Static-Voltage Gauge

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1 Problem

Show that it is also possible to (re)define the scalar potential V of electrodynamics to have no time dependence, such that the time-varying part of the electric field is entirely due to the vector potential $\mathbf{A}$.

2 Solution

In electrostatics the electric field $\mathbf{E}$ can be related to a (static) scalar potential V according to

$$\mathbf{E} = -\boldsymbol{\nabla} V_0, \tag{1}$$

and inversely,

$$V_a - V_b = -\int_b^a \mathbf{E} \cdot d\mathbf{l} \tag{2}$$

expresses the fact that a unique voltage difference $V_a - V_b$ can be defined for any pair of points a and b independent of the path of integration between them. The static electric field is said to be conservative, and eqs. (1)-(2) are equivalent to the vector calculus relation

$$\boldsymbol{\nabla} \times \mathbf{E} = 0. \tag{3}$$

In electrodynamics Faraday discovered (as later interpreted by Maxwell) that eq. (3) must be generalized to

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

in SI units, which implies that time-dependent magnetic fields $\mathbf{B}$ lead to additional electric fields beyond those associated with the scalar potential V. The nonexistence (so far as we know) of isolated magnetic charges (monopoles) implies that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{5}$$

and hence that the magnetic field can be related to a vector potential $\mathbf{A}$ according to

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{6}$$

Using eq. (6) in (4), we can write

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0, \tag{7}$$

which implies that $\mathbf{E} + \partial \mathbf{A}/\partial t$ can be related to a scalar potential V as $-\boldsymbol{\nabla} V$, *i.e.*,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} . \tag{8}$$

We restrict our discussion to media for which the dielectric permittivity is ϵ_0 and the magnetic permeability is μ_0. Then, using eq. (8) in the Maxwell equation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{9}$$

leads to

$$\boldsymbol{\nabla}^2 V + \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0} , \tag{10}$$

and using eqs. (6) and (8) in the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \tag{11}$$

leads to

$$\boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) . \tag{12}$$

Suppose that the charge and current densities ρ and $\mathbf{J}$ consist of time-independent terms plus terms with time dependence $e^{-i\omega t}$. That is,

$$\rho = \rho_0 + \rho_\omega e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{J} = \mathbf{J}_0 + \mathbf{J}_\omega e^{-i\omega t}. \tag{13}$$

Then, eq. (10) indicates that we can choose that the scalar potential $V = V_0 + V_\omega e^{-i\omega t}$ obeys the static relation

$$\boldsymbol{\nabla}^2 V = -\frac{\rho_0}{\epsilon_0} , \qquad V_\omega = 0, \tag{14}$$

provided the vector potential $\mathbf{A} + \mathbf{A}_0 + \mathbf{A}_\omega e^{-i\omega t}$ obeys the gauge condition

$$\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A} = -i\omega \boldsymbol{\nabla} \cdot \mathbf{A}_\omega e^{-i\omega t} = -\frac{\rho_\omega e^{-i\omega t}}{\epsilon_0} , \tag{15}$$

i.e.,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_\omega = -\frac{i\rho_\omega}{\epsilon_0 \omega} . \tag{16}$$

We also choose that the time-independent part $\mathbf{A}_0$ of the vector potential satisfies the usual condition of magnetostatics,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_0 = 0, \tag{17}$$

in which case eq. (12) shows that the vector potentials obeys the relations

$$\boldsymbol{\nabla}^2 \mathbf{A}_0 = -\mu_0 \mathbf{J}_0, \qquad \text{and} \qquad \boldsymbol{\nabla}^2 \mathbf{A}_\omega + k^2 \mathbf{A}_\omega = -\mu_0 \mathbf{J}_\omega - \frac{i \boldsymbol{\nabla} \rho_\omega}{\epsilon_0 \omega} . \tag{18}$$

The formal solutions to equations (14) and (18) are

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_0(\mathbf{r}')}{R}\, d\mathrm{Vol}', \tag{19}$$

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_0(\mathbf{r}')}{R}\, d\mathrm{Vol}', \tag{20}$$

and

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_\omega(\mathbf{r}')e^{ikR}}{R}\, d\mathrm{Vol}' + \frac{i}{4\pi\epsilon_0\omega}\int \frac{\boldsymbol{\nabla}\rho_\omega(\mathbf{r}')e^{ikR}}{R}\, d\mathrm{Vol}'\,, \tag{21}$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

While the forms (19)-(21) are not used in practice, they show how it is possible to define the scalar potential V to be purely static, such that the time-dependent voltage V_ω is always zero.

The conditions (16)-(17) on $\boldsymbol{\nabla}\cdot\mathbf{A}$ are the so-called gauge conditions of the static-voltage gauge. An interesting review of other gauge conditions is given in [1].

In electrostatics, one can invert the present problem and set the scalar potential to zero and derive the electric field from the vector potential $\mathbf{A} = t\boldsymbol{\nabla}V$, *where* V *would be the scalar potential when* $\mathbf{A} = 0$ *[2].*

References

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