e^{j\omega t} = Re\tilde{\mathbf{B}} \cos \omega t - Im\tilde{\mathbf{B}} \sin \omega t + j(Im\tilde{\mathbf{B}} \cos \omega t + Re\tilde{\mathbf{B}} \sin \omega t), \quad (17)$$

eq. (15) becomes

$$u_{\text{reactive}} = u_B - u_E = \frac{(Re\tilde{\mathbf{B}})^2 \cos^2 \omega t + (Im\tilde{\mathbf{B}})^2 \sin^2 \omega t - Re\tilde{\mathbf{B}} \cdot Im\tilde{\mathbf{B}} \sin 2\omega t}{2\mu_0} - \epsilon_0 \frac{(Re\tilde{\mathbf{E}})^2 \cos^2 \omega t + (Im\tilde{\mathbf{E}})^2 \sin^2 \omega t - Re\tilde{\mathbf{E}} \cdot Im\tilde{\mathbf{E}} \sin 2\omega t}{2}, \quad (18)$$

whose time average is

$$\langle u_{\text{reactive}} \rangle = \frac{|\tilde{\mathbf{B}}|^2}{4\mu_0} - \frac{\epsilon_0 |\tilde{\mathbf{E}}|^2}{4} = \frac{|\mathbf{B}|^2}{4\mu_0} - \frac{\epsilon_0 |\mathbf{E}|^2}{4}. \quad (19)$$

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<sup>4</sup>The relation  $X_{\text{antenna}} = \omega L - 1/\omega C$  could be satisfied by defining  $L|I|^2/4 = \langle U_{B_{\text{nonrad}}} \rangle + \alpha(\langle U_{B_{\text{rad}}} \rangle - \langle U_{E_{\text{rad}}} \rangle)$  and  $|I|^2/4\omega^2 C = \langle U_{E_{\text{nonrad}}} \rangle + (\alpha - 1)(\langle U_{B_{\text{rad}}} \rangle - \langle U_{E_{\text{rad}}} \rangle)$  for any  $\alpha$ . One possible prescription is to take  $\alpha = 1$  if  $\langle U_{B_{\text{rad}}} \rangle - \langle U_{E_{\text{rad}}} \rangle \geq 0$ , and  $\alpha = 0$  if  $\langle U_{B_{\text{rad}}} \rangle - \langle U_{E_{\text{rad}}} \rangle < 0$ . An alternative [8] is to define  $L = (dX/d\omega + X/\omega)/2$  and  $C = 2/\omega^2(dX/d\omega - X/\omega)$ . However, the utility of any such choice is limited in that the capacitance and inductance could only be calculated via integrals of the fields, which requires knowledge of the charges and currents in the system. If these are known (via a computer program such as NEC4 [9] given the terminal voltage  $V$ ), the (complex) terminal current  $I$  is known and the terminal impedance can be directly calculated as  $Z_{\text{antenna}} = V/I$ .

<sup>5</sup>See [11] for discussion of the relation between reactive field energy and the  $Q$  of a resonant circuit.

<sup>6</sup>The so-called radiation fields contribute to reactive field energy, as shown in the Appendix.

For definition (15) to be consistent we should also be able to identify a part,  $\mathbf{S}_{\text{reactive}}$ , of the (real) Poynting vector  $\mathbf{S} = \text{Re}\mathbf{E} \times \text{Re}\mathbf{B}/\mu_0$  that obeys Poynting's theorem in the form

$$\nabla \cdot \mathbf{S}_{\text{reactive}} = -\frac{\partial u_{\text{reactive}}}{\partial t} - \text{Re}\mathbf{E} \cdot \text{Re}\mathbf{J}, \quad (20)$$

where  $\mathbf{J}$  is the current density. From (18) we find

$$-\frac{\partial u_{\text{reactive}}}{\partial t} = \omega \frac{[(\text{Re}\tilde{\mathbf{B}})^2 - (\text{Im}\tilde{\mathbf{B}})^2] \sin 2\omega t + 2\text{Re}\tilde{\mathbf{B}} \cdot \text{Im}\tilde{\mathbf{B}} \cos 2\omega t}{2\mu_0} - \epsilon_0 \omega \frac{[(\text{Re}\tilde{\mathbf{E}})^2 - (\text{Im}\tilde{\mathbf{E}})^2] \sin 2\omega t - 2\text{Re}\tilde{\mathbf{E}} \cdot \text{Im}\tilde{\mathbf{E}} \cos 2\omega t}{2}, \quad (21)$$

which has terms of time dependence  $\cos 2\omega t$  and  $\sin 2\omega t$ .

If we apply eq. (4) to a volume that does not contain any currents or the power source we obtain

$$\oint \text{Im}\tilde{\mathbf{S}} \cdot d\mathbf{Area} = -2\omega(\langle U_B \rangle - \langle U_E \rangle). \quad (22)$$

Equation (22) suggests that the imaginary part of the complex Poynting vector (1) is related to the flow of reactive field energy, in that we have come to associate the Poynting vector with flow of electromagnetic field energy. This relation is not immediately evident in that the reactive field energy  $\langle U_B \rangle - \langle U_E \rangle$  is constant in time, and cannot be said to flow.

The (real) Poynting vector is, recalling eqs. (16)-(17),

$$\begin{aligned} \mathbf{S} &= \frac{\text{Re}\mathbf{E} \times \text{Re}\mathbf{B}}{\mu_0} = \frac{(\text{Re}\tilde{\mathbf{E}} \cos \omega t - \text{Im}\tilde{\mathbf{E}} \sin \omega t) \times (\text{Re}\tilde{\mathbf{B}} \cos \omega t - \text{Im}\tilde{\mathbf{B}} \sin \omega t)}{\mu_0} \\ &= \frac{\text{Re}\tilde{\mathbf{E}} \times \text{Re}\tilde{\mathbf{B}} + \text{Im}\tilde{\mathbf{E}} \times \text{Im}\tilde{\mathbf{B}}}{2\mu_0} + \frac{\text{Re}\tilde{\mathbf{E}} \times \text{Re}\tilde{\mathbf{B}} - \text{Im}\tilde{\mathbf{E}} \times \text{Im}\tilde{\mathbf{B}}}{\mu_0} \cos 2\omega t \\ &\quad - \frac{\text{Re}\tilde{\mathbf{E}} \times \text{Im}\tilde{\mathbf{B}} + \text{Im}\tilde{\mathbf{E}} \times \text{Re}\tilde{\mathbf{B}}}{2\mu_0} \sin 2\omega t, \end{aligned} \quad (23)$$

and the complex Poynting vector is

$$\begin{aligned} \tilde{\mathbf{S}} &= \frac{\mathbf{E} \times \mathbf{B}^*}{2\mu_0} = [\text{Re}\tilde{\mathbf{E}} \cos \omega t - \text{Im}\tilde{\mathbf{E}} \sin \omega t + j(\text{Im}\tilde{\mathbf{E}} \cos \omega t + \text{Re}\tilde{\mathbf{E}} \sin \omega t)] \\ &\quad \times \frac{\text{Re}\tilde{\mathbf{B}} \cos \omega t - \text{Im}\tilde{\mathbf{B}} \sin \omega t - j(\text{Im}\tilde{\mathbf{B}} \cos \omega t + \text{Re}\tilde{\mathbf{B}} \sin \omega t)}{2\mu_0} \\ &= \frac{\text{Re}\tilde{\mathbf{E}} \times \text{Re}\tilde{\mathbf{B}} + \text{Im}\tilde{\mathbf{E}} \times \text{Im}\tilde{\mathbf{B}}}{2\mu_0} - j \frac{\text{Re}\tilde{\mathbf{E}} \times \text{Im}\tilde{\mathbf{B}} + \text{Im}\tilde{\mathbf{E}} \times \text{Re}\tilde{\mathbf{B}}}{2\mu_0}. \end{aligned} \quad (24)$$

Hence, the imaginary part of the complex Poynting vector,  $\text{Im}\tilde{\mathbf{S}}$ , is the coefficient of the term in the real Poynting vector  $\mathbf{S}$  that varies as  $\sin 2\omega t$ . However, we cannot consistently identify this term with the “reactive” part of the (real) Poynting vector, in that its divergence would also have time dependence  $\sin 2\omega t$  while  $\partial u_{\text{reactive}}/\partial t$  of eq. (21) has a term of time dependence  $\cos 2\omega t$ .

This leaves us without a crisp physical interpretation of  $\text{Im}\tilde{\mathbf{S}}$ , and without a fully consistent identification of reactive field energy. Such difficulties are typical of attempts to partition quadratic quantities like field energy and Poynting flux. See also the discussion in [12] and in the Appendix.

# A Appendix: Radiation Field Energy of a Pair of Oscillating Point Dipoles

We consider a system of only two charges,  $q_1$  and  $q_2$ , at (average) positions  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , with (complex) oscillating dipole moments  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . Then,

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) = & k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \times \hat{\mathbf{r}}_1 \frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0 r_1} + p_1 [3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_1)\hat{\mathbf{r}}_1 - \hat{\mathbf{p}}_1] \left( \frac{1}{r_1^3} + \frac{jk}{r_1^2} \right) \frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0} \\ & + k^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \times \hat{\mathbf{r}}_2 \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0 r_2} + p_2 [3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_2)\hat{\mathbf{r}}_2 - \hat{\mathbf{p}}_2] \left( \frac{1}{r_2^3} + \frac{jk}{r_2^2} \right) \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0}, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) = & \frac{\mu_0 ck^2 p_1}{4\pi} (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \left( \frac{1}{r_1} + \frac{1}{jkr_1^2} \right) e^{j(\omega t - kr_1)} \\ & + \frac{\mu_0 ck^2 p_2}{4\pi} (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \left( \frac{1}{r_2} + \frac{1}{jkr_2^2} \right) e^{j(\omega t - kr_2)}, \end{aligned} \quad (26)$$

where  $\mathbf{r}_j = \mathbf{x} - \mathbf{x}_j$ . The radiation fields are

$$\mathbf{E}_{\text{rad}}(\mathbf{x}, t) = k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \times \hat{\mathbf{r}}_1 \frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0 r_1} + k^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \times \hat{\mathbf{r}}_2 \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0 r_2}, \quad (27)$$

$$\mathbf{B}_{\text{rad}}(\mathbf{x}, t) = \mu_0 ck^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \frac{e^{j(\omega t - kr_1)}}{4\pi r_1} + \mu_0 ck^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \frac{e^{j(\omega t - kr_2)}}{4\pi r_2}. \quad (28)$$

The corresponding time-average field-energy densities are

$$\begin{aligned} \langle u_{E_{\text{rad}}} \rangle = & \frac{\epsilon_0 |E_{\text{rad}}|^2}{4} = \frac{k^4 p_1^2 (1 - |\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1|^2)}{64\pi^2 \epsilon_0 r_1^2} + \frac{k^4 p_2^2 (1 - |\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2|^2)}{64\pi^2 \epsilon_0 r_2^2} \\ & + \frac{k^4}{32\pi^2 \epsilon_0 r_1 r_2} \text{Re} \left\{ p_1 p_2^* e^{jk(r_2 - r_1)} [\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^* - (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_2^*) - (\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2^*) \right. \\ & \left. + (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2^*)(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2)] \right\}, \end{aligned} \quad (29)$$

$$\begin{aligned} \langle u_{B_{\text{rad}}} \rangle = & \frac{|B_{\text{rad}}|^2}{4\mu_0} = \frac{k^4 p_1^2 (1 - |\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1|^2)}{64\pi^2 \epsilon_0 r_1^2} + \frac{k^4 p_2^2 (1 - |\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2|^2)}{64\pi^2 \epsilon_0 r_2^2} \\ & + \frac{k^4}{32\pi^2 \epsilon_0 r_1 r_2} \text{Re} \left\{ p_1 p_2^* e^{jk(r_2 - r_1)} [(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^*) - (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_2^*)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1)] \right\}. \end{aligned} \quad (30)$$

In general,  $\langle u_{B_{\text{rad}}} \rangle \neq \langle u_{E_{\text{rad}}} \rangle$ , but in the far zone they become equal since  $\hat{\mathbf{r}}_1 = \hat{\mathbf{r}}_2$  there. Hence, radiation field energy contributes to reactive field energy according to the definition of eq. (15).

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