Circular Orbits Inside the Globe of Death

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Abstract

A disk or sphere rolling without slipping on the inside of a sphere in a uniform gravitational field can have stable circular orbits that lie wholly above the "equator", while a particle sliding freely cannot.

1 Introduction

In a recent article [1], Abramowicz and Szuszkiewicz remarked on an interesting analogy between orbits above the equator of a "wall of death" and orbits near a black hole; namely that the centrifugal force in both cases appears to point towards rather than away from the center of an appropriate coordinate system. Here we take a "wall of death" to be a hollow sphere on the Earth's surface large enough that a motorcycle can be driven on the inside of the sphere. The intriguing question is whether there exist stable orbits for the motorcycle that lie entirely above the equator (horizontal great circle) of the sphere.

In ref. [1] the authors stated that no such orbits are possible, perhaps recalling the well-known result for a particle sliding freely on the inside of a sphere in a uniform gravitational field. However, the extra degrees of freedom associated with a rolling disk (or sphere) actually do permit such orbits, in apparent defiance of intuition. In particular, the friction associated with the condition of rolling without slipping can in some circumstances have an upward component large enough to balance all other downward forces.

In this paper we examine the character of all circular orbits inside a fixed sphere, for both disks and spheres that roll without slipping. The rolling constraint is velocity dependent (non-holonomous), so explicit use of a Lagrangian is not especially effective [2]. Instead, we follow a vectorial approach as advocated by Milne (Chap. 17) [3]. This approach does utilize the rolling constraint (sec. 2.1), a careful choice of coordinates, and the elimination of the constraint force from the equations of motion (sec. 2.2), all of which are implicit in Lagrange's method. The vector approach is, of course, a convenient codification of earlier methods in which individual components were explicitly written out. Compare with classic works such as those of Lamb (Chap. 9) [4], Deimel (Chap. 7) [5] and Routh (Chap. 5) [6]. The related problem of a disk rolling without friction on the inside of a sphere has been treated by Kholmskaya [7].

Once the possible types of circular motion for disks rolling inside spheres are obtained in sec. 2.3, we make a numerical evaluation of the magnitude of the acceleration in g's, and of the required coefficient of static friction on some representative orbits. The resulting parameters are rather extreme, and the circus name "globe of death" [8, 9] seems apt.

Almost all circular orbits of disks inside a sphere are found to be stable against small oscillations (sec. 2.4), in support of the achievements of motorcycle stunt riders [8].

Discussions of disks and spheres rolling outside a fixed sphere are given in secs. 3 and 5, respectively. The case of a sphere rolling inside another sphere is reviewed in sec. 4.

2 Disk Rolling Inside a Fixed Sphere

2.1 Kinematics

We consider a thin disk of radius a rolling without slipping on the inner surface of a sphere of radius $r > a$. The analysis is performed in the lab frame, in which the sphere is fixed. The z-axis is vertical and upwards with origin at the center of the sphere as shown in Fig. 1. As the disk rolls on the sphere, the point of contact traces a path that is an arc of a circle during any short interval. We therefore introduce a set of axes (x', y', z') that are related to the circular motion of the point of contact.

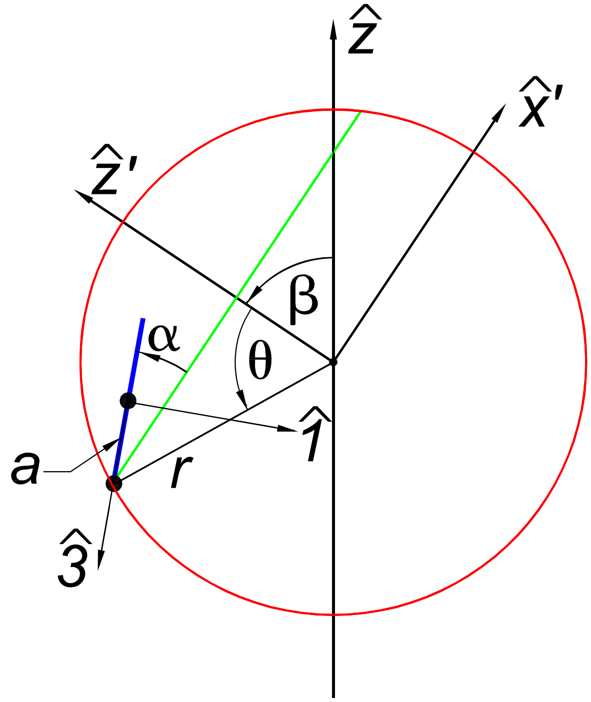

Figure 1: A thin disk of radius a rolls without slipping on a circular arc inside a fixed sphere of radius r. The arc sweeps out a cone of angle θ about the z'-axis, which axis makes angle β to the vertical. The x'-axis is orthogonal to the z'-axis in the z-z' plane The angle between the plane of the arc and diameter of the disk that includes the point of contact with the sphere is denoted by α. A right-handed triad of unit vectors, $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, is defined with $\hat{\mathbf{1}}$ along the axis of the disk and $\hat{\mathbf{3}}$ pointing from the center of the disk to the point of contact.

The normal to the plane of the arc traced by the point of contact that passes through the center of the sphere (and also through the center of the circle) is labeled z'. The angle between axes z and z' is β with $0 \le \beta \le \pi/2$. A radius from the center of the sphere to the point of contact of the disk sweeps out a cone of angle θ about the z' axis, where $0 \le \theta \le \pi$. The azimuthal angle of the point of contact on this cone is called ϕ, with $\phi = 0$ defined by the direction of the x'-axis, which is along the projection of the z-axis onto the plane of the arc, as shown in Fig. 2. Unit vectors are labeled with a superscript ˆ, so that $\hat{\mathbf{y}}' = \hat{\mathbf{z}}' \times \hat{\mathbf{x}}'$ completes the definition of the ′-coordinate system.

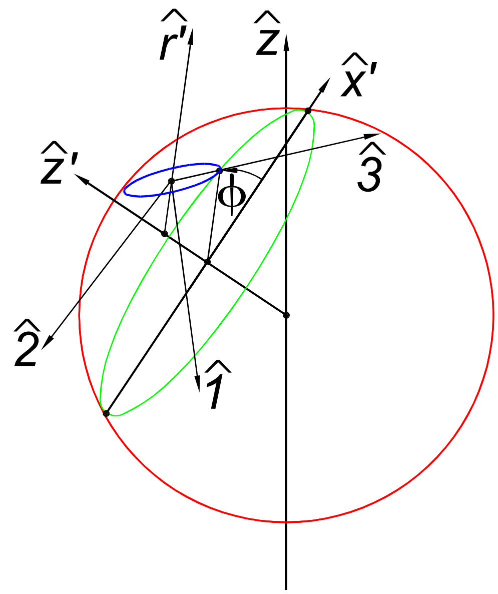

Figure 2: The azimuth of the point of contact of the disk with the sphere to the x'-axis is ϕ. The unit vector $\hat{\mathbf{r}}'$ is orthogonal to the z'-axis and points towards the center of the disk (or equivalently, towards the point of contact). Unit vector $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}} = \hat{\mathbf{z}}' \times \hat{\mathbf{r}}'$.

For a particle sliding freely, the only stationary orbits have $\beta = 0$ (horizontal circles) or $\beta = \pi/2$ (vertical great circles). For disks and spheres rolling inside a sphere it turns out that $\beta = 0$ or $\pi/2$ also, as we will demonstrate. However, the friction at the point of contact in the rolling cases permits orbits with a larger range of θ than in the sliding case. If $\beta = 0$ or $\pi/2$ were accepted as an assumption the derivation could be shortened somewhat.

We also introduce a right-handed coordinate triad of unit vectors $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ related to the geometry of the disk. Axis $\hat{\mathbf{1}}$ lies along the symmetry axis of the disk as shown in Fig. 1. Axis $\hat{\mathbf{3}}$ is directed from the center of the disk to the point of contact of the wheel with the sphere. The vector from the center of the wheel to the point of contact is then

$$\mathbf{a} = a\,\hat{\mathbf{3}}. \tag{1}$$

Axis $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}}$ lies in the plane of the disk, and also in the plane of the orbit (parallel to the x'-y' plane). The sense of axis $\hat{\mathbf{1}}$ is chosen so that the component ω_1 of the angular velocity vector $\boldsymbol{\omega}$ of the wheel about this axis is positive. Consequently, axis $\hat{\mathbf{2}}$ points in the direction of the velocity of the point of contact, and therefore is parallel to the tangent to the orbit.

The disk does not necessarily lie in the plane of the orbit. Indeed, it is the freedom to "bank" the disk that makes the "death-defying" orbits possible. The diameter of the disk through the point of contact (*i.e.*, axis $\hat{\mathbf{3}}$) makes angle α to the plane of the orbit. In general, a disk can have an arbitrary rotation about the $\hat{\mathbf{3}}$-axis, but the disk will roll without slipping only if angular velocity component ω_3 is such that the plane of the disk intersects the plane of the orbit along the tangent to the orbit at the point of contact. This will permit us to deduce a constraint on ω_3. The case of a rolling sphere is distinguished by the absence of this constraint, as considered later.

Since the wheel lies inside the sphere, as shown in Fig. 3, we can readily deduce the geometric relation that

$$\theta - \pi + \sin^{-1}(a/r) < \alpha < \theta - \sin^{-1}(a/r). \tag{2}$$

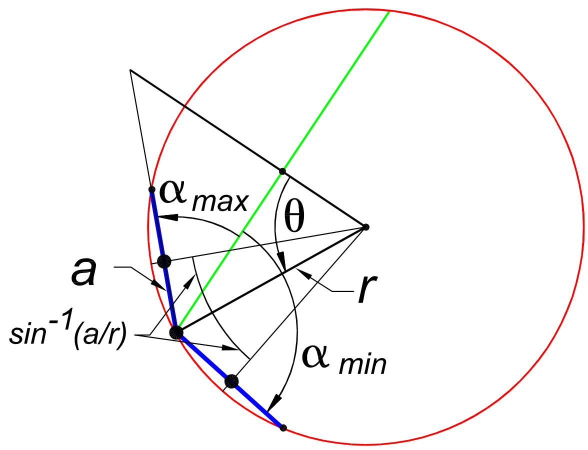

Figure 3: Geometry illustrating the extremes of angle α.

The vector from the center of the sphere to the center of mass of the disk is

$$\mathbf{r}_{\rm cm} = \mathbf{r} - a\ \hat{\mathbf{3}}, \tag{3}$$

where the vector $\mathbf{r}$ from the center of the sphere to the point of contact of the disk can be written as

$$\mathbf{r} = -r\cos(\theta - \alpha)\ \hat{\mathbf{1}} + r\sin(\theta - \alpha)\ \hat{\mathbf{3}}, \tag{4}$$

referring to Fig. 1. It is also useful to write this as

$$\mathbf{r}_{\rm cm} = r'\ \hat{\mathbf{r}}' + z'\ \hat{\mathbf{z}}', \tag{5}$$

where

$$r' = r\sin\theta - a\cos\alpha, \tag{6}$$

and

$$z' = r\cos\theta + a\sin\alpha, \tag{7}$$

as shown in Fig. 4. The vector $\hat{\mathbf{z}}' \times \hat{\mathbf{r}}'$ is in the direction of motion of the point of contact, which was defined previously to be direction $\hat{\mathbf{2}}$. That is, $(\hat{\mathbf{r}}', \hat{\mathbf{2}}, \hat{\mathbf{z}}')$ form a right-handed unit triad, which is related to the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ by

$$\hat{\mathbf{r}}' = \hat{\mathbf{2}} \times \hat{\mathbf{z}}' = -\sin\alpha\ \hat{\mathbf{1}} + \cos\alpha\ \hat{\mathbf{3}}, \tag{8}$$

and

$$\hat{\mathbf{z}}' = -\cos\alpha\ \hat{\mathbf{1}} - \sin\alpha\ \hat{\mathbf{3}}, \tag{9}$$

as can be seen from Fig. 1.

The length r' is negative when the center of the disk is on the opposite side of the z'-axis from the point of contact. This can occur for large enough a/r when the point of contact is near the z'-axis, such as when $\theta \approx 0$ and $\alpha < 0$ or $\theta \approx \pi$ and $\alpha > 0$.

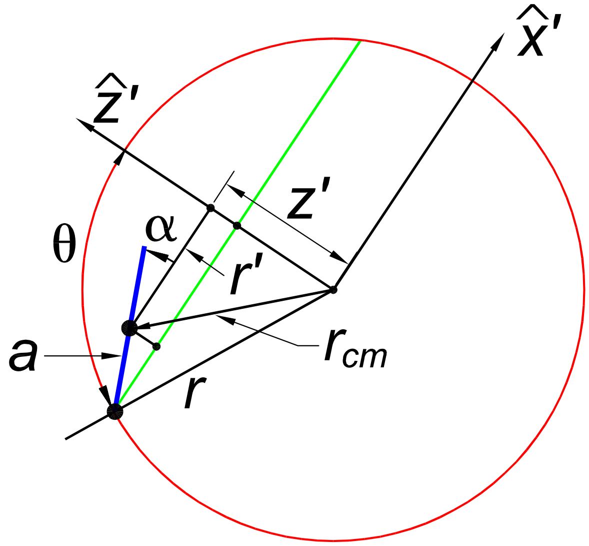

Figure 4: Geometry illustrating the vector $\mathbf{r}_{\rm cm} = r'\ \hat{\mathbf{r}}' + z'\ \hat{\mathbf{z}}'$ from the center of the sphere to the center of the disk.

Before discussing the dynamics of the problem, a considerable amount can be deduced from kinematics. The total angular velocity $\boldsymbol{\omega}$ can be thought of as composed of two parts,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{123} + \omega_{\rm rel}\ \hat{\mathbf{1}}, \tag{10}$$

where $\boldsymbol{\omega}_{123}$ is the angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, and $\omega_{\rm rel}\ \hat{\mathbf{1}}$ is the angular velocity of the disk relative to the triad; the relative angular velocity can only have a component along $\hat{\mathbf{1}}$ by definition. The angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ has component $\dot{\alpha}$ about the axis $\hat{\mathbf{2}}$ (where the dot indicates differentiation with respect to time), and has component $\dot{\phi}$ about the axis $\hat{\mathbf{z}}'$.

A distinction between a disk rolling without slipping on a sphere and on a plane [10] is that the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ can also rotate about the axis $\hat{\mathbf{2}} \times \hat{\mathbf{z}}' = \hat{\mathbf{r}}'$ in the case of the sphere. For rolling on a plane, axis $\hat{\mathbf{2}}$ is parallel to the plane, which precludes it from rotating about $\hat{\mathbf{r}}'$. We define the angular velocity of rotation of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ about the $\hat{\mathbf{r}}'$ axis to be $\dot{\psi}$.

The total angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ is thus,

$$\boldsymbol{\omega}_{123} = \dot{\psi}\ \hat{\mathbf{r}}' + \dot{\alpha}\ \hat{\mathbf{2}} + \dot{\phi}\ \hat{\mathbf{z}}' = -(\dot{\psi}\sin\alpha + \dot{\phi}\cos\alpha)\ \hat{\mathbf{1}} + \dot{\alpha}\ \hat{\mathbf{2}} + (\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)\ \hat{\mathbf{3}}, \qquad (11)$$

using eqs. (8)-(9). The time rates of change of the axes are therefore,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{1}} = (\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)\ \hat{\mathbf{2}} - \dot{\alpha}\ \hat{\mathbf{3}}, \qquad (12)$$

$$\frac{d\hat{\mathbf{2}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{2}} = (-\dot{\psi}\cos\alpha + \dot{\phi}\sin\alpha)\ \hat{\mathbf{1}} - (\dot{\psi}\sin\alpha + \dot{\phi}\cos\alpha)\ \hat{\mathbf{3}}, \qquad (13)$$

$$\frac{d\hat{\mathbf{3}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{3}} = \dot{\alpha}\ \hat{\mathbf{1}} + (\dot{\psi}\sin\alpha + \dot{\phi}\cos\alpha)\ \hat{\mathbf{2}}, \qquad (14)$$

Combining eqs. (10) and (11), we write the total angular velocity as

$$\boldsymbol{\omega} = \omega_1\ \hat{\mathbf{1}} + \dot{\alpha}\ \hat{\mathbf{2}} + (\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)\ \hat{\mathbf{3}}, \qquad (15)$$

where

$$\omega_1 = \omega_{\rm rel} - \dot{\psi}\sin\alpha - \dot{\phi}\cos\alpha. \qquad (16)$$

The constraint that the disk rolls without slipping relates the velocity of the center of mass to the angular velocity vector $\boldsymbol{\omega}$ of the disk. In particular, the instantaneous velocity of the point contact of the disk with the sphere is zero,

$$\mathbf{v}_{\rm contact} = \mathbf{v}_{\rm cm} + \boldsymbol{\omega} \times \mathbf{a} = 0. \qquad (17)$$

Hence,

$$\mathbf{v}_{\rm cm} = \frac{d\mathbf{r}_{\rm cm}}{dt} = a\hat{\mathbf{3}} \times \boldsymbol{\omega} = -a\dot{\alpha}\ \hat{\mathbf{1}} + a\omega_1\ \hat{\mathbf{2}}, \qquad (18)$$

using eqs. (1) and (15).

Additional kinematic relations can be deduced by taking the time derivative of eqs. (3)-(4),

$$\frac{d\mathbf{r}_{\rm cm}}{dt} = [r\dot{\theta}\sin(\theta-\alpha) - a\dot{\alpha}]\ \hat{\mathbf{1}} + [\dot{\phi}(r\sin\theta - a\cos\alpha) - \dot{\psi}(r\cos\theta + a\sin\alpha)]\ \hat{\mathbf{2}} + r\dot{\theta}\cos(\theta-\alpha)\ \hat{\mathbf{3}}. \quad (19)$$

Comparing this with eq. (18), we see that

$$\dot{\theta} = 0, \qquad (20)$$

and

$$a\omega_1 = (r\sin\theta - a\cos\alpha)\dot{\phi} - (r\cos\theta + a\sin\alpha)\dot{\psi} = r'\dot{\phi} - z'\dot{\psi}, \qquad (21)$$

recalling eqs. (6)-(7). Thus, we learn that the condition of rolling without slipping requires the point of contact to move at a constant distance $r\sin\theta$ from the axis z' of its instantaneous circular arc on the sphere. In addition, we find that the "spin" angular velocity ω_1 is a known function of the three variables α, $\dot{\phi}$ and $\dot{\psi}$ (and the three constants a, r and θ).

Except for axis $\hat{\mathbf{1}}$, the rotating axes $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ are not body axes, but the inertia tensor $\mathbf{I}$ is diagonal with respect to them in view of the symmetry of the disk. We write

$$I_{11} = 2kma^2, \qquad I_{22} = kma^2 = I_{33}, \tag{22}$$

which holds for any thin, circularly symmetric disk according to the perpendicular axis theorem; $k = 1/2$ for a wheel of radius a with mass m concentrated at the rim, $k = 1/4$ for a uniform disc, *etc.* The angular momentum $\mathbf{L}_{\rm cm}$ of the disk with respect to its center of mass can now be written as

$$\mathbf{L}_{\rm cm} = \mathbf{I} \cdot \boldsymbol{\omega} = kma[2\omega_1\ \hat{\mathbf{1}} + a\dot{\alpha}\ \hat{\mathbf{2}} + a(\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)\ \hat{\mathbf{3}}]. \tag{23}$$

We also note that the kinetic energy T of the disk is

$$\begin{aligned} T &= \frac{1}{2}mv_{\rm cm}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \\ &= \frac{1}{2}ma^2[(2k+1)\omega_1^2 + (k+1)\dot{\alpha}^2 + k(\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)^2], \end{aligned} \tag{24}$$

and the potential energy V is

$$\begin{aligned} V &= mgz_{\rm cm} = mg(z'\cos\beta + r'\cos\phi\sin\beta) \\ &= mg[r(\cos\theta\cos\beta + \sin\theta\sin\beta\cos\phi) + a(\sin\alpha\cos\beta - \cos\alpha\sin\beta\cos\phi)], \end{aligned} \tag{25}$$

referring to Fig. 2. Under the assumption of rolling without slipping, and ignoring other dissipative forces, the energy

$$E = T + V \tag{26}$$

is a constant of the motion.

2.2 Equations of Motion

The force of contact of the sphere on the wheel is labeled $\mathbf{F}$. For the wheel to be in contact with the sphere the force $\mathbf{F}$ must have a component towards the center of the sphere, which will be verified after the motion is obtained.

The equation of motion of the center of mass of the wheel is

$$m\frac{d^2\mathbf{r}_{\rm cm}}{dt^2} = \mathbf{F} - mg\hat{\mathbf{z}}, \tag{27}$$

where g is the acceleration due to gravity. The equation of motion for the angular momentum $\mathbf{L}_{\rm cm}$ about the center of mass is

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = \mathbf{N}_{\rm cm} = \mathbf{a} \times \mathbf{F}. \tag{28}$$

We eliminate the unknown force $\mathbf{F}$ in eq. (28) via eqs. (1) and (27) to find

$$\frac{1}{ma}\frac{d\mathbf{L}_{\rm cm}}{dt} + \frac{d^2\mathbf{r}_{\rm cm}}{dt^2} \times \hat{\mathbf{3}} = g\ \hat{\mathbf{3}} \times \hat{\mathbf{z}}. \tag{29}$$

Using eqs. (12)-(14) and (23), we have

$$\frac{1}{ma}\frac{d\mathbf{L}}{dt} = 2ka\dot{\omega}_1\ \hat{\mathbf{1}} + ka[\ddot{\alpha} + (\dot{\psi}\cos\alpha - \dot{\phi}\sin\alpha)(2\omega_1 + \dot{\psi}\sin\alpha + \dot{\phi}\cos\alpha)]\cos\alpha)\ \hat{\mathbf{2}}$$
$$+ka[\ddot{\psi}\cos\alpha - \ddot{\phi}\sin\alpha - 2\dot{\alpha}(\omega_1 + \dot{\psi}\sin\alpha + \dot{\phi}\cos\alpha)\ \hat{\mathbf{3}}. \qquad (30)$$

By differentiating eq. (18) we find

$$\frac{d^2\mathbf{r}_{\rm cm}}{dt^2} = a(-\ddot{\alpha} - \omega_1\dot{\psi}\cos\alpha + \omega_1\dot{\phi}\sin\alpha)\ \hat{\mathbf{1}} + a(\dot{\omega}_1 - \dot{\alpha}\dot{\psi}\cos\alpha + \dot{\alpha}\dot{\phi}\sin\alpha)\ \hat{\mathbf{2}}$$
$$+a(\dot{\alpha}^2 - \omega_1\dot{\psi}\sin\alpha - \omega_1\dot{\phi}\cos\alpha)\ \hat{\mathbf{3}}, \qquad (31)$$

so that

$$\frac{d^2\mathbf{r}_{\rm cm}}{dt^2} \times \hat{\mathbf{3}} = a(\dot{\omega}_1 - \dot{\alpha}\dot{\psi}\cos\alpha + \dot{\alpha}\dot{\phi}\sin\alpha)\ \hat{\mathbf{1}} + a(\ddot{\alpha} + \omega_1\dot{\psi}\cos\alpha - \omega_1\dot{\phi}\sin\alpha)\ \hat{\mathbf{2}}. \qquad (32)$$

To evaluate $\hat{\mathbf{3}} \times \hat{\mathbf{z}}$, we first express $\hat{z}$ in terms of the triad $(\hat{\mathbf{r}}', \hat{\mathbf{2}}, \hat{\mathbf{z}}')$, and then transform to triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$. When the point of contact of the wheel (and hence the $\hat{\mathbf{r}}'$-axis) has azimuth ϕ relative to the $\hat{\mathbf{x}}'$ axis, the $\hat{\mathbf{z}}$ axis has azimuth $-\phi$ relative to the $\hat{\mathbf{r}}'$ axis. Hence,

$$\hat{\mathbf{z}} = \sin\beta\cos\phi\ \hat{\mathbf{r}}' - \sin\beta\sin\phi\ \hat{\mathbf{2}} + \cos\beta\ \hat{\mathbf{z}}' \qquad (33)$$
$$= -(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi)\ \hat{\mathbf{1}} - \sin\beta\sin\phi\ \hat{\mathbf{2}} - (\sin\alpha\cos\beta - \cos\alpha\sin\beta\cos\phi)\ \hat{\mathbf{3}},$$

using eqs. (9)-(8). Thus,

$$\hat{\mathbf{3}} \times \hat{\mathbf{z}} = \sin\beta\sin\phi\ \hat{\mathbf{1}} - (\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi)\ \hat{\mathbf{2}}. \qquad (34)$$

The $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ components of the equation of motion are thus,

$$(2k+1)\dot{\omega}_1 + \dot{\alpha}(\dot{\phi}\sin\alpha - \dot{\psi}\cos\alpha) = \frac{g}{a}\sin\beta\sin\phi, \qquad (35)$$

$$[(2k+1)\omega_1 + k(\dot{\phi}\cos\alpha + \dot{\psi}\sin\alpha)](\dot{\phi}\sin\alpha - \dot{\psi}\cos\alpha) - (k+1)\ddot{\alpha} = \frac{g}{a}(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi), \qquad (36)$$

and

$$\ddot{\phi}\sin\alpha - \ddot{\psi}\cos\alpha + 2\dot{\alpha}\dot{\phi}\cos\alpha + 2\dot{\alpha}\dot{\psi}\sin\alpha + 2\omega_1\dot{\alpha} = 0. \qquad (37)$$

The angular velocity ω_1 is a function of α, $\dot{\phi}$ and $\dot{\psi}$ according to eq. (21), so that

$$a\dot{\omega}_1 = r'\ddot{\phi} - z'\ddot{\psi} + a\dot{\alpha}(\dot{\phi}\sin\alpha - \dot{\psi}\cos\alpha), \qquad (38)$$

and the equation of motion (35) can also be written as

$$(2k+1)(r'\ddot{\phi} - z'\ddot{\psi}) + 2(k+1)a\dot{\alpha}(\dot{\phi}\sin\alpha - \dot{\psi}\cos\alpha) = g\sin\beta\sin\phi, \qquad (39)$$

While eq. (20) indicates that angle θ is constant, we do not expect that angle β is constant also. The three equations of motion (36)-(37) can be supplemented by conservation of energy (26) to determine the time dependence of α, β, $\dot{\phi}$ and $\dot{\psi}$.

2.3 Circular Motion of the Center of Mass

We begin by considering the case that tilt angle α is constant, and that the angular velocity $\dot{\psi}$ of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ about axis $\hat{\mathbf{r}}'$ vanishes. However, the angular velocity $\dot{\phi}$ about the z' axis can vary for circular motion of the center of mass of the disk. Then, $\omega_1 = r'\dot{\phi}/a$, and the equations of motion (36)-(37) and (39) become

$$(2k+1)a\dot{\omega}_1 = (2k+1)r'\ddot{\phi} = g \sin\beta \sin\phi, \tag{40}$$

$$(2k+1)\omega_1\dot{\phi}\sin\alpha + k\dot{\phi}^2 \sin\alpha\cos\alpha = \frac{g}{a}(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi), \tag{41}$$

and

$$\ddot{\phi}\sin\alpha = 0. \tag{42}$$

From eq. (42) we learn that for constant α either $\sin\alpha = 0$ or $\ddot{\phi} = 0$.

2.3.1 Vertical Orbits

We first consider the simpler case that $\sin\alpha = 0$, which implies that the plane of the wheel lies in the plane of the orbit. For a disk inside the sphere with $\sin\alpha = 0$, we must have $\alpha = 0$ to satisfy the geometric constraint (2). Then eq. (41) can only be satisfied if $\cos\beta = 0$; *i.e.*, $\beta = \pi/2$ and the plane of the orbit is vertical. The remaining equation of motion (40) now reads

$$(2k+1)r'_0\ddot{\phi} = g\sin\phi, \tag{43}$$

with $r'_0 = r\sin\theta - a > 0$, which integrates to

$$\frac{2k+1}{2}mr_0'^2(\dot{\phi}^2 - \dot{\phi}_0^2) = mgr'_0(1-\cos\phi), \tag{44}$$

where $\dot{\phi}_0$ is the angular velocity at the top of the orbit, where $\phi_0 = 0$. Equation (44) expresses conservation of energy. The angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}_{\rm cm}$ vary in magnitude but are always perpendicular to the plane of the orbit.

The requirement that the disk stay in contact with the sphere is that the contact force $\mathbf{F}$ have component $F_\perp$ that points to the center of the sphere. On combining eqs. (27), (33), (31) and (43) we find

$$\mathbf{F} = -\frac{2k}{2k+1}mg\sin\phi\ \hat{\mathbf{2}} + m(g\cos\phi - r'_0\dot{\phi}^2)\ \hat{\mathbf{3}}. \tag{45}$$

The contact force is in the plane of the orbit, so the resulting torque about the center of mass of the disk changes the magnitude but not the direction of the angular momentum. On the vertical orbits, axis $\hat{\mathbf{2}}$ is tangent to the sphere, and axis $\hat{\mathbf{3}}$ makes angle $\pi/2 - \theta$ to the radius from the center of the sphere to the point of contact. Hence,

$$F_\perp = -F_3\sin\theta \tag{46}$$

is positive and the orbit is physical so long as the angular velocity $\dot{\phi}_0$ at the peak of the orbit obeys

$$\dot{\phi}_0^2 > \frac{g}{r'_0}, \tag{47}$$

as readily deduced from elementary considerations as well.

The required coefficient μ of static friction is given by $\mu = F_\parallel / F_\perp$ where

$$F_\parallel = \sqrt{F_3^2 \cos\theta^2 + F_2^2} \tag{48}$$

is the component of the contact force parallel to the local surface of the sphere. We see that

$$\mu = \cot\theta \sqrt{1 + (F_2/F_3\cos\theta)^2}, \tag{49}$$

which must be greater than $\cot\theta$, but only much greater if the disk nearly loses contact at the top of the orbit. Hence, orbits with $\pi/4 \lesssim \theta \le \pi/2$ are consistent with the friction of typical rubber wheels, namely $\mu \lesssim 1$.

Because a wheel experiences friction at the point of contact, vertical orbits are possible with $\theta < \pi/2$. This is in contrast to the case of a particle sliding freely on the inside of a sphere for which the only vertical orbits are great circles ($\theta = \pi/2$). The only restriction in the present case is that the wheel fits inside the sphere, *i.e.*, $r\sin\theta > a$, and that the minimum angular velocity satisfy eq. (47).

2.3.2 Horizontal Orbits

The second class of orbits is defined by $\ddot{\phi} = 0$, so that the condition (42) for circular motion implies that the precession angular velocity is constant, say $\dot{\phi} = \Omega_0$. The motion in horizontal circular orbits is steady. Then, $\omega_1 = r_0'\Omega_0/a$ is constant also, and from eq. (40) we see that $\sin\beta = 0$, and hence $\beta = 0$ for these orbits, which implies that they are horizontal. The angle α has constant value α_0. Then, eq. (41) gives the relation between the required angular velocity Ω_0 and the geometrical parameters of the orbit:

$$\Omega_0^2 = \frac{g\cot\alpha_0}{(2k+1)r_0' + ka\cos\alpha_0} = \frac{g\cot\alpha_0}{(2k+1)r\sin\theta - (k+1)a\cos\alpha_0}, \tag{50}$$

recalling eq. (6). In effect, this steady motion is the same as for a disk rolling on a horizontal plane, for which condition (50) holds as well [3, 6, 10].

There are no steady horizontal orbits for which $\alpha_0 = 0$, *i.e.*, for which the disk lies in the plane of the orbit. For such an orbit the angular momentum would be constant, but the torque on the disk would be nonzero in contradiction.

In the following we will find that horizontal orbits are possible only for $0 < \alpha_0 < \pi/2$.

First, the requirement that $\Omega_0^2 > 0$ for real orbits puts various restrictions on the parameters of the problem. We examine these for the four quadrants of angle α_0.

1. $0 < \alpha_0 < \pi/2$. Then $\cot\alpha_0 > 0$ so we must have

$$r_0' > -\frac{ka\cos\alpha_0}{2k+1}. \tag{51}$$

   This is satisfied by all $r_0' > 0$ and some $r_0' < 0$. However, for the disk to fit inside the sphere with $0 < \alpha_0 < \pi/2$, we can have $r_0' < 0$ only for $\theta > \pi/2$ according to eqs. (2) and (6).

2. $\pi/2 < \alpha_0 < \pi$. Then $\cos\alpha_0 < 0$ and $\cot\alpha_0 < 0$ so the numerator of (50) is negative and the denominator is positive. Hence, Ω_0 is imaginary and there are no circular orbits in this quadrant.

3. $-\pi < \alpha_0 < -\pi/2$. Then $\cos\alpha_0 < 0$, but $\cot\alpha_0 > 0$, so $\Omega_0^2 > 0$ and $r_0' > 0$ and eq. (50) imposes no to restriction. For the disk to fit inside the sphere with α_0 in this quadrant we must have $\theta < \pi/2$.

4. $-\pi/2 < \alpha_0 < 0$. Then $\cot\alpha_0 < 0$ so we must have

$$r_0' < -\frac{ka\cos\alpha_0}{2k+1} < 0. \tag{52}$$

   For the disk to be inside the sphere with $r_0' < 0$ and α_0 in this quadrant we must have $\theta < \pi/2$.

To obtain further restrictions on the parameters we examine under what conditions the disk remains in contact with the sphere. The contact force $\mathbf{F}$ is deduced from eqs. (27), (31) and (33) to be

$$\mathbf{F}/m = (-g\cos\alpha_0 + r_0'\Omega_0^2\sin\alpha_0)\ \hat{\mathbf{1}} - (g\sin\alpha_0 + r_0'\Omega_0^2\cos\alpha_0)\ \hat{\mathbf{3}}. \tag{53}$$

It is more useful to express $\mathbf{F}$ in components along the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ axes where $\hat{\mathbf{r}}$ points away from the center of the sphere and $\hat{\boldsymbol{\theta}}$ points towards increasing angle θ. The two sets of axes are related by a rotation about axis $\hat{\mathbf{2}}$:

$$\hat{\mathbf{1}} = -\cos(\theta-\alpha_0)\ \hat{\mathbf{r}} + \sin(\theta-\alpha_0)\ \hat{\boldsymbol{\theta}}, \qquad \hat{\mathbf{3}} = \sin(\theta-\alpha_0)\ \hat{\mathbf{r}} + \cos(\theta-\alpha_0)\ \hat{\boldsymbol{\theta}}, \tag{54}$$

so that

$$\begin{aligned} \mathbf{F}/m &= -(r_0'\Omega_0^2\sin\theta - g\cos\theta)\ \hat{\mathbf{r}} - (r_0'\Omega_0^2\cos\theta + g\sin\theta)\ \hat{\boldsymbol{\theta}} \\ &= -r_0'\Omega_0^2\ \hat{\mathbf{r}}' + g\ \hat{\mathbf{z}}. \end{aligned} \tag{55}$$

The second form of eq. (55) follows directly from elementary considerations. The inward component of the contact force, $F_\perp = -F_r$, is positive and the orbits are physical provided

$$r_0'\Omega_0^2 > g\cot\theta. \tag{56}$$

There can be no orbits with $r_0' < 0$ and $\theta < \pi/2$, which rules out orbits in quadrant 4 of α_0, *i.e.*, for $-\pi/2 > \alpha_0 < 0$.

Using eq. (50) for Ω^2 in eq. (56) we deduce that contact is maintained for orbits with $r_0' > 0$ only if

$$\cot\alpha_0 > [2k + 1 + k(a/r_0')\cos\alpha_0]\cot\theta. \tag{57}$$

For $r_0' < 0$ the sign of the inequality is reversed.

In the third quadrant of α_0 we have $\cos\alpha_0 < 0$, so inequality (57) can be rewritten with the aid of (6) as

$$\cot\alpha_0 > \left(1 + 2k - \frac{k}{1 + r\sin\theta/a\,|\cos\alpha_0|}\right)\cot\theta > \cot\theta. \tag{58}$$

However, in this quadrant inequality (2) tells us

$$\cot \alpha_0 < \cot \left[\theta + \sin^{-1}(a/r) \right] < \cot \theta. \tag{59}$$

Hence, there can be no circular orbits with $-\pi < \alpha_0 < -\pi/2$.

Thus, steady horizontal orbits are possible only for $0 < \alpha_0 < \pi/2$. Furthermore, since the factor in brackets of inequality (57) is roughly 2 for a uniform disk, this kinematic constraint is somewhat stronger than the purely geometric relation (2). However, a large class of orbits remains with $\theta < \pi/2$ as well as $\theta > \pi/2$.

The coefficient of friction μ at the point of contact must be at least $F_\parallel / F_\perp$ where $F_\parallel = |F_\theta|$ from eq. (55). (For $\theta > \pi/2$ and α_0 near zero the tangential friction F_θ can sometimes point in the $+\theta$ direction.) Hence, we need

$$\mu \geq \frac{|r_0' \Omega_0^2 \cos\theta + g \sin\theta|}{r_0' \Omega_0^2 \sin\theta - g\cos\theta}. \tag{60}$$

For example, a coefficient of friction of unity is required for

$$\cot \alpha_0 \approx (2k+1) \frac{1 + \cot\theta}{1 - \cot\theta}, \tag{61}$$

using eq. (50) in the approximation that $(2k+1) r_0' \gg ka \cos \alpha_0$. Then, $\mu = 1$ is required for orbits such that

$$\alpha_0 \approx \theta - \pi/4, \tag{62}$$

as illustrated in Fig. 5.

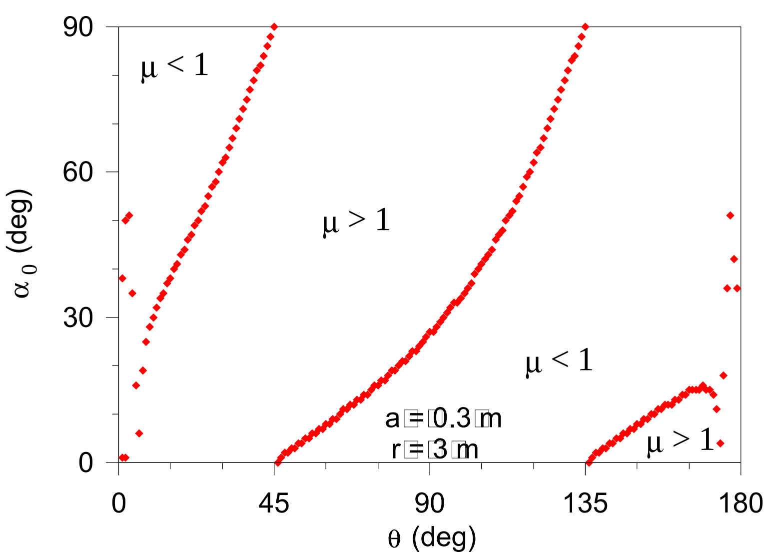

Figure 5: The regions of the α_0-θ plane for which the required coefficient of friction μ is greater than or less than 1 according to eq. (60), for the conditions of Table 1.

The acceleration of the center of mass of the disk is $r'_0\Omega_0^2$, so according to eq. (50) the corresponding number of g's is

$$\frac{\cot\alpha_0}{2k + 1 + k(a/r'_0)\cos\alpha_0}. \tag{63}$$

Table 1 lists parameters of several horizontal orbits for a sphere of size as might be found in a motorcycle circus. The coefficient of friction of rubber tires is of order one, so orbits more than a few degrees above the equator involve very strong accelerations. The head of the motorcycle rider is closer to the vertical axis of the sphere than is the center of the disk, so the number of g's experienced by the rider is somewhat less than that given in the Table.

Table 1: Parameters for horizontal circular orbits of a hoop of radius $a = 0.3$ m rolling inside a sphere of radius $r = 3$ m. The hoop has coefficient $k = 1/2$ pertaining to its moment of inertia. The minimum coefficient of friction required to support the motion is μ. The magnitude of the horizontal acceleration of the center of mass is reported as the No. of g's.

θ (deg.)	α_0 (deg.)	μ	$v_{\rm cm}$ (m/s)	No. of g's
15	5	16.1	4.8	48
30	5	2.82	8.0	53
45	10	2.15	7.0	27
60	10	1.19	7.9	27
60	25	3.45	4.9	10
75	15	0.96	6.8	18
75	30	2.13	4.7	8
90	25	0.96	5.3	10
90	45	2.04	3.7	5
135	60	0.56	2.3	3

Figure 6 illustrates the allowed values of the tilt angle α_0 as a function of the angle θ of the plane of the orbit, for $a/r = 0.1$ as in Table 1.

From eq. (50) we see that $\alpha_0 = \pi/2$, $\Omega_0 = 0$ is a candidate "orbit" in the lower hemisphere. On such an "orbit" the disk is standing vertically at rest, and is not stable against falling over. We infer that stability will only occur for Ω_0 greater than some minimum value not revealed by the analysis thus far.

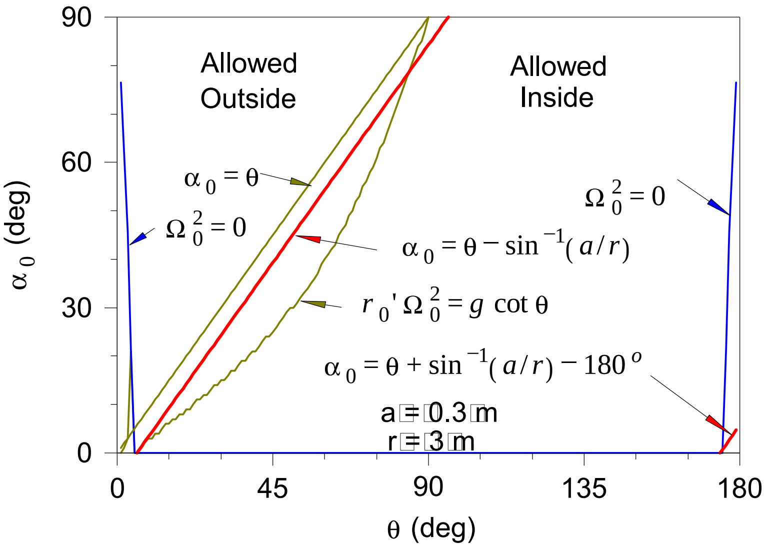

Figure 6: The allowed values of the tilt angle α_0 as a function of the angle θ of horizontal orbits for the conditions of Table 1. The allowed region is bounded by four curves, derived from expressions (2), (50) and (56).

2.4 Stability Analysis

We analyze the cases of horizontal and vertical orbits separately.

2.4.1 Vertical Orbits

For unperturbed vertical orbits, the values of angle α and angular velocity $\dot{\psi}$ are zero, $\beta = \pi/2$, and we label the unperturbed angular velocity $\dot{\phi}$ as Ω. We consider small oscillatory perturbations of the form

$$\alpha = \epsilon \cos \varpi t, \tag{64}$$

$$\beta = \pi/2 + \eta \cos \varpi t, \tag{65}$$

$$\dot{\phi} = \Omega + \delta \cos \varpi t, \tag{66}$$

$$\dot{\psi} = \gamma \cos \varpi t, \tag{67}$$

where ϵ, η, δ and γ are small constants.

Inserting the perturbations (64)-(67) into eq. (37), we find

$$\dot{\Omega}\epsilon \cos \varpi t + \left[\gamma - 2\left(1 + \frac{r_0'}{a}\right)\Omega\epsilon\right]\varpi \sin \varpi t = 0. \tag{68}$$

This equation cannot be satisfied in general, which indicates that a stability analysis based on the hypothesis of small oscillations is not adequate for vertical orbits. However, we note that at the top of the orbit ($\phi = 0$), $\dot{\Omega} = 0$ according to eq. (43), and $\Omega \equiv \Omega(0)$ is a minimum, so gyroscopic stability will be the least here. Hence, we restrict our attention to the top of the orbit, where an analysis of small oscillations can be carried out.

At $\phi = 0$, eq. (68) tell us that

$$\gamma = 2\frac{r \sin\theta}{a}\Omega(0)\epsilon, \tag{69}$$

recalling that $r'_0 = r \sin\theta - a$ for vertical orbits. Inserting the perturbations (64)-(67) into eq. (39), we find that

$$\delta = \frac{z'_0}{r'_0}\gamma = \frac{r^2 \sin 2\theta}{ar'_0}\Omega(0)\epsilon. \tag{70}$$

Inserting the perturbations (64)-(67) into the conserved energy (26) and using eq. (70), we find that the perturbation η of angle β vanishes. Then, inserting the perturbations (64)-(67) into eq. (36) and using eq. (69), we find that

$$(k+1)\varpi^2 = \frac{g}{a} + \frac{\Omega^2(0)}{a^2}(2r'_0 + a)[(2k+1)r'_0 + ka] > 0. \tag{71}$$

Thus, ϖ is always real, and the motion near $\phi = 0$ (and by extension for all ϕ) is always stable against small perturbations.

However, condition (47) that the disk stay in contact with the sphere must still be satisfied. All vertical orbits for which the disk remains in contact with the sphere are stable against small perturbations.

The stability analysis yields the result that if $(\phi, \dot\phi) = (0, 0)$ then $\varpi = \sqrt{g/(k+1)a}$. We recognize this as the frequency of oscillation of a simple pendulum formed by suspending the disk from a point on its rim, the motion being perpendicular to the plane of the disk.

2.4.2 Horizontal Orbits

We now suppose that α, β, $\dot\phi$ and $\dot\psi$ undergo oscillations at angular frequency ϖ about their values α_0, 0, Ω_0 and 0, respectively, during steady motion on horizontal orbits with the form

$$\begin{aligned} \alpha &= \alpha_0 + \epsilon \cos \varpi t, & (72)\\ \beta &= \eta \cos \varpi t, & (73)\\ \dot\phi &= \Omega_0(1 + \delta \cos \varpi t), & (74)\\ \dot\psi &= \gamma \Omega_0 \cos \varpi t, & (75) \end{aligned}$$

where ϵ, δ and γ are small constants.

Inserting the perturbations (72)-(75) into the conserved energy (26), we find that the term in η multiplies $\sin\phi \approx \sin\Omega_0 t$, but that no other terms have this time dependence. We conclude that the pertubation η of angle β must vanish, and set $\eta = 0$ hereafter.

Inserting the perturbations (72)-(75) into eq. (39), we find that

$$-\delta \sin\alpha_0 + \gamma \cos\alpha_0 = 2\epsilon\frac{r \sin\theta}{a}. \tag{76}$$

Inserting the perturbations (72)-(75) into eq. (37), we find that

$$-\delta(2k+1)r'_0 + \gamma(2k+1)z'_0 = 2\epsilon(k+1)a \sin\alpha_0. \tag{77}$$

Equations (76)-(77) yield δ and γ in terms of ϵ as

$$\frac{\delta}{\epsilon} = 2\frac{(2k+1)z_0' r\sin\theta - (k+1)a^2\sin\alpha_0\cos\alpha_0}{(2k+1)a[r\sin(\theta-\alpha_0)-a]} \tag{78}$$

$$\frac{\gamma}{\epsilon} = 2\frac{(2k+1)r_0' r\sin\theta - (k+1)a^2\sin^2\alpha_0}{(2k+1)a[r\sin(\theta-\alpha_0)-a]} \tag{79}$$

The amplitudes δ and γ diverge relative to ϵ when $r\sin(\theta-\alpha_0) = a$, which occurs when α_0 is maximal for a given θ and the disk touches the sphere at two points, as illustrated in Fig. 3. In this case, oscillatory perturbations in α are not possible, and ϵ would be zero.

Inserting the perturbations (72)-(75) into eq. (36), the first-order terms yield the relation

$$\begin{aligned}(k+1)\varpi^2 = & -\frac{g}{a}\sin\alpha_0 - \frac{\Omega_0^2}{a}[(2k+1)r\sin\theta\cos\alpha_0 + (k+1)a(1-2\cos^2\alpha_0)] \\ & -2\frac{\delta}{\epsilon}\frac{\Omega_0^2\sin\alpha_0}{a}[(2k+1)r\sin\theta - (k+1)a\cos\alpha_0] \\ & +\frac{\gamma}{\epsilon}\frac{\Omega_0^2}{a}[(2k+1)r\sin(\theta+\alpha_0) + (k+1)a(1-2\cos^2\alpha_0)].\end{aligned} \tag{80}$$

Using eqs. (78)-(79) for δ and γ, and relating g to Ω_0^2 via eq. (50), we obtain a lengthy expression for ϖ^2 as a function of a, r, k, α_0, Ω_0 and θ. Stability against small oscillations occurs when this function is positive.

The region of stability for a hoop ($k = 1/2$) of radius $a = 0.3$ m inside a sphere of radius $r = 3$ m is shown in Fig. 7. Comparing with Fig. 6, we see that nearly all possible orbits found in sec. 2.3.2 are stable, the exception being orbits in the lower hemisphere ($\pi/2 < \theta < \pi$) where the disk is nearly vertical ($\alpha_0 \approx \pi/2$). The region of physically plausible orbits, for which the required coefficient of friction is less than one, is also indicated in Fig. 7.

For the special case of a disk spinning about a vertical axis (and not rolling) at the bottom of the sphere, $\alpha_0 = \pi/2$ and $\theta = \pi$, and the stability condition $\varpi^2 > 0$ requires that

$$\Omega_0^2 > \frac{g}{(k+1)a}\frac{(2k+1)(r-a)}{(2k+1)r-a}. \tag{81}$$

In the limit that $r \gg a$, the sphere is in effect a horizontal surface, and condition (81) becomes the well-known condition [6] that $\Omega_0^2 > g/(k+1)a$ for stability of a disk spinning about a vertical axis on a horizontal plane. As the radius of the sphere approaches that of the disk, the spinning is stable at lower and lower angular velocity Ω_0.

3 Disk Rolling Outside a Fixed Sphere

The analysis of sec. 2 holds for a disk rolling without slipping on the outside of a sphere with only minor changes. First, the geometric relation (2) for a disk outside a sphere is

$$\theta < \alpha < \pi + \theta. \tag{82}$$

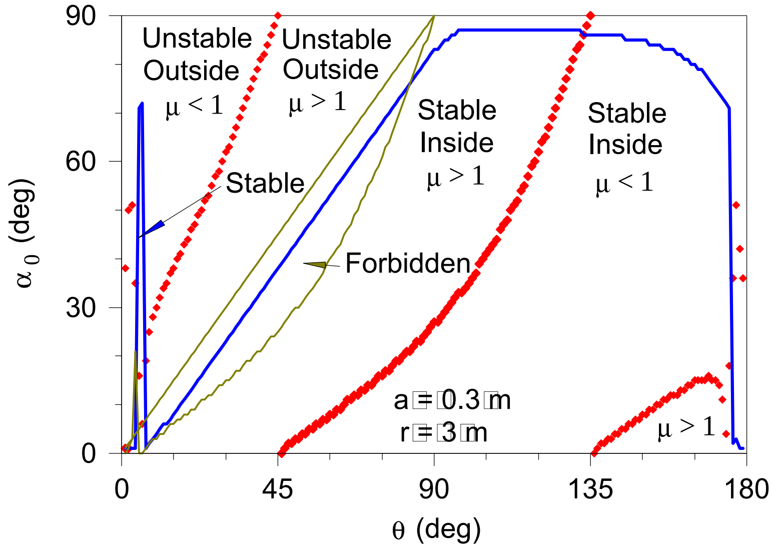

Figure 7: The regions of stability in the α_0-θ plane of small oscillations of a hoop of radius $a = 0.3$ m inside or outside a sphere of radius $r = 3$ m, according to eq. (80). Orbits are forbidden inside the crescentlike region near $\alpha_0 = \theta$, according to eqs. (60) and (82). The regions where a coefficient of friction μ greater or less than unity according to eq. (60) are also shown.

We expect no vertical orbits as the disk will lose contact with the sphere at some point. To verify this, note that the condition $\sin\alpha = 0$ (from eq. (37)) implies that $\alpha = \pi$ when the wheel is outside the sphere. Then eqs. (44)-(46) indicate, for example, that if the disk starts from rest at the top of the sphere it loses contact with the sphere when

$$\cos\phi = \frac{2}{3+2k}. \tag{83}$$

The result for a particle sliding on a sphere ($k = 0$) is well known.

For horizontal orbits, eqs. (50)-(55) are still valid, but the condition that the disk stay in contact with the sphere is that the frictional force have an outward component, which requires

$$r_0'\Omega_0^2 < g\cot\theta. \tag{84}$$

and hence,

$$\cot\alpha_0 < (2k + 1 + k(a/r_0')\cos\alpha_0)\cot\theta. \tag{85}$$

Equation (50) can be satisfied for $\alpha_0 < \pi/2$ so long at the radius of the disk is small enough that $(2k+1)r_0' + ka\cos\alpha_0$ is positive. We must have $\theta < \pi/2$ to have $\alpha_0 < \pi/2$ since $\alpha_0 > \theta$. Hence, horizontal orbits exist only on the upper hemisphere, as expected.

The stability analysis of horizontal orbits in the preceding section holds formally for disks outside spheres. We find that the are no stable orbits for a disk rolling on spheres that are small relative to the radius of the disk. For example, with a hoop of radius $a = 0.3$ m, stable orbits exist for a very restricted range of θ once $r > 3$ m (Fig. 7), and over a wide range of

θ only for $r > 10.7$ m. The rapid change in the region of stability with radius of the sphere is illustrated in Figs. 8 and 9.

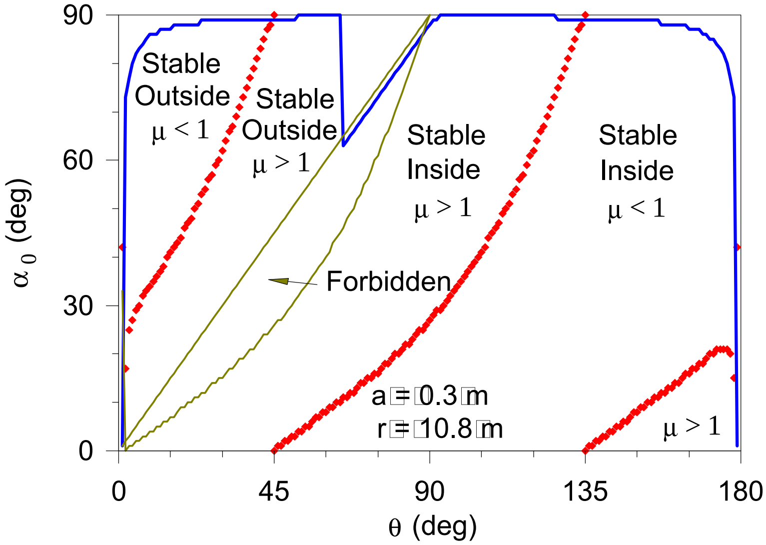

Figure 8: The regions of the α_0-θ plane for which horizontal orbits are stable, and for which the required coefficient of friction is greater or less than 1, for a hoop of radius 0.3 m either inside or outside a sphere of radius 10.8 m.

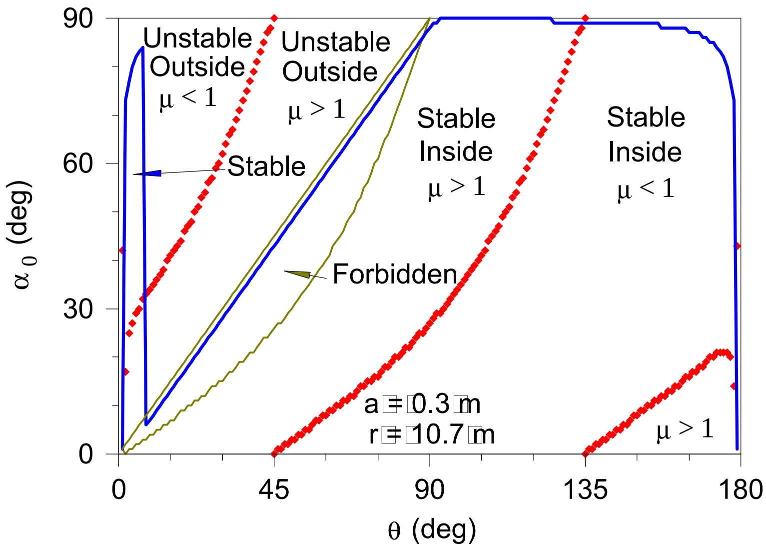

Figure 9: The regions of the α_0-θ plane for which horizontal orbits are stable for a hoop of radius 0.3 m either inside or outside a sphere of radius 10.7 m.

In addition to rolling motion of a disk on the sphere, there is the special case of a disk spinning about a vertical axis (and not rolling) at the top of the sphere, $\alpha_0 = \pi/2$ and $\theta = 0$.

For this, and the stability condition $\varpi^2 > 0$ requires that

$$\Omega_0^2 > \frac{g}{(k+1)a} \frac{(2k+1)(r+a)}{(2k+1)r+a}. \tag{86}$$

Again, in the limit that $r \gg a$, condition (86) becomes that for stability of a disk spinning about a vertical axis on a horizontal plane. A larger spin is required for stability on the top of a sphere than on a plane, but stable spinning is possible even in the limit $r \to 0$.

4 Sphere Rolling Inside a Fixed Sphere

The case of a sphere rolling on horizontal orbits inside a fixed sphere has been treated by Milne [3], Lamb [4] and Routh [6]. For completeness, we give an analysis for orbits of arbitrary inclination to compare and contrast with the case of a disk.

The radius of the rolling sphere is a and that of the fixed sphere is $r > a$. Again the axis normal to the circular arc traced by the point of contact is called $\hat{\mathbf{z}}'$,as shown in Fig. 10, which makes angle β to the vertical $\hat{\mathbf{z}}$. The polar angle of the arc about $\hat{\mathbf{z}}'$ is θ, and ϕ is the azimuth of the point of contact between the two spheres.

The diameter of the rolling sphere that passes through the point of contact must always be normal to the fixed sphere. That is, the "bank" angle of the rolling sphere is always $\theta - \pi/2$ with respect to the plane of the orbit.

We again introduce a right-handed triad of unit vectors $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ centered on the rolling sphere. For consistency with the notation used for the disk, axis $\hat{\mathbf{3}}$ is directed towards the point of contact, axis $\hat{\mathbf{2}}$ is parallel to the plane of the orbit, and axis $\hat{\mathbf{1}}$ is in the $\hat{\mathbf{3}}$-$\hat{\mathbf{z}}'$ plane, as shown in Fig. 10. In general, none of these vectors are body vectors for the rolling sphere. The center of mass of the rolling sphere lies on the line joining the center of the fixed sphere to the point of contact, and so

$$\mathbf{r}_{\rm cm} = (r-a)\ \hat{\mathbf{3}} \equiv r'\ \hat{\mathbf{3}}, \tag{87}$$

We cannot use the decomposition (10) of the total angular velocity into that of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ plus that relative to the triad, because no body vector is readily identifiable to aid in characterizing the relative angular velocity.

Instead, we note that the first form of the rolling constraint (18) still applies to a sphere,

$$\mathbf{v}_{\rm cm} = \frac{d\mathbf{r}_{\rm cm}}{dt} = a\ \hat{\mathbf{3}} \times \boldsymbol{\omega}. \tag{88}$$

Multiplying this by $\hat{\mathbf{3}}$, we find

$$\boldsymbol{\omega} = -\hat{\mathbf{3}} \times \frac{\mathbf{v}_{\rm cm}}{a} + \omega_3\ \hat{\mathbf{3}} \tag{89}$$

Using eq. (87) in this, we have

$$\boldsymbol{\omega} = -\frac{r'}{a}\ \hat{\mathbf{3}} \times \frac{d\hat{\mathbf{3}}}{dt} + \omega_3\ \hat{\mathbf{3}}. \tag{90}$$

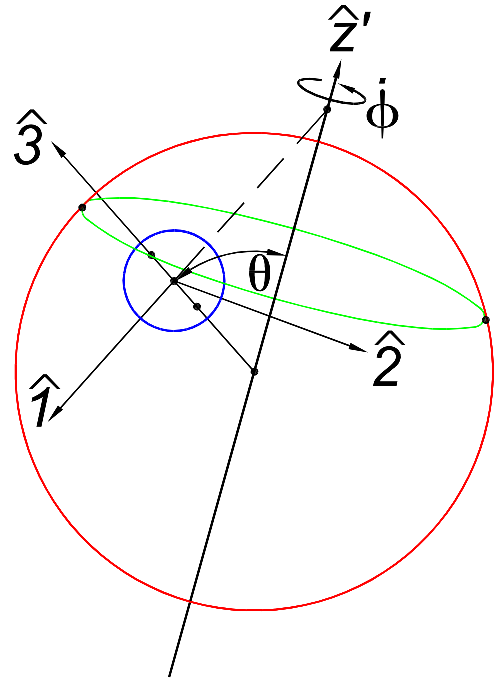

Figure 10: Geometry illustrating the case of a sphere rolling without slipping on a circular orbit perpendicular to the $\hat{\mathbf{z}}'$-axis inside a fixed sphere. The $\hat{\mathbf{3}}$-axis is along the line of centers of the two spheres, which passes through the point of contact and makes angle θ to the z' axis. The $\hat{\mathbf{2}}$-axis lies in the plane of the orbit along the direction of motion of the center of the rolling sphere, and axis $\hat{\mathbf{1}} = \hat{\mathbf{2}} \times \hat{\mathbf{3}}$ is in the $\hat{\mathbf{3}}$-$\hat{\mathbf{z}}'$ plane. The angular velocity of the center of the rolling sphere about the z' axis is $\dot{\phi}$.

We can also characterize the anglular velocity $\boldsymbol{\omega}_{123}$ of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ as

$$\boldsymbol{\omega}_{123} = \dot{\theta}\ \hat{\mathbf{2}} + \dot{\phi}\ \hat{\mathbf{z}}' = -\dot{\phi}\sin\theta\ \hat{\mathbf{2}} + \dot{\theta}\ \hat{\mathbf{2}} + \dot{\phi}\cos\theta\ \hat{\mathbf{3}}, \tag{91}$$

using

$$\hat{\mathbf{z}}' = -\sin\theta\ \hat{\mathbf{1}} + \cos\theta\ \hat{\mathbf{3}}, \tag{92}$$

By definition, the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ does not rotate about the $\hat{\mathbf{2}} \times \hat{\mathbf{z}}$ axis. The time rates of change of the axes are therefore,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{1}} = \dot{\phi}\cos\theta\ \hat{\mathbf{2}} - \dot{\theta}\ \hat{\mathbf{3}}, \tag{93}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{2}} = -\dot{\phi}\cos\theta\ \hat{\mathbf{1}} - \dot{\phi}\sin\theta\ \hat{\mathbf{3}}, \tag{94}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{3}} = \dot{\theta}\ \hat{\mathbf{1}} + \dot{\phi}\sin\theta\ \hat{\mathbf{2}}, \tag{95}$$

The rolling sphere has mass m and moment of inertia kma^2 about any diameter. The angular momentum is, of course,

$$\mathbf{L}_{\rm cm} = kma^2\ \boldsymbol{\omega}, \tag{96}$$

where $\boldsymbol{\omega}$ is the angular velocity of the rolling sphere.

The equations of motion (27)-(29) hold for a rolling sphere as well as for a disk. Using eqs. (87) and (96) we can write eq. (29) as

$$ka\frac{d\boldsymbol{\omega}}{dt} + r'\frac{d^2\hat{\mathbf{3}}}{dt^2} \times \hat{\mathbf{3}} = g\ \hat{\mathbf{3}} \times \hat{\mathbf{z}}. \tag{97}$$

We can now see that $\omega_3 = \boldsymbol{\omega} \cdot \hat{\mathbf{3}}$ is a constant by noting that $\boldsymbol{\omega} \cdot d\hat{\mathbf{3}}/dt = 0$ from eq. (90), and also $\hat{\mathbf{3}} \cdot d\boldsymbol{\omega}/dt = 0$ from eq. (97). The freedom to chose the constant angular velocity ω_3 for a rolling sphere permits stable orbits above the equator of the fixed sphere, just as the freedom to adjust the bank angle α allows such orbits for a disk.

Taking the derivative of eq. (90) we find

$$\frac{d\boldsymbol{\omega}}{dt} = -\frac{r'}{a}\ \hat{\mathbf{3}} \times \frac{d^2\hat{\mathbf{3}}}{dt^2} + \omega_3\frac{d\hat{\mathbf{3}}}{dt}, \tag{98}$$

so the equation of motion (97) can be written

$$(k+1)r'\hat{\mathbf{3}} \times \frac{d^2\hat{\mathbf{3}}}{dt^2} - ka\omega_3\frac{d\hat{\mathbf{3}}}{dt} = g\ \hat{\mathbf{z}} \times \hat{\mathbf{3}}. \tag{99}$$

Routh notes that (the component form of) this equation has the same form as that for a symmetric top with one point fixed [6], and so the usual extensive analysis of nutations about the stable orbits follows if desired.

From eqs. (93)-(95) we find that

$$\hat{\mathbf{3}} \times \frac{d^2\hat{\mathbf{3}}}{dt^2} = (\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta)\ \hat{\mathbf{1}} + (-\ddot{\theta} + \dot{\phi}^2\sin\theta\cos\theta)\ \hat{\mathbf{2}}, \tag{100}$$

and we can use eq. (34) for $\hat{\mathbf{z}} \times \hat{\mathbf{3}}$ if we substitute $\alpha = \theta - \pi/2$ for the rolling sphere:

$$\hat{\mathbf{z}} \times \hat{\mathbf{3}} = -\sin\beta\sin\phi\ \hat{\mathbf{1}} + (\sin\theta\cos\beta - \cos\theta\sin\beta\cos\phi)\ \hat{\mathbf{2}}. \tag{101}$$

The equation of motion (99) has only $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ components, which are

$$(k+1)r'(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) + ka\omega_3\dot{\theta} = g\sin\beta\sin\phi, \tag{102}$$

$$(k+1)r'(-\ddot{\theta} + \dot{\phi}^2\sin\theta\cos\theta) + ka\omega_3\dot{\phi}\sin\theta = g\cos\theta\sin\beta\cos\phi - g\sin\theta\cos\beta. \tag{103}$$

Here, we only consider circular orbits, for which $\dot{\theta} = 0 = \ddot{\theta}$ and the component equations of motion become

$$(k+1)r'\ddot{\phi}\sin\theta = g\sin\beta\sin\phi, \tag{104}$$

and

$$[(k+1)r'\dot{\phi}^2\cos\theta) + ka\omega_3\dot{\phi}]\sin\theta = g\cos\theta\sin\beta\cos\phi - g\sin\theta\cos\beta. \tag{105}$$

The equations (104) and (105) are not consistent in general. To see this, take the derivative of eq. (105) and substitute $\ddot{\phi}$ from eq. (104):

$$ka\omega_3\sin\beta\sin\phi = -3(k+1)r'\dot{\phi}\cos\theta\sin\beta\sin\phi. \tag{106}$$

While this is certainly true for $\beta = 0$ (horizontal orbits), for nonzero β we must have $\dot\phi\cos\theta$ constant since ω_3 is constant. Equation (106) is satisfied for $\theta = \pi/2$ (great circles), but for arbitrary θ we would need $\dot\phi$ constant which is inconsistent with eq. (104). Further, on a great circle eq. (105) becomes $ka\omega_3\dot\phi = -g\cos\beta$. This is inconsistent with eq. (104) unless $\beta = \pi/2$ (vertical great circles) and $\omega_3 = 0$.

In summary, the only possible circular orbits for a sphere rolling within a fixed sphere are horizontal circles and vertical great circles.

We remark further only on the horizontal orbits. For these $\dot\phi \equiv \Omega_0$ is constant according to eq. (104). Equation (105) then yields a quadratic equation for Ω_0:

$$(k+1)r'\Omega_0^2\cos\theta + ka\omega_3\Omega_0 + g = 0, \tag{107}$$

so that there are orbits with real values of Ω_0 provided

$$(ka\omega_3)^2 \ge 4(k+1)gr'\cos\theta. \tag{108}$$

This is satisfied for orbits below the equator ($\theta > \pi/2$) for any value of the "spin" ω_3 of the sphere (including zero), but places a lower limit on $|\omega_3|$ for orbits above the equator. For the orbit on the equator we must have $\Omega_0 = -g/(ka\omega_3)$ so a nonzero ω_3 is required here as well.

The contact force $\mathbf{F}$ is given by

$$\mathbf{F}/m = (g + r'\Omega_0^2\cos\theta)\sin\theta\ \hat{\mathbf{1}} - (r'\Omega_0^2\sin^2\theta - g\cos\theta)\ \hat{\mathbf{3}}, \tag{109}$$

using eqs. (27) and (106). For the rolling sphere to remain in contact with the fixed sphere there must be a positive component of $\mathbf{F}$ pointing toward the center of the fixed sphere. Since axis $\hat{\mathbf{3}}$ is radial outward from the fixed sphere, we require that F_3 be negative, and hence,

$$r'\Omega_0^2\sin^2\theta > g\cos\theta. \tag{110}$$

This is always satisfied for orbits below the equator. For orbits well above the equator this requires a larger value of $|\omega_3|$ than does eq. (108). To see this, suppose ω_3 is exactly at the minimum value allowed by eq. (108), which implies that $\Omega_0 = -ka\omega_3/(2(k+1)r'\cos\theta)$. Then eq. (110) requires that $\tan^2\theta > k+1$. So for $k = 2/5$ and at angles $\theta < 50°$ larger values of $|\omega_3|$ are needed to satisfy eq. (108) than to satisfy eq. (108). However, there are horizontal orbits at any $\theta > 0$ for $|\omega_3|$ large enough.

5 Sphere Rolling Outside a Fixed Sphere

This case has also been treated by Milne [3]. A popular example is spinning a basketball on one's fingertip.

Equations eq. (107) and (108) hold with the substitution that $r' = r + a$. The condition on the contact force becomes

$$r'\Omega_0^2\sin^2\theta < g\cos\theta, \tag{111}$$

which can only be satisfied for $\theta < \pi/2$. While eq. (108) requires a large spin $|\omega_3|$, if it is too large eq. (111) can no longer be satisfied in view of the relation (107). For any case in which the orbit exists a perturbation analysis shows that the motion is stable against small nutations [3].

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