Energy Balance in an Electrostatic Accelerator

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1 Problem

The principle of an electrostatic accelerator is that when a charge e escapes from a conducting plane that supports a uniform electric field of strength E_0, then the charge gains energy eE_0d as it moves distance d from the plane. Where does this energy come from?

Show that the mechanical energy gain of the electron is balanced by the decrease in the electrostatic field energy of the system.

2 Solution

Once the charge has reached distance d from the plane, the static electric field $\mathbf{E}_e$ at an arbitrary point $\mathbf{r}$ due to the charge can be calculated by summing the field of the charge plus its image charge,

$$\mathbf{E}_e(\mathbf{r}, d) = \frac{e\mathbf{r}_1}{r_1^3} - \frac{e\mathbf{r}_2}{r_2^3}, \tag{1}$$

where $\mathbf{r}_1$ ($\mathbf{r}_2$) points from the charge (image) to the observation point $\mathbf{r}$, as illustrated in Fig. 1. The total electric field is then $E_0\hat{\mathbf{z}} + \mathbf{E}_e$.

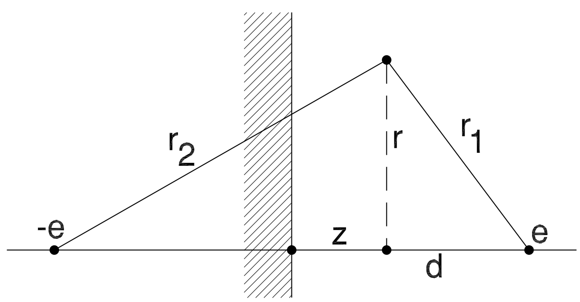

Figure 1: The charge e and its image charge $-e$ at positions $(r, \theta, z) = (0, 0, \pm d)$ with respect to a conducting plane at $z = 0$. Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed from the charges to the observation point $(r, 0, z)$.

It turns out to be convenient to use a cylindrical coordinate system, where the observation point is $\mathbf{r} = (r, \theta, z) = (r, 0, z)$, and the charge is at $(0, 0, d)$. Then,

$$r_{1,2}^2 = r^2 + (z \mp d)^2. \tag{2}$$

The part of the electrostatic field energy that varies with the position of the charge is the interaction term (in Gaussian units),

$$
\begin{aligned}
U_{\rm int} &= \int \frac{E_0 \hat{\mathbf{z}} \cdot \mathbf{E_e}}{4\pi} d{\rm Vol} \\
&= \frac{eE_0}{4\pi} \int_0^\infty dz \int_0^\infty \pi dr^2 \left(\frac{z-d}{[r^2+(z-d)^2]^{3/2}} - \frac{z+d}{[r^2+(z+d)^2]^{3/2}} \right) \\
&= \frac{eE_0}{4} \int_0^\infty dz \left(\left\{ \begin{array}{ll} 2 & \text{if } z > d \\ -2 & \text{if } z < d \end{array} \right\} - 2 \right) \\
&= -eE_0 \int_0^d dz = -eE_0 d. \qquad (3)
\end{aligned}
$$

When the particle has traversed a potential difference $V = E_0 d$, it has gained energy eV and the electromagnetic field has lost the same energy.[1]

3 Comments

In a practical "electrostatic" accelerator, electrons (of charge $-e$) are freed from rest on an electrode at potential $-V$ and emerge with energy eV into a region of zero potential beyond the ground electrode. However, the electrons can not be brought to the negative electrode from a region of zero potential by purely electrostatic forces, since these forces oppose the desired transport. An "electrostatic" accelerator must have an essential component (such as a battery) that has a nonelectrostatic force that can do work against the electrostatic field while moving the electron from potential 0, so as to put the charge at rest at potential $-V$ prior to acceleration.

The nonelectrostatic component also provides the energy that is stored at potential energy eV when an electron has been placed on the negative electrode. Can we say more precisely where this potential energy is stored?

In a pair of interesting papers [1, 2], Leon Brillouin discusses the theme of mass renormalization in classical electrodynamics.[2] He argues that the electrostatic potential energy

[1] In general, the uniform electric field $\mathbf{E}_0$ terminates at a planar electrode at $z = D$. In this case, the electric field E_e associated with the charge e at $z = d < D$ can be deduced from an infinite set of image charges; $+e$ at $z = 2nD + d$ and $-e$ at $z = 2nD - d$, where $n = 0, \pm 1, \pm 2, ...$ To calculate the interaction energy $U_{\rm int}$ it is convenient to group these charges into pairs whose positions are symmetric about $z = 0$. Pairs that have charge $+e$ at $z > 0$ have z coordinates $\pm(2nD + d)$ where now the integer n takes on only the values 0,1,2,3,..., while pairs that have charge $-e$ at $z > 0$ have z coordinates $\pm(2mD - d)$ where $m = 1, 2, 3, ...$ The interaction energy for each pair can be calculated as in eq. (3), with the integration in z only between 0 and D. For $n = 0$ the integral is $-eE_0 d$ as above. For pairs with $+e$ at $z > 0$ and $n > 0$ the integral is $-eE_0 D$, and for pairs with $-e$ at $z > 0$ (and $m > 0$) the integral is $eE_0 D$. The interaction energies cancel for each pair of pairs with $n = m$, and the total interaction energy remains $U_{\rm int} = -eE_0 d$.

[2] Brillouin's consideration of classical mass renormalization was closely, but apparently independently, followed by a different vision. Namely, that if a charge e is in potential V in such a way that its total energy is the same as if V were zero, then the mechanical mass of the charge is reduced by amount eV/c^2. If in addition, the charge has a velocity $\mathbf{v}$, and hence a mechanical momentum, then that momentum is lower than when $V = 0$ by $e\mathbf{v}V/c^2$. This phenomenon is sometimes called "hidden momentum" [3].

eV contributes an amount eV/c^2 to the total "relativistic mass" of the charge, according to Einstein's insight that $E = mc^2$. In his second paper [2], Brillouin considers the example of an electrostatic accelerator (where the potential energy eV can exceed the rest energy m_0c^2 of an electron). If Brillouin were correct, the relativistic mass of an electron would be greater than γm_0 when it has been accelerated to velocity $v = c\sqrt{1 - 1/\gamma^2}$ and is still within the region of nonzero electric potential. Therefore, the magnitude of the acceleration would be less than if the relativistic mass of the electron were γm_0, and the acceleration would be slower than expected is the usual analysis. Once the electron has left the accelerator and is in a region of zero potential, its velocity obeys the usual relation $\gamma m_0c^2 = eV$.

But, is Brillouin correct? He implies that the Lorentz force law is not

$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m_0 \mathbf{v})}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(Lorentz)}, \tag{4}$$

but rather

$$\frac{d\mathbf{p}'}{dt} = \frac{d(\gamma m_0 + eV/c^2)\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(Brillouin)}. \tag{5}$$

The great success in the application of the usual Lorentz force law to relativistic particle accelerators argues against the validity of Brillouin's proposed classical mass renormalization.

Furthermore, the present problem shows that there is a decrease in the interaction field energy as the electron is accelerated. In the spirit of Brillouin, should we consider this energy to contribute to the mass of the electron? If so, the electron could acquire a negative mass while still inside an accelerator that uses sufficiently high voltage, which seems preposterous.

References

[1] L. Brillouin, *The Actual Mass of Potential Energy. A Correction to Classical Relativity*, Proc. Nat. Acad. Sci. **53**, 475 (1965),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brillouin_pnas_53_475_65.pdf

[2] L. Brillouin, *The Actual Mass of Potential Energy, II*, Proc. Nat. Acad. Sci. **53**, 1280 (1965),
http://puhep1.princeton.edu/~mcdonald/examples/EM/brillouin_pnas_53_1280_65.pdf

[3] K.T. McDonald, *"Hidden" Momentum of a Steady Current Distribution in a System at "Rest"* (Apr. 21, 2009), http://puhep1.princeton.edu/~mcdonald/examples/current.pdf