

Why Doesn't a Steady Current Loop Radiate?

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(December 1, 2001)

1 Problem

A steady current in a circular loop presumably involves a large number of electrons in uniform circular motion. Each electron undergoes accelerated motion, and individual accelerated charges emit radiation. Yet, the current density \mathbf{J} is independent of time in the limit of a continuous current distribution, and therefore does not radiate. How can we reconcile these two views?

1.1 Comments and Hints

The answer must be that the radiation is canceled by destructive interference between the radiation fields of the large number N of electrons that make up the steady current.

A single electron in uniform circular motion emits electric dipole radiation, whose power is proportional to the square of the acceleration $a = v^2/r$, and hence to $(v/c)^4$. But, the electric dipole moment vanishes for two electrons in uniform circular motion at opposite ends of a common diameter; quadrupole radiation is the highest multipole in this case, with power proportional to $(v/c)^6$. It is suggestive that in case of 3 electrons 120° apart in uniform circular motion the (time-dependent) quadrupole moment vanishes, and the highest multipole radiation is octupole. For N electrons evenly spaced around a ring, the highest multipole that radiates is the N th, and the power of this radiation is proportional to $(v/c)^{2N+2}$. Then, for steady motion with $v/c \ll 1$, the radiated power of a ring of N electrons is very small.

Verify this argument with a detailed calculation.

Recall the basic expression for the vector potential of the radiation fields,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{[\mathbf{J}]}{r} d\text{Vol}' \approx \frac{1}{cR} \int [\mathbf{J}] d\text{Vol}' = \frac{1}{cR} \int \mathbf{J}(\mathbf{r}', t' = t - r/c) d\text{Vol}', \quad (1)$$

where R is the (large) distance from the observer to the center of the ring of radius a . For uniform circular motion of N electrons with angular frequency ω , the current density \mathbf{J} is a periodic function with period $T = 2\pi/\omega$, so a Fourier analysis can be made where

$$\mathbf{J}(\mathbf{r}', t') = \sum_{l=-\infty}^{\infty} \mathbf{J}_l(\mathbf{r}') e^{-il\omega t'}, \quad (2)$$

with

$$\mathbf{J}_l(\mathbf{r}') = \frac{1}{T} \int_0^T \mathbf{J}(\mathbf{r}', t') e^{il\omega t'} dt'. \quad (3)$$

Then,

$$\mathbf{A}(\mathbf{r}, t) = \sum_l \mathbf{A}_l(\mathbf{r}) e^{-il\omega t}, \quad (4)$$

etc.

The radiated power follows from the Poynting vector [1],

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} R^2 |\mathbf{B}|^2 = \frac{c}{4\pi} R^2 |\nabla \times \mathbf{A}|^2. \quad (5)$$

One must be careful in going from a Fourier analysis of an amplitude, such as \mathbf{B} , to a Fourier analysis of an intensity that depends on the square of the amplitude. A Fourier analysis of the average power radiated during one period T can be given as

$$\begin{aligned} \frac{d\langle P \rangle}{d\Omega} &= \frac{1}{T} \int_0^T \frac{dP}{d\Omega} dt = \frac{cR^2}{4\pi T} \int_0^T |\mathbf{B}|^2 dt = \frac{cR^2}{4\pi T} \int_0^T \mathbf{B}^* \sum_l \mathbf{B}_l e^{-il\omega t} dt \\ &= \frac{cR^2}{4\pi} \sum_l \mathbf{B}_l \frac{1}{T} \int_0^T \mathbf{B}^* e^{-il\omega t} dt = \frac{cR^2}{4\pi} \sum_{l=-\infty}^{\infty} \mathbf{B}_l \mathbf{B}_l^* \\ &= \frac{cR^2}{2\pi} \sum_{l=0}^{\infty} |\mathbf{B}_l|^2 \equiv \sum_{l=0}^{\infty} \frac{dP_l}{d\Omega}. \end{aligned} \quad (6)$$

That is, the Fourier components of the time-averaged radiated power can be written

$$\frac{dP_l}{d\Omega} = \frac{cR^2}{2\pi} |\mathbf{B}_l|^2 = \frac{cR^2}{2\pi} |\nabla \times \mathbf{A}_l|^2 = \frac{cR^2}{2\pi} |ilk\hat{\mathbf{n}} \times \mathbf{A}_l|^2, \quad (7)$$

where $k = \omega/c$ and $\hat{\mathbf{n}}$ points from the center of the ring to the observer.

Evaluate the Fourier components of the vector potential and of the radiated power first for a single electron, and then for N electrons evenly spaced around the ring. It will come as no surprise that a 3-dimensional problem with charges distributed on a ring leads to Bessel functions, and we must be aware of the integral representation

$$J_l(s) = \frac{i^l}{2\pi} \int_0^{2\pi} e^{il\phi - is \cos \phi} d\phi. \quad (8)$$

Use the asymptotic expansion for large index and small argument,

$$J_l(lx) \approx \frac{(ex/2)^l}{\sqrt{2\pi l}} \quad (l \gg 1, x \ll 1), \quad (9)$$

to verify the suppression of the radiation for large N .

This problem was first posed (and solved via series expansions without explicit mention of Bessel functions) by J.J. Thomson [2]. He knew that atoms (in what we now call their ground state) don't radiate, and used this calculation to support his model that the electric charge in an atom must be smoothly distributed. This was a classical precursor to the view of a continuous probability distribution for the electron's position in an atom.

Thomson's work was followed shortly by an extensive treatise by G.A. Schott [3], that included analyses in term of Bessel functions correct for any value of v/c .

These pioneering works were largely forgotten during the following era of nonrelativistic quantum mechanics, and were reinvented around 1945 when interest emerged in relativistic particle accelerators. See Arzimovitch and Pomeranchuk [4], and Schwinger [5].

2 Solution

The solution given here follows the succinct treatment by Landau, sec. 74 of [6].

For charges in steady motion at angular frequency ω in a ring of radius a , the current density \mathbf{J} is periodic with period $T = 2\pi/\omega$, so the Fourier analysis (2) at the retarded time t' can be evaluated via the usual approximation that $r \approx R - \mathbf{r}' \cdot \hat{\mathbf{n}}$, where R is the distance from the center of the ring to the observer, \mathbf{r}' points from the center of the ring to the electron, and $\hat{\mathbf{n}}$ is the unit vector pointing from the center of the ring to the observer. Then,

$$\begin{aligned} [\mathbf{J}] &= \mathbf{J}(\mathbf{r}', t' = t - r/c) = \sum_l \mathbf{J}_l(\mathbf{r}') e^{-il\omega(t - R/c + \mathbf{r}' \cdot \hat{\mathbf{n}}/c)} \\ &= \sum_l e^{il(kR - \omega t)} \mathbf{J}_k(\mathbf{r}') e^{-ik\omega \mathbf{r}' \cdot \hat{\mathbf{n}}/c}, \end{aligned} \quad (10)$$

where $k = \omega/c$.

The ring lies in the plane $z = 0$, centered on the origin. We use rectangular coordinates (x, y, z) , cylindrical coordinates (ρ, ϕ, z) , and spherical coordinates (r, θ, ϕ) with angle θ measured with respect to the $+z$ axis. Then,

$$\mathbf{r}' = \rho(\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}), \quad \hat{\mathbf{n}} = \sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{z}}, \quad \text{and} \quad \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}. \quad (11)$$

We first consider a single electron, whose azimuth varies as $\phi = \omega t + \phi_0$, and whose velocity is, of course, $v = a\omega$. The current density of a point electron of charge q can be written in a cylindrical coordinate system (with volume element $\rho d\rho d\phi dz$) using Dirac delta functions as

$$\mathbf{J} = \rho_{\text{charge}} v \hat{\boldsymbol{\phi}} = qv \delta(\rho - a) \delta(z) \delta(\phi - \omega t - \phi_0) \hat{\boldsymbol{\phi}}. \quad (12)$$

The Fourier components \mathbf{J}_l are given by

$$\mathbf{J}_l = \frac{1}{T} \int_0^T \mathbf{J}(\mathbf{r}, t) e^{ilt} dt = qv \delta(\rho - a) \delta(z) \frac{e^{il(\phi - \phi_0)}}{\rho \omega T} \hat{\boldsymbol{\phi}}. \quad (13)$$

Using eqs. (11) and (13) in (10) and noting that $\omega T = 2\pi$, we find

$$[\mathbf{J}] = \frac{qv}{2\pi\rho} \sum_l e^{il(kR - \omega t)} e^{il(\phi - \phi_0 - \omega\rho \sin \theta \cos \phi/c)} \delta(\rho - a) \delta(z) \hat{\boldsymbol{\phi}}. \quad (14)$$

Inserting this in eq. (1), we have

$$\begin{aligned} \mathbf{A} &\approx \frac{1}{cR} \int [\mathbf{J}] \rho d\rho d\phi dz = \frac{qv}{2\pi cR} \sum_l e^{il(kR - \omega t - \phi_0)} \int_0^{2\pi} e^{il(\phi - \omega a \sin \theta \cos \phi/c)} \hat{\boldsymbol{\phi}} d\phi \\ &= \sum_l \mathbf{A}_l e^{-il\omega t}, \end{aligned} \quad (15)$$

so that the Fourier components of the vector potential are

$$\mathbf{A}_l = \frac{qv}{2\pi cR} e^{il(kR - \phi_0)} \int_0^{2\pi} e^{il(\phi - v \sin \theta \cos \phi/c)} (-\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}) d\phi. \quad (16)$$

The integrals yield Bessel functions with the aid of the integral representation (8). The $\hat{\mathbf{y}}$ part of eq. (16) can be found by taking the derivative of this relation with respect to s :

$$J'_l(s) = -\frac{i^{l+1}}{2\pi} \int_0^{2\pi} e^{il\phi - is \cos \phi} \cos \phi \, d\phi, \quad (17)$$

For the $\hat{\mathbf{x}}$ part of eq. (16) we play the trick

$$\begin{aligned} 0 &= \int_0^{2\pi} e^{i(l\phi - s \cos \phi)} d(l\phi - z \cos \phi) \\ &= l \int_0^{2\pi} e^{il\phi - is \cos \phi} \, d\phi + s \int_0^{2\pi} e^{il\phi - is \cos \phi} \sin \phi \, d\phi, \end{aligned} \quad (18)$$

so that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{il\phi - is \cos \phi} \sin \phi \, d\phi = -\frac{l}{s} \frac{1}{2\pi} \int_0^{2\pi} e^{il\phi - is \cos \phi} \, d\phi = -\frac{l}{i^l s} J_l(s). \quad (19)$$

Using eqs. (17) and (19) with $s = lv \sin \theta / c$ in (16) we have

$$\mathbf{A}_l = \frac{qv}{cR} e^{i l(kR - \phi_0)} \left(\frac{1}{i^l v \sin \theta / c} J_l(lv \sin \theta / c) \hat{\mathbf{x}} - \frac{1}{i^{l+1}} J'_l(lv \sin \theta / c) \hat{\mathbf{y}} \right). \quad (20)$$

We skip the calculation of the electric and magnetic fields from the vector potential, and proceed immediately to the angular distribution of the radiated power according to eq. (7),

$$\begin{aligned} \frac{dP_l}{d\Omega} &= \frac{cR^2}{2\pi} |ilk\hat{\mathbf{n}} \times \mathbf{A}_l|^2 = \frac{ck^2 l^2 R^2}{2\pi} |\hat{\mathbf{n}} \times \mathbf{A}_l|^2 \\ &= \frac{ck^2 l^2 R^2}{2\pi} \left(\cos^2 \theta |A_{l,x}|^2 + |A_{l,y}|^2 \right) \\ &= \frac{cq^2 k^2 l^2}{2\pi} \left(\cot^2 \theta J_l^2(lv \sin \theta / c) + \frac{v^2}{c^2} J_l'^2(lv \sin \theta / c) \right). \end{aligned} \quad (21)$$

The present interest in this result is for $v/c \ll 1$, but in fact it holds for any value of v/c . As such, it can be used for a detailed discussion of the radiation from a relativistic electron that moves in a circle, which emits so-called synchrotron radiation. This topic is discussed further in Lecture 20 of the Notes [7].¹ Furthermore, eq. (21) holds even if the velocity v exceeds the speed of light c/n in a medium of index of refraction n , in which case a kind of synchrotron-Čerenkov radiation is emitted [8]. Since each amplitude \mathbf{A}_l varies as $1/R$ at large distance R from the source, the total radiated power varies as $1/R^2$.²

We now turn to the case of N electrons uniformly spaced around the ring. The initial azimuth of the n th electron can be written

$$\phi_n = \frac{2\pi n}{N}. \quad (22)$$

¹If the electron of mass m moves in a circle due to static magnetic field B , then the angular velocity is given by $\omega = kc = v/a = qB/\gamma mc$, such that $k^2 = (q^2 B^2 / m^2 c^4)(1 - v^2/c^2)$, and eq. (21) agrees with eq. (74.8) of [6], noting that our θ is $\pi/2 - \theta$ there.

²This is in contrast to claims [9] that the power varies as $1/R$ for Čerenkov radiation emitted by a particle in uniform circular motion.

The l th Fourier component of the total vector potential is simply the sum of components (20) inserting ϕ_n in place of ϕ_0 :

$$\begin{aligned} \mathbf{A}_l &= \sum_{n=1}^N \frac{qv}{cR} e^{i l(kR - \phi_n)} \left(\frac{1}{i^l v \sin \theta / c} J_l(lv \sin \theta / c) \hat{\mathbf{x}} - \frac{1}{i^{l+1}} J'_l(lv \sin \theta / c) \hat{\mathbf{y}} \right) \\ &= \frac{qv e^{i l k R}}{cR} \left(\frac{1}{i^l v \sin \theta / c} J_l(lv \sin \theta / c) \hat{\mathbf{x}} - \frac{1}{i^{l+1}} J'_l(lv \sin \theta / c) \hat{\mathbf{y}} \right) \sum_{n=1}^N e^{-i 2\pi l n / N}. \end{aligned} \quad (23)$$

This sum vanishes unless l is a multiple of N , in which case the sum is just N . The lowest nonvanishing Fourier component has order N , and the radiation is at frequency $N\omega$. We recognize this as N th-order multipole radiation, whose radiated power follows from eq. (21) as

$$\frac{dP_N}{d\Omega} = \frac{c q^2 k^2 N^2}{2\pi} \left(\cot^2 \theta J_N^2(Nv \sin \theta / c) + \frac{v^2}{c^2} J_N'^2(Nv \sin \theta / c) \right). \quad (24)$$

For large N but $v/c \ll 1$ we can use the asymptotic expansion (9), and its derivative,

$$J'_l(lx) \approx \frac{(ex/2)^l}{\sqrt{2\pi l} x} \quad (l \gg 1, x \ll 1), \quad (25)$$

to write eq. (24) as

$$\frac{dP_N}{d\Omega} \approx \frac{c q^2 k^2 N}{4\pi^2 \sin^2 \theta} \left(\frac{e v}{2 c} \sin \theta \right)^{2N} (1 + \cos^2 \theta) \ll N \frac{dP_{E1}}{d\Omega} \quad (N \gg 1, v/c \ll 1). \quad (26)$$

In eqs. (25) and (26) the symbol e inside the parentheses is not the charge but rather the base of natural logarithms, 2.718...

For currents in, say, a loop of copper wire, $v \approx 1$ cm/s, so $v/c \approx 10^{-10}$, while $N \approx 10^{23}$. The radiated power predicted by eq. (26) is extraordinarily small!

Note, however, that this nearly complete destructive interference depends on the electrons being uniformly distributed around the ring. Suppose instead that they were distributed with random azimuths ϕ_n . Then the square of the magnetic field at order m has the form

$$|\mathbf{B}_m|^2 \propto \left| \sum_{n=1}^N e^{-im\phi_n} \right|^2 = N + \sum_{l \neq n} e^{-im(\phi_l - \phi_n)} = N. \quad (27)$$

Thus, for random azimuths the power radiated by N electrons (at any order) is just N times that radiated by one electron.

If the charge carriers in a wire were localized to distances much smaller than their separation, radiation of “steady” currents could occur. However, in the quantum view of metallic conduction, such localization does not occur.

The random-phase approximation is relevant for electrons in a so-called storage ring, for which the radiated power is a major loss of energy – or source of desirable photon beams of synchrotron radiation, depending on one’s point of view. We do not expound here on the interesting topic of the “formation length” for radiation by relativistic electrons, which length sets the scale for interference of multiple electrons. See, for example, [10].

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