

# Orbital and Spin Angular Momentum of the Fields of a Turnstile Antenna

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## 1 Problem

A “turnstile” antenna [1, 2] consists of a pair of linear dipole antennas oriented at  $90^\circ$  to each other, and driven  $90^\circ$  out of phase, as shown in Fig. 1.

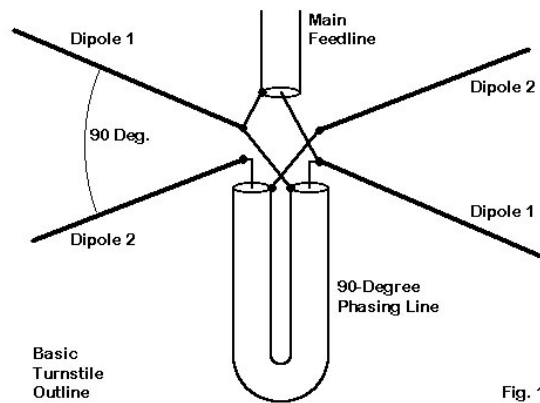


Figure 1: A “turnstile” antenna. From [2].

The linear antennas could be either dipoles as shown in the figure, or simply monopoles. If a pair of loops antennas is used instead, the configuration is called an “eggbeater” antenna.

Consider the case that the length of the linear antennas is small compared to a wavelength, so that it suffices to characterize each antenna by its electric dipole  $\mathbf{p}_{1,2}e^{-i\omega t}$ , where the magnitudes  $p_1$  and  $p_2$  are equal but their phases differ by  $90^\circ$ , the directions of the two moment differs by  $90^\circ$ , *i.e.*,  $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$ , and  $\omega$  is the angular frequency.

Discuss the angular momentum of the fields of a turnstile antenna.

## 2 Solution

We consider a basic turnstile antenna whose component antennas lie in the  $x$ - $y$  plane at a common point. Then, we can write the total electric dipole moment of the antenna system as

$$\mathbf{p} = \mathbf{p}_0 e^{-i\omega t} = p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega t}, \quad (1)$$

In Gaussian units (and in vacuum) these fields can be written as (See, for example, sec. 9.1 of [3])

$$\mathbf{E}(\mathbf{r}, t) = \left( \frac{k^2}{r} (\hat{\mathbf{r}} \times \mathbf{p}_0) \times \hat{\mathbf{r}} + \left( -\frac{ik}{r^2} + \frac{1}{r^3} \right) (3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_0) \right) e^{i(kr - \omega t)}, \quad (2)$$

$$\mathbf{B}(\mathbf{r}, t) = \left( \frac{k^2}{r} + \frac{ik}{r^2} \right) \hat{\mathbf{r}} \times \mathbf{p}_0 e^{i(kr - \omega t)}. \quad (3)$$

where  $\hat{\mathbf{r}} = \mathbf{r}/r$  is the unit vector from the center of the dipole to the observer,  $c$  is the speed of light, and  $k = \omega/c$ . The time-average Poynting vector is, in spherical coordinates  $(r, \theta, \phi)$ ,

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{c \text{Re}(\mathbf{E} \times \mathbf{B}^*)}{8\pi} = \frac{ck^4(1 + \cos^2 \theta)}{8\pi r^2} \hat{\mathbf{r}} \\ &\quad - \frac{c}{8\pi} \text{Re} \left[ \left( \frac{2k^2}{r^4} - \frac{ik^3}{r^3} - \frac{ik}{r^5} \right) (2(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\mathbf{p}_0^* + (3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})(\mathbf{p}_0^* \cdot \hat{\mathbf{r}}) + 2p_0^2) \hat{\mathbf{r}}) \right]. \end{aligned} \quad (4)$$

The time average density of field momentum is  $\langle \mathbf{S} \rangle / c^2$ , so the time-average density of field angular momentum is

$$\langle \mathbf{l} \rangle = \mathbf{r} \times \frac{\langle \mathbf{S} \rangle}{c^2} = -\frac{1}{4\pi c} \text{Re} \left[ \left( \frac{2k^2}{r^3} - \frac{ik^3}{r^2} - \frac{ik}{r^4} \right) (\mathbf{p}_0 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \times \mathbf{p}_0^* \right]. \quad (5)$$

We now restrict our attention to the far zone where the electromagnetic fields are

$$\mathbf{B} = k^2 \frac{e^{i(kr - \omega t)}}{r} \hat{\mathbf{r}} \times \mathbf{p}_0, \quad \mathbf{E} = \mathbf{B} \times \hat{\mathbf{r}}, \quad (6)$$

whose components in spherical coordinates are

$$E_r = B_r = \hat{\mathbf{r}} \cdot \mathbf{B} = 0, \quad (7)$$

$$E_\theta = B_\phi = p_0 k^2 \frac{e^{i(kr - \omega t)}}{r} \cos \theta (\cos \phi + i \sin \phi), \quad (8)$$

$$E_\phi = -B_\theta = -p_0 k^2 \frac{e^{i(kr - \omega t)}}{r} (\sin \phi - i \cos \phi), \quad (9)$$

noting that  $\hat{\mathbf{r}} \times \hat{\mathbf{x}} = \sin \phi \hat{\boldsymbol{\theta}} + \cos \theta \cos \phi \hat{\boldsymbol{\phi}}$ , and  $\hat{\mathbf{r}} \times \hat{\mathbf{y}} = -\cos \phi \hat{\boldsymbol{\theta}} + \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$ . In the plane of the antenna,  $\theta = 90^\circ$ , the electric field has no  $\theta$  component, and hence no  $z$  component; the turnstile radiation in the horizontal plane is horizontally polarized. In the vertical direction,  $\theta = 0^\circ$  or  $180^\circ$ , the radiation is circularly polarized. For intermediate angles  $\theta$  the radiation is elliptically polarized.

The magnitudes of the fields are

$$E = B = \frac{p_0 k^2}{r} \sqrt{1 + \cos^2 \theta}, \quad (10)$$

so the time-averaged radiation pattern is

$$\frac{d\langle P \rangle}{d\Omega} = \frac{cr^2}{8\pi} B^2 = r^2 \langle S_{\text{far}, r} \rangle = \frac{cp_0^2 k^4}{8\pi} (1 + \cos^2 \theta). \quad (11)$$

The intensity of the radiation varies by a factor of 2 over the sphere. Compared to other simple antennas, this pattern is remarkably isotropic. The radiated power is greatest for  $\theta = 0$  or  $180^\circ$  in which directions the polarization is purely circular. The total time-average radiated power is

$$\langle P \rangle = \frac{2cp_0^2 k^4}{3}. \quad (12)$$

The density of angular momentum in the far zone is

$$\langle \mathbf{l}_{\text{far}} \rangle = \frac{k^3}{4\pi cr^2} \text{Re} [i(\mathbf{p}_0 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \times \mathbf{p}_0^*] = -\frac{p_0^2 k^3}{4\pi cr^2} \sin \theta \hat{\boldsymbol{\theta}}. \quad (13)$$

This density flows radially outward at the speed of light, and we can speak of the rate of radiation of angular momentum in the far zone as

$$\frac{d\langle \mathbf{L} \rangle}{dt d\Omega} = cr^2 \langle \mathbf{l}_{\text{far}} \rangle = -\frac{cp_0^2 k^3}{4\pi} \sin \theta \hat{\boldsymbol{\theta}} = \frac{cp_0^2 k^3}{4\pi} (\sin^2 \theta \hat{\mathbf{z}} - \sin \theta \cos \theta \hat{\boldsymbol{\rho}}), \quad (14)$$

where  $\hat{\boldsymbol{\rho}}$  is the radial unit vector in cylindrical coordinates  $(\rho, \phi, z)$ . Integrating over solid angle, we find that

$$\frac{d\langle \mathbf{L} \rangle}{dt} = \frac{2cp_0^2 k^3}{3} \hat{\mathbf{z}} = \frac{\langle P \rangle}{\omega} \hat{\mathbf{z}}, \quad (15)$$

which seems consistent with the notion that photons have angular momentum  $J = \hbar = U/\omega$ .

## 2.1 Orbital and Spin Angular Momentum

The formalism that  $\mathbf{l} = \mathbf{r} \times \mathbf{S}/c^2$  implies that the density (and flow) of angular momentum has no radial component. However, we say that the fields along the  $z$ -axis are circularly polarized (and elliptically polarized in general). This suggests that there should be a description in which we can identify an angular momentum with a radial component for the fields along the  $z$ -axis.

A decomposition has been given in [4, 5] whereby the total field angular momentum can be written in three terms,

$$\int \mathbf{l} d\text{Vol} = \sum_i \mathbf{l}_{\text{canonical}, i} + \int \mathbf{l}_{\text{EM}, \text{orbital}} d\text{Vol} + \int \mathbf{l}_{\text{EM}, \text{spin}} d\text{Vol}, \quad (16)$$

with

$$\mathbf{l}_{\text{canonical}, i} = \mathbf{r} \times \mathbf{p}_{\text{canonical}, i}, \quad \mathbf{l}_{\text{EM}, \text{orbital}} = \mathbf{r} \times \mathbf{p}_{\text{EM}, \text{orbital}}, \quad \mathbf{l}_{\text{EM}, \text{spin}} = \frac{\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}}{4\pi c}, \quad (17)$$

and

$$\mathbf{p}_{\text{EM}, \text{orbital}} = \frac{\sum_{j=1}^3 E_{\text{rot}, j} \nabla A_{\text{rot}, j}}{4\pi c}, \quad (18)$$

where  $\mathbf{A}_{\text{rot}}$  is the (gauge-invariant) rotational part of the (gauge-dependent) vector potential  $\mathbf{A}$ . That is,  $\mathbf{A}_{\text{rot}}$  is the vector potential in the Coulomb gauge.

The electric field is related to the potentials by

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}_{\text{rot}}}{\partial t} \approx ik\mathbf{A}_{\text{rot}}, \quad (19)$$

where the approximation hold in the far zone, noting the Coulomb-gauge scalar potential  $V$  is the instantaneous electric dipole field, which falls off as  $1/r^3$ . Then, in the far zone we have that

$$\begin{aligned} \langle \mathbf{l}_{\text{EM,spin}} \rangle &= \frac{\text{Re}(\mathbf{E}_{\text{rot}} \times \mathbf{A}_{\text{rot}}^*)}{8\pi c} = \frac{\text{Re}(i\mathbf{E}_{\text{rot}} \times \mathbf{E}^*)}{8\pi ck} \\ &= \frac{-p_0^2 k^3}{8\pi cr^2} \text{Im} [((\hat{\mathbf{r}} \times \mathbf{p}_0) \times \hat{\mathbf{r}}) \times ((\hat{\mathbf{r}} \times \mathbf{p}_0^*) \times \hat{\mathbf{r}})] \\ &= \frac{-p_0^2 k^3}{8\pi cr^2} \text{Im} [\mathbf{p}_0 \times \mathbf{p}_0^* + 2i \text{Im}[(\mathbf{p}_0 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \times \mathbf{p}_0^*]] = \frac{p_0^2 k^3}{4\pi cr^2} (\hat{\mathbf{z}} + \sin \theta \hat{\boldsymbol{\theta}}) \\ &= \frac{p_0^2 k^3}{4\pi cr^2} \cos \theta \hat{\mathbf{r}}, \end{aligned} \quad (20)$$

noting that  $\hat{\mathbf{z}} = \cos \theta \mathbf{r} - \sin \theta \hat{\boldsymbol{\theta}}$ . This is an appealing result, in that the spin angular momentum is full strength along the  $z$ -axis and vanishing in the  $x$ - $y$  plane where the polarization is linear.

The orbital angular momentum in the far zone is

$$\langle \mathbf{l}_{\text{EM,orbital}} \rangle = \mathbf{r} \times \frac{\text{Re} \left( \sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}^* \right)}{8\pi c}. \quad (21)$$

Keeping terms only of order  $1/r$  in the far zone, we have that

$$\mathbf{E}_{\text{far,rot}} = ik\mathbf{A}_{\text{far,rot}} = \frac{k^2 e^{i(kr-\omega t)}}{r} (\mathbf{p}_0 - (\mathbf{p}_0 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}) \quad (22)$$

$$\nabla A_{\text{far,rot},j} = -\frac{i}{k} \nabla E_{\text{far,rot},j} = \hat{\mathbf{r}} E_{\text{far,rot},j}, \quad (23)$$

$$\frac{\text{Re} \left( \sum_{j=1}^3 E_{\text{rot},j} \nabla A_{\text{rot},j}^* \right)}{8\pi c} = \frac{|\mathbf{E}_{\text{far,rot}}|^2}{8\pi c} \hat{\mathbf{r}}, \quad (24)$$

$$\langle \mathbf{l}_{\text{EM,orbital}} \rangle = 0. \quad (25)$$

This result is also appealing. The difficulty is that  $\langle \mathbf{l}_{\text{EM,spin}} \rangle + \langle \mathbf{l}_{\text{EM,orbital}} \rangle$  does not equal  $\langle \mathbf{l}_{\text{far}} \rangle$  of eq. (13).

*It is tempting to consider that the formalism of [4, 5] is superior.*

## References

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