Wagon in the Rain

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Discuss the motion of a wagon that moves on a horizontal plane without friction but which collects falling rain at a constant rate $dm/dt = k$.

2 Solution

This problem was discussed by Tiersten [1] in a thoughtful commentary of variable-mass problems, and may be the inspiration of the subtler example of the motion of a leaky tank car [2].

We suppose for simplicity that the rain falls vertically, and the water collected in the wagon has no motion relative to the latter, and that the wagon moves along a straight line (or in a circler of constant radius). The rate of collection of rain in the wagon is independent of its speed v. The horizontal momentum p (or angular momentum rp in case of motion in a circle of radius r) is constant,

$$p = m_0 v_0 = mv = (m_0 + kt)v, \tag{1}$$

which immediately tells us that the wagon slows down according to

$$v(t) = \frac{m_0}{m_0 + kt} v_0 = \frac{m_0}{m(t)} v_0. \tag{2}$$

The wagon takes an infinite time to come to rest, and travels logarithmically infinite distance,

$$x(t) = \int_0^t v\,dt = m_0 v_0 \int_0^t \frac{dt}{m_0 + kt} = \frac{m_0 v_0}{k} \ln \frac{m_0 + kt}{m_0} = \frac{m_0 v_0}{k} \ln \frac{m(t)}{m_0}\,. \tag{3}$$

If the rain lasts only for time T, the final velocity of, and distance x traveled during time T by, the wagon are obtained from the above with $t = T$.

3 Comment

Taking the time derivative of eq. (1) we find

$$(m_0 + kt)\frac{dv}{dt} = m(t)a = -k\,v, \tag{4}$$

where a is the acceleration of the wagon plus collected water. This form is suggestive, but its interpretation is problematic, as noted by Tiersten [1].

One can note that the velocity of the rain relative to the wagon is $-v$, so that in the instantaneous rest frame of the wagon the rain transfers momentum to the wagon at rate $dp'/dt' = -kv$, which effect can be interpreted as a force $F' = -kv$. The frame of the wagon is accelerated, so the total force on the wagon in this frame is $F'_{\rm total} = -m(t')a - kv = 0$. In the "lab" frame (in which the rain falls vertically), we are led to say that the rain exerts horizontal force $F = F' = -kv$ on the wagon plus the collected rain, which is a reaction to the wagon giving momentum to the collected rain at rate kv.

The difficulty with this association is that the total horizontal force on the total system (wagon + water) is zero. We can say that there are two equal and opposite forces, kv and $-kv$, but it is delicate to say what these forces act on. The force kv acts on the water that is "newly collected" in the wagon, while the force $-kv$ acts on the wagon plus water already collected. However, the notion of "newly collected" water is ill defined in the continuum limit; if the mass of the wagon plus collected water is $m(t) = m_0 + kt$, then the mass of the "newly collected" water must be zero, and its acceleration is infinite during the collection process. Instead, if we think of individual drops of mass Δm of rain hitting the wagon and being accelerated from horizontal velocity zero to v during some small but finite time Δt, each drop experiences a large but finite force $v\Delta m/\Delta t$ from the water already collected in the wagon; and if N drops hit the wagon per second, the average force on the drops is $N\ \Delta t \cdot v\Delta m/\Delta t = kv$ with $k = N\Delta m$. These statements are logically consistent, but they invoke an implausible physical limit as $N \to \infty$ and $\Delta m,\ \Delta t \to 0$ with $N\Delta m = k$.

It seems more straightforward just to do the analysis (1)-(3) based on the fact that in the lab frame $F_{\rm total} = 0 = dp_{\rm total}/dt$ without interpreting the results in terms of $F = ma$.

References

[1] M.S. Tiersten, *Force, Momentum Change and Motion*, Am. J. Phys. **37**, 82 (1969), http://puhep1.princeton.edu/~mcdonald/examples/mechanics/tiersten_ajp_37_82_69.pdf

[2] K.T. McDonald, *Motion of a Leaky Tank Car* (Dec. 4, 1989), http://www.hep.princeton.edu/~mcdonald/examples/tankcar.pdf