

Lorentz Invariance of the Number of Photons in a Rectangular Cavity

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1 Problem

In sec. 8 of his 1905 paper introducing special relativity [1], Einstein noted that “It is remarkable that the energy and the frequency of a light complex vary with the state of motion of the observer in accordance with the same law.” In that paper he did not interpret the ratio of energy to frequency as the number of photons in the “light complex”, but this notion was likely on his mind. We are led to the appealing conclusion that the number of photons associated with a “light complex” is Lorentz invariant.

Einstein’s argument was for a finite volume in which the fields were approximated as a monochromatic plane wave. Details of the fields near the boundary of the volume were not considered. A variant of Einstein’s argument was given by Zeldovich [2], but again its application to a well-defined volume that contained the fields was not pursued. Avron *et al.* [3] considered a box filled with monochromatic waves in its rest frame, and noted that one can obtain conflicting results as to the number of photons according to an observer moving with respect to the box.

Consider a hollow rectangular box of superconducting material with inner dimensions $d_x \times d_y \times d_z$ in its rest frame, where it contains electromagnetic energy U a single cavity mode of angular frequency ω . The number of photons in this classical electromagnetic field is said to be $N = U/\hbar\omega$, where \hbar is Planck’s constant. Show that if the box is observed to have uniform velocity \mathbf{v} , the observer reports the same number of photons. You may assume that the box is large compared to the wavelength of the fields within, and that the motion is parallel to an edge of the box.

2 Solution

2.1 The Cavity Is at Rest

The standing-wave solutions for electromagnetic fields \mathbf{E} and \mathbf{B} of angular frequency ω inside a superconducting, rectangular cavity of extent $0 < x < d_x$, $0 < y < d_y$, $0 < z < d_z$ can be written (in Gaussian units) as¹

¹See, for example, <http://puhep1.princeton.edu/~mcdonald/examples/ph501/ph501lecture14.pdf>

$$\begin{aligned} E_x &= E_{0x} \cos k_x x \sin k_y y \sin k_z z e^{-i\omega t}, \\ E_y &= E_{0y} \sin k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \end{aligned} \quad (1)$$

$$\begin{aligned} E_z &= E_{0z} \sin k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \\ B_x &= B_{0x} \sin k_x x \cos k_y y \cos k_z z e^{-i\omega t}, \\ B_y &= B_{0y} \cos k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \\ B_z &= B_{0z} \cos k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \end{aligned} \quad (2)$$

where the wave vector is^{2,3}

$$\mathbf{k} = (k_x, k_y, k_z) = \left(\frac{l\pi}{d_x}, \frac{m\pi}{d_y}, \frac{n\pi}{d_z} \right), \quad \text{and} \quad k = \frac{\omega}{c}, \quad (3)$$

for any set of integers $\{l, m, n\}$, and c is the speed of light in vacuum (and inside the cavity). Further, Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \text{implies} \quad \mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0, \quad (4)$$

so that

$$\mathbf{B}_0 \cdot \mathbf{E}_0 = 0, \quad |B_0| = |E_0|, \quad \text{and} \quad \mathbf{B}_0 = \frac{(k_y E_{0z} - k_z E_{0y}, k_z E_{0x} - k_x E_{0z}, k_x E_{0y} - k_y E_{0x})}{k}. \quad (5)$$

The free-space Maxwell equations $\nabla \cdot \mathbf{B} = 0 = \nabla \cdot \mathbf{E}$ imply that

$$\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 = \hat{\mathbf{k}} \cdot \mathbf{B}_0. \quad (6)$$

For each set of integers $\{l, m, n\}$ there are two orthogonal polarizations of the vectors \mathbf{E}_0 and \mathbf{B}_0 . As usual, the physical fields are the real parts of eqs. (1)-(2).

The electromagnetic energy stored in the cavity is

$$U = \frac{1}{2} \int \frac{|E|^2 + |B|^2}{8\pi} d\text{Vol} = \frac{d_x d_y d_z |E_0|^2}{64\pi}, \quad (7)$$

independent of the polarization of the fields, and of the mode indices $\{l, m, n\}$. The classical number of photons in the cavity (at rest) is therefore

$$N = \frac{U}{\hbar\omega} = \frac{d_x d_y d_z |E_0|^2}{64\pi \hbar\omega}. \quad (8)$$

²The fields (1)-(2) can be decomposed into eight plane waves with the eight wave vectors $(\pm l\pi/d_x, \pm m\pi/d_y, \pm n\pi/d_z)$.

³Nontrivial cavity modes exist only with two or three of $\{l, m, n\}$ nonzero, so there is no cavity mode in which the wave vector \mathbf{k} is parallel to a wall of the box. Hence, the arguments of Einstein [1] and of Avron *et al.* [3] are not strictly applicable to the present example.

2.2 The Cavity Has Velocity $\mathbf{v} = v \hat{\mathbf{x}}'$.

We now consider the cavity as observed in the ' frame, in which it has velocity $\mathbf{v} = v \hat{\mathbf{x}}'$.

The fields inside the cavity are no longer perceived as standing waves, but rather as the sum of waves of two different frequencies. To prepare for this, we re-express the standing wave (1)-(2) in the rest frame of the cavity as a sum of waves traveling in the $+x$ and $-x$ directions: $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, $\mathbf{B} = \mathbf{B}_+ + \mathbf{B}_-$, where

$$\mathbf{E}_+ = (E_{0x} \sin k_y y \sin k_z z, -iE_{0y} \cos k_y y \sin k_z z, -iE_{0z} \sin k_y y \sin k_z z) \frac{e^{i(k_x x - \omega t)}}{2}, \quad (9)$$

$$\mathbf{E}_- = (-E_{0x} \sin k_y y \sin k_z z, iE_{0y} \cos k_y y \sin k_z z, iE_{0z} \sin k_y y \sin k_z z) \frac{e^{-i(k_x x + \omega t)}}{2}, \quad (10)$$

$$\mathbf{B}_+ = (-iB_{0x} \cos k_y y \cos k_z z, B_{0y} \sin k_y y \cos k_z z, B_{0z} \cos k_y y \sin k_z z) \frac{e^{i(k_x x - \omega t)}}{2}, \quad (11)$$

$$\mathbf{B}_- = (iB_{0x} \cos k_y y \cos k_z z, -B_{0y} \sin k_y y \cos k_z z, -B_{0z} \cos k_y y \sin k_z z) \frac{e^{-i(k_x x + \omega t)}}{2}. \quad (12)$$

We now need expressions for the fields (9)-(12) in the ' frame. The phase of a plane wave is a Lorentz invariant, so we can write

$$\mathbf{E}' = \mathbf{E}'_+ + \mathbf{E}'_- = \mathbf{E}'_+(y, z) e^{i(k'_x x' - \omega'_+ t')} + \mathbf{E}'_-(y, z) e^{i(k'_x x' - \omega'_- t')}, \quad (13)$$

$$\mathbf{B}' = \mathbf{B}'_+ + \mathbf{B}'_- = \mathbf{B}'_+(y, z) e^{i(k'_x x' - \omega'_+ t')} + \mathbf{B}'_-(y, z) e^{i(k'_x x' - \omega'_- t')}. \quad (14)$$

The frequency transformations are

$$\omega'_\pm = \gamma(\omega \mp k_x v) = \gamma\omega \left(1 \mp \frac{k_x v}{k c}\right). \quad (15)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Waves that move in the $+x'$ (right) direction have the lower frequency ω'_+ .

The transformations of the electromagnetic fields are

$$E'_x = E_x, \quad E'_y = \gamma \left(E_y + \frac{v}{c} B_z\right), \quad E'_z = \gamma \left(E_z - \frac{v}{c} B_y\right), \quad (16)$$

and

$$B'_x = B_x, \quad B'_y = \gamma \left(B_y - \frac{v}{c} E_z\right), \quad B'_z = \gamma \left(B_z + \frac{v}{c} E_y\right), \quad (17)$$

To calculate the electromagnetic energy inside the moving cavity at time t' we need

$$\begin{aligned} & |E'|^2 + |B'|^2 \\ &= |E_x|^2 + |B_x|^2 + \gamma^2 \left[\left(1 + \frac{v^2}{c^2}\right) (|E_y|^2 + |E_z|^2 + |B_y|^2 + |B_z|^2) + 4\frac{v}{c} \text{Re}(E_y B_z^* - E_z B_y^*) \right]. \end{aligned} \quad (18)$$

The volume of the moving cavity is $d_x d_y d_z / \gamma$, due to the Lorentz contraction. Also, the cross terms due to fields with different frequencies will vanish in the calculation the time-average energy stored in the moving cavity, so we can write

$$\langle U' \rangle = \langle U'_+ \rangle + \langle U'_- \rangle, \quad (19)$$

where

$$\begin{aligned}
\langle U'_{\pm} \rangle &= \frac{1}{2} \int \frac{|E_{\pm}|^2 + |B_{\pm}|^2}{8\pi} d\text{Vol}' \\
&= \frac{d_x d_y d_z}{256\pi\gamma} \left\{ |E_{0x}|^2 + |B_{0x}|^2 \right. \\
&\quad \left. + \gamma^2 \left[\left(1 + \frac{v^2}{c^2} \right) (|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2) + 4\frac{v}{c} \text{Re}(iE_{0z}B_{0y}^* - iE_{0y}B_{0z}^*) \right] \right\} \\
&= \frac{\gamma d_x d_y d_z}{256\pi} \left\{ \left(1 - \frac{v^2}{c^2} \right) (|E_{0x}|^2 + |B_{0x}|^2) + \left(1 + \frac{v^2}{c^2} \right) (|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2) \right\} \\
&= \frac{\gamma d_x d_y d_z}{256\pi} \left\{ 2|E_0|^2 - \frac{v^2}{c^2} (|E_{0x}|^2 + |B_{0x}|^2 - |E_{0y}|^2 - |E_{0z}|^2 - |B_{0y}|^2 - |B_{0z}|^2) \right\}, \quad (20)
\end{aligned}$$

recalling from eq. (4) that vectors \mathbf{E}_0 and \mathbf{B}_0 are relatively real. Using eq. (5), we have

$$\begin{aligned}
&|E_{0x}|^2 + |B_{0x}|^2 - |E_{0y}|^2 - |E_{0z}|^2 - |B_{0y}|^2 - |B_{0z}|^2 \\
&= 2|E_0|^2 - 2|E_{0y}|^2 - 2|E_{0z}|^2 - 2|B_{0y}|^2 - 2|B_{0z}|^2 \\
&= 2|E_0|^2 - 2|E_{0y}|^2 - 2|E_{0z}|^2 \\
&\quad - \frac{2}{k^2} (k_z^2 |E_{0x}|^2 + k_x^2 |E_{0z}|^2 - 2k_x k_z \text{Re}(E_{0x} E_{0z}^*)) \\
&\quad - \frac{2}{k^2} (k_y^2 |E_{0x}|^2 + k_x^2 |E_{0y}|^2 - 2k_x k_y \text{Re}(E_{0x} E_{0y}^*)) \\
&= 2|E_0|^2 - 2|E_{0y}|^2 - 2|E_{0z}|^2 \\
&\quad - \frac{2}{k^2} (k^2 |E_{0x}|^2 + k_x^2 |E_0|^2 - 2k_x \text{Re}[E_{0x}(k_x E_{0x}^* + k_y E_{0y}^* + k_z E_{0z}^*)]) \\
&= -\frac{2k_x^2}{k^2} |E_0|^2. \quad (21)
\end{aligned}$$

Thus,

$$\langle U'_{\pm} \rangle = \frac{\gamma d_x d_y d_z}{128\pi} \left(1 - \frac{k_x^2 v^2}{k^2 c^2} \right) |E_0|^2. \quad (22)$$

The energies are the same in the waves moving in the $+x'$ and $-x'$ directions.

The numbers of classical photons in the moving cavity are, recalling eq. (15),

$$N'_{\pm} = \frac{\langle U'_{\pm} \rangle}{\hbar\omega_{\pm}} = \frac{\gamma d_x d_y d_z}{128\pi\hbar} \left(1 - \frac{k_x^2 v^2}{k^2 c^2} \right) \frac{|E_0|^2}{\gamma\omega(1 \mp k_x v/kc)} = \frac{N}{2} \left(1 \pm \frac{k_x v}{k c} \right). \quad (23)$$

While there are more photons of lower frequency (right-moving) than higher (left-moving) in the moving box, the total number $N' = N'_+ + N'_- = N$ of photons observed at a time t' is the same as when the box is at rest (at a time t).⁴

⁴For observations in the $'$ frame at time $t' = 0$, the corresponding times of observation in the rest frame are $t = 0$ for the left wall at $x = 0$ and $t = -vd_x/c^2$ for the right wall at $x = d_x$. The $'$ observer counts some photons near the right wall as being right-moving at time $t' = 0$ ($t < 0$), while the rest frame observer will count these as left-moving at the later time $t = 0$ after they have reflected off the right wall. That is, the $'$ observer counts more right-moving (lower-energy) than left-moving photons.

2.3 Is the Field Energy Part of the Rest Energy of the System?

A side issue of possible interest is the question of whether the field energy U can/should be considered as contributing to the “rest” energy of the system, such that

$$U_0 = Mc^2 + U, \quad (24)$$

in the rest frame of the box, whose rest mass without fields inside is M .

If so, then in the frame where the box has velocity $\mathbf{v} = v \hat{\mathbf{x}}$, the energy of the system should be

$$U' = \gamma U_0 = \gamma(Mc^2 + U). \quad (25)$$

However, in this frame the box itself has energy

$$U'_{\text{box}} = \gamma Mc^2, \quad (26)$$

and the fields have energy

$$U' = \gamma U \left(1 - \frac{k_x^2 v^2}{k^2 c^2} \right), \quad (27)$$

so that the actual energy of the system is

$$U' = \gamma \left[Mc^2 + U \left(1 - \frac{k_x^2 v^2}{k^2 c^2} \right) \right], \quad (28)$$

rather than expression (25).

Missing from the above analysis is the energy associated with the stress on the walls of the box due to the “radiation pressure” of the fields inside. Presumably, if these stresses, and their Lorentz transformations, are included in the analysis, a meaningful rest energy can be defined for the system. For a simpler example of this, see [4].

References

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