Laser Tweezers

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1 Problem

It is well known that a charged particle cannot be held a rest by purely electrostatic fields (Earnshaw's theorem [1, 2]). Give a simple classical explanation of how a neutral atom of polarizability α can be "trapped" at the focus of a laser beam.

a) First, ignore magnetic interactions, and deduce that there is a (time-averaged) trapping force dependent on the electric field of the laser.

b) Atoms have some probability of absorbing photons from the laser beam, thereby being kicked along the direction of the beam. This processes can be modelled classically by supposing that the polarizability of the atom has an imaginary part: $\alpha = \alpha' + i\alpha''$. Deduce the (time-averaged) force on an atom along the direction of propagation of a linearly polarized plane electromagnetic wave in terms of α'', the imaginary (absorptive) part of the polarizability.

c) For an idealized atom with a single natural frequency ω_0, deduce the ratio α'/α'' at the frequency ω for which the real part, α', of the polarizibility is a maximum. For this, you may use a classical model of an atom as an electron on a spring of frequency ω_0, subject to a damping force $-\gamma m\dot{\mathbf{x}}$, where $\gamma \ll \omega_0$ is the reciprocal of the lifetime of the 'excited state'.

d) In practice, the trapping force a) must be larger than the longitudinal force b). This requires the laser beam to be tightly focused. Deduce the $f_\#$ of the lens needed for trapping under the conditions of part c).

2 Solution

This concept was proposed by Ashkin in 1977 [3], and first realized by Ashkin *et al.* in 1985 [4]. Aspects of this problem have been discussed in the Journal from a semiclassical view [5]. Here, a completely classical approach displays the essential results quickly.

Undergraduate laboratory experiments with laser tweezers have been described in [6].

a) The important hint is that the atom is polarizable, so it takes on an induced dipole moment $\mathbf{p} = \alpha\mathbf{E}$ where E is the electric field strength, and α is the atomic polarizability. We suppose this simple relation holds for a wave field as well as for a static field.

We now want the force $\mathbf{F}$ on the dipole. This is obtained from the form $(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$ which holds even when $\nabla\times\mathbf{E} \neq 0$,

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} = \alpha(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E} = \frac{\alpha}{2}\boldsymbol{\nabla}E^2. \tag{1}$$

So the polarizable atom can be "trapped" at a point where E^2 takes on a local maximum, since the force (1) is restoring for departures in any direction from that point.

There is no such place in a charge-free region of an electrostatic field, as demonstrated in the Appendix. Thus, Earnshaw's theorem can be extended to include the case of a polarizable atom.

However, E^2 can have a local maximum in a nonelectrostatic field, such as at the focus of a laser beam.

For oscillatory fields, it is appropriate to restrict the discussion to the time-averaged behavior of the atom. The time-average of the trapping force (1) is written

$$\langle \mathbf{F} \rangle = \frac{\alpha}{2} \boldsymbol{\nabla} \left\langle E^2 \right\rangle . \tag{2}$$

b) The induced, oscillating dipole is equivalent to an oscillating charge with velocity along the direction of $\mathbf{E}$. Then the $\mathbf{v} \times \mathbf{B}$ force is in the direction of $\mathbf{E} \times \mathbf{B}$, *i.e.*, along the direction of propagation of the wave. This is consistent with the force being due to absorption of photons from the wave.

Since it is stated that the polarizability has an imaginary part, we describe the plane wave using complex notation. The wave is polarized in, say, the x direction: $\mathbf{E}(\mathbf{x}, t) = E_0 \hat{\mathbf{x}} e^{i(kz - \omega t)}$. It suffices to consider the dipole as being at the origin, where

$$\mathbf{E} = E_0 \hat{\mathbf{x}} e^{-i\omega t}, \qquad \mathbf{B} = E_0 \hat{\mathbf{y}} e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{p} = \alpha E_0 \hat{\mathbf{x}} e^{-i\omega t} = e\mathbf{x}(t). \tag{3}$$

The oscillating dipole is equivalent to a charge e at distance $\mathbf{x}(t)$ from the nucleus. The velocity of the charge is, of course, $\dot{\mathbf{x}}$. Then, the Lorentz force on the moving charge is (in Gaussian units)

$$\mathbf{F} = e\frac{\dot{\mathbf{x}}}{c} \times \mathbf{B} = \frac{\dot{\mathbf{p}}}{c} \times \mathbf{B} \tag{4}$$

More precisely, since we are using complex notation, the time-average force is

$$\langle \mathbf{F} \rangle = \frac{1}{2} Re \left(\frac{\dot{\mathbf{p}}}{c} \times \mathbf{B}^{\star} \right) = \frac{1}{2} Re \left(-i\frac{\omega}{c} \alpha E_0^2 \right) \hat{\mathbf{z}} = \frac{k}{2} \alpha'' E_0^2 \hat{\mathbf{z}}, \tag{5}$$

where $k = 2\pi/\lambda$ is the wave number.

c) To model the polarizability, it suffices to consider the response of the model atom to the electric field. The equation of motion is then

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{eE_0}{m} e^{-i\omega t}. \tag{6}$$

The trial solution $x = x_0 e^{-i\omega t}$ leads to

$$x_0 = \left(\frac{eE_0}{m} \right) \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}. \tag{7}$$

Since the magnitude of the dipole moment is $p_0 = ex_0 = \alpha E_0$, the polarizability $\alpha = \alpha' + i\alpha''$ is

$$\alpha' = \left(\frac{e^2}{m}\right) \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \qquad \alpha'' = \left(\frac{e^2}{m}\right) \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{8}$$

Since $\alpha'' > 0$, the force (5) is in the $+z$ direction, as expected for photon absorption from a wave that moves in the $+z$ direction.

To find the frequency at which α' is maximum, we take the derivative:

$$0 = \frac{-2\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} - \frac{(\omega_0^2 - \omega^2)[-4\omega(\omega_0^2 - \omega^2) + 2\omega\gamma^2]}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^2}. \tag{9}$$

Thus, $\gamma\omega_0 = \omega_0^2 - \omega^2$ at the maximum, and α' is maximal for ω slightly less than ω_0;

$$\omega_{\max} \approx \omega_0 - \gamma/2. \tag{10}$$

Approximating $\gamma\omega_{\max}$ by $\gamma\omega_0$ since $\gamma \ll \omega_0$, we have

$$\alpha'(\omega_{\max}) = \frac{e^2}{m}\frac{1}{2\gamma\omega_0} = \alpha''(\omega_{\max}). \tag{11}$$

The desired ratio is $\alpha'/\alpha'' = 1$.

d) Comparing the results for a) and b), for trapping to work we must have

$$\frac{\alpha'}{2}\frac{\partial \langle E^2 \rangle}{\partial z} > \frac{\pi}{\lambda}\alpha'' E_0^2 = \frac{2\pi}{\lambda}\alpha'' \left\langle E^2 \right\rangle. \tag{12}$$

From optics [7], we know that the characteristic length for changes in the electric field along the axis near the focus is the Rayleigh range,

$$z_0 \approx \pi f_{\#}^2 \lambda, \tag{13}$$

where the $f_{\#}$ is the usual f/D ratio of the focusing lens. That is, we can approximate

$$\frac{\partial \langle E^2 \rangle}{\partial z} \approx \frac{\langle E^2 \rangle}{z_0} \approx \frac{\langle E^2 \rangle}{\pi f_{\#}^2 \lambda}. \tag{14}$$

The maximum trapping force occurs about one Rayleigh range away from the focus. Using (14) in (12), we find the requirement

$$f_{\#} < \frac{1}{2\pi}\sqrt{\frac{\alpha'}{\alpha''}}. \tag{15}$$

From part c), if we run the trap at the frequency $\omega = \omega_0 - \gamma/2$ where the trapping term is maximal, the detrapping term is large: $\alpha'' = \alpha'$. In this case, we need

$$f_{\#} < \frac{1}{2\pi} = 0.16, \tag{16}$$

which is a very strong focus! For $\omega < \omega_0 - \gamma/2$, the ratio α'/α'' grows rapidly, and a softer focus can be used. But the trapping is not so strong, so other detrapping effects become important.

3 Appendix

We show that an electrostatic field $\mathbf{E}$ (or magnetostatic field $\mathbf{B}$) cannot have a local maximum of $E^2 = |\mathbf{E}|^2$ at a charge-free point.

The demonstration makes use of the mean-value theorem [8], that the average value of the electrostatic field in a charge-free sphere is equal to the value of the field at the center of the sphere.

If E^2 has a local maximum at some point P in a charge-free region, then there is a nonzero r such that $E^2 < E^2(P)$ for all points (other than P) within a sphere of radius r about P. Consequently, $E < E(P)$ in that sphere.

Let $\hat{\mathbf{z}}$ point along $\mathbf{E}(P)$. Then the mean-value theorem can be written

$$\int E_z \; d\mathrm{Vol} = \frac{4\pi r^3}{3} E(P), \tag{17}$$

for the sphere about P. In general, $E_z \le E$, and by assumption $E < E(P)$ for all points other than P within the sphere, so

$$\int E_z \; d\mathrm{Vol} \le \int E \; d\mathrm{Vol} < \int E(P) \; d\mathrm{Vol} = \frac{4\pi r^3}{3} E(P), \tag{18}$$

which contradicts eq. (17). Hence, E^2 cannot be locally maximal at P.

However, E^2 can take on a local minimum. This has been shown by explicit examples in Ref. [9], which also provides an alternative demonstration that E^2 cannot have a local maximum. A brief discussion of this issue in Ref. [10] shows that $\nabla^2 E^2 \ge 0$, which does not exclude the possibility that at a maximum both the first and second derivatives of E^2 vanish. However, as noted in Refs. [10] and [11], the condition that $\nabla^2 E^2 \ge 0$ is sufficient to exclude the possibility of trapping.

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