

# Transverse Waves on an Inelastic Vertical String

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## 1 Problem

What are the frequencies of small transverse oscillations in a vertical plane of an inelastic string of length  $l$  and linear mass density  $\lambda$  whose upper point is fixed at a point in a uniform gravitational field of strength  $g$ ?

Estimate the lowest oscillation frequency via Rayleigh's energy method using, say, a trial waveform  $s(y) = l^p - y^p$  for  $y$  measured upwards from the lower end of the string, where  $p$  is to be optimized.

## 2 Solution

The equilibrium state of the string is, of course, that it hangs vertically, with its lower end at  $y = 0$  and its upper end at  $y = l$ .

The tension in the string is

$$T(y) = \lambda gy. \quad (1)$$

The equation of motion for a transverse displacement  $s(y, t)$  in a vertical plane of a segment  $dy$  of the string is

$$\lambda dx \ddot{s} = T(y + dy)s'(y + dy) - T(y)s'(y) = \frac{\partial T s'}{\partial y} dy = \lambda g \frac{\partial(y s')}{\partial y} dy \quad (2)$$

For oscillations at angular frequency  $\omega$  of the form  $s(y, t) = s(y)e^{i\omega t}$ , eq. (2) reduces to

$$\frac{d(y s')}{dy} + \frac{\omega^2}{g} s = y \frac{d^2 s}{dy^2} + \frac{ds}{dy} + \frac{\omega^2}{g} s = 0. \quad (3)$$

This is a form of Bessel's equation of order zero, as can be seen using the substitution  $x = \sqrt{y}$ , with which eq. (3) becomes

$$x^2 \frac{d^2 s}{dx^2} + x \frac{ds}{dx} + \frac{4\omega^2}{g} x^2 s = 0, \quad (4)$$

whose solutions are

$$s(y) = s_0 J_0(2\omega \sqrt{y/g}). \quad (5)$$

The condition that  $s(y = l) = 0$  determine a series of frequencies of small oscillation,

$$2\omega \sqrt{\frac{l}{g}} = 2.405, 5.520, 8.654, \dots, \quad (6)$$

or

$$\omega = 1.202\sqrt{\frac{g}{l}}, 2.760\sqrt{\frac{g}{l}}, 4.318\sqrt{\frac{g}{l}}, \dots \quad (7)$$

Rayleigh notes that for a springlike system,  $\langle \text{KE} \rangle = \langle \text{PE} \rangle$  (virial theorem), so that a trial waveform with parameter  $p$  can be used to estimate the frequency  $\omega(p)$  using this constraint. Then the lowest frequency is obtained by minimizing  $\omega(p)$  with respect to the parameter  $p$ .

We consider the form

$$s(y, t) = (l^p - y^p)e^{i\omega t}, \quad (8)$$

for which the time-average kinetic energy is

$$\begin{aligned} \langle \text{KE} \rangle &= \left\langle \int_0^l \frac{\lambda s'^2}{2} dy \right\rangle = \frac{\lambda \omega^2}{4} \int_0^l (l^p - y^p)^2 dy = \frac{\lambda \omega^2}{4} l^{2p+1} \left( 1 - \frac{2}{p+1} + \frac{1}{2p+1} \right) \\ &= \frac{\lambda \omega^2}{4} l^{2p+1} \frac{2p^2}{(p+1)(2p+1)}, \end{aligned} \quad (9)$$

and the time-average potential energy (= work done in stretching the string) is

$$\langle \text{PE} \rangle = \left\langle \int_0^l T(\sqrt{1 + s'^2} - 1) dy \right\rangle \approx \left\langle \int_0^l \frac{T s'^2}{2} dy \right\rangle = \frac{\lambda g}{4} \int_0^l y(-py^{p-1})^2 dy = \frac{\lambda g}{4} l^{2p} \frac{p}{2}. \quad (10)$$

Equating the kinetic and potential energies, we have that

$$\omega^2(p) = \frac{g(p+1)(2p+1)}{l \cdot 4p}. \quad (11)$$

The minimum frequency occurs for  $p = 1/\sqrt{2}$ , which implies that its value is

$$\omega \approx \sqrt{\frac{g}{l}} \sqrt{\frac{1.707 \cdot 2.414}{2.828}} = 1.207\sqrt{\frac{g}{l}}, \quad (12)$$

which compares well with the “exact” value of  $1.202\sqrt{g/l}$ .